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Topology, Domain Theory and Theoretical Computer Science

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Abstract

In this paper, we survey the use of order-theoretic topology in theoretical computer science, with an emphasis on applications of domain theory. Our focus is on the uses of order-theoretic topology in programming language semantics, and on problems of potential interest to topologists that stem from concerns that semantics generates.

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1 Introduction

Topology has proved to be an essential tool for certain aspects of theoretical computer science. Conversely, the problems that arise in the computational setting have provided new and interesting stimuli for topology. These problems also have increased the interaction between topology and related areas of mathematics such as order theory and topological algebra. In this paper, we outline some of these interactions between topology and theoretical computer science, focusing on those aspects that have been most useful to one particular area of theoretical computation – denotational semantics.

This paper began with the goal of highlighting how the interaction of order and topology plays a fundamental role in programming semantics and related areas. It also started with the viewpoint that there are many purely topological notions that are useful in theoretical computer science that *could* be highlighted and which could attract the attention of topologists to this area. And, to be sure, there are many interesting and appealing applications of "pure topology" – certainly in the form of metric space arguments – that have been made to theoretical computer science. But, as the work evolved, it became clear that the main applications of topology to the area of programming

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semantics involve not just topology alone, but also involve order in an essential way. And, this approach places domain theory at center stage, since it is the area that has combined order and topology for application to theoretical computation most effectively. The goal now is to show why this is so, and why it is order-theoretic topology that has had such a large impact on theoretical computer science. In particular, we highlight those aspects of domain theory and its relationship to topology that have proved to be of greatest utility and importance. At the same time, we document the advantages domain theory enjoys in this area of application, and the "standard" that domain theory has set in providing solutions to the problems this area of application poses.

2 Topology versus Order

Let's begin with a simple but illustrative example.

Example 2.1 Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers, and let $(\mathbb{N} \rightharpoonup \mathbb{N})$ denote the set of *partial functions* from $\mathbb{N}$ to itself. Consider the family of partial functions $f_n \in (\mathbb{N} \rightharpoonup \mathbb{N})$ defined by

$$f_n(m) = \begin{cases} m! & \text{if } m < n, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We would like to assert that the functions $\{f_n\}_{n\in\mathbb{N}}$ converge to the function $\mathrm{FAC}\colon \mathbb{N} \to \mathbb{N}$ by $\mathrm{FAC}(m) = m!$ $(\forall m \in \mathbb{N})$.

To find a suitable topology on $(\mathbb{N} \rightharpoonup \mathbb{N})$ to express this convergence, we first identify $(\mathbb{N} \rightharpoonup \mathbb{N})$ with a space of total functions. Let $\perp$ be an element not in $\mathbb{N}$, and define $\mathbb{N}_\perp = \mathbb{N} \cup \{\perp\}$. We interpret $\perp$ as *undefined*, and we define an injection

$$f \mapsto f_\perp\colon (\mathbb{N} \rightharpoonup \mathbb{N}) \to (\mathbb{N}_\perp \to \mathbb{N}_\perp) \text{ by } f_\perp(x) = \begin{cases} f(x) & \text{if } f(x) \text{ is defined,} \\ \perp & \text{otherwise.} \end{cases}$$

Then it is clear that $\{f_\perp \mid f \in (\mathbb{N} \rightharpoonup \mathbb{N})\}$ is precisely the set of selfmaps of $\mathbb{N}_\perp$ that are *strict* – i.e., those that take $\perp$ to itself. If we endow $\mathbb{N}_\perp$ with the discrete topology, then all the functions in $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$ are continuous.

Proposition 2.2 *In the compact-open topology on* $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$*, the sequence* $\{f_{n\perp}\}_{n\in\mathbb{N}}$ *converges to* $\mathrm{FAC}_\perp$.

Proof. Since $\mathbb{N}_\perp$ is discrete, the compact-open topology on $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$ is the same as the topology of pointwise convergence, so the result is clear. □

We also note that by endowing $\mathbb{N}_\perp$ with the discrete metric and giving $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$ the Frechet metric, the Proposition remains true for the metric topology on $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$.

But even though we have convergence of $\{f_n\}_{n\in\mathbb{N}}$ to FAC (after suitable identification with $\{f_{n\perp}\}_{n\in\mathbb{N}}$), something is lost in this assertion. Namely, the functions $\{f_n\}_{n\in\mathbb{N}}$ represent *increasing* approximations to FAC; indeed, as n increases, so does the amount of information we have about the limit function FAC. In fact, there is a natural order on $(\mathbb{N} \rightharpoonup \mathbb{N})$ that makes this idea precise.

Definition 2.3 Define the *extensional order* on the space $(\mathbb{N} \rightharpoonup \mathbb{N})$ by

$$f \sqsubseteq g \quad \text{iff} \quad \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \ \& \ g|_{\mathrm{dom}(f)} = f,$$

where $\mathrm{dom}(f) = f^{-1}(\mathbb{N})$.

Clearly, in this order any increasing family of partial functions has a supremum – the union. Moreover, the function FAC is the supremum of the family $\{f_n\}_{n\in\mathbb{N}}$. Our next goal is to capture this as a "convergence in order." This leads us to the *Scott topology*.

Definition 2.4 Let P be a partially ordered set.

- A subset $D \subseteq P$ is *directed* if $(\forall F \subseteq D \text{ finite})(\exists x \in D)\ y \sqsubseteq x\ (\forall y \in F)$.
- P is a *complete partial order* (*cpo* for short) if P has a least element – usually denoted $\perp$ – and if every directed subset of P has a least upper bound in P.

Note that a directed set must be non-empty, since the empty set is a finite subset of every set.

For example, the family $(\mathbb{N} \rightharpoonup \mathbb{N})$ is a cpo: the nowhere-defined function is the least element, and the supremum of a directed family of functions is just their union. Similarly, we can give $\mathbb{N}_\perp$ the *flat order*, whereby $x \sqsubseteq y$ if and only if $x = \perp$ or $x = y$ for all $x, y \in \mathbb{N}_\perp$. This corresponds to the pointwise order on the space $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$ of monotone selfmaps of $\mathbb{N}_\perp$, and in this order the supremum of a directed family of functions is the pointwise supremum, and the constant function with value $\perp$ is the least element.

Definition 2.5 Let P be a partially ordered set. A subset $U \subseteq P$ is *Scott open* if

- $U = \uparrow U = \{y \in P \mid (\exists x \in U)\, u \sqsubseteq y\}$ is an upper set, and
- $(\forall D \subseteq P \text{ directed})\ \bigsqcup D \in U \ \Rightarrow\ D \cap U \neq \emptyset$.

Proposition 2.6 *Let P be a partially ordered set.*

(i) *The family of Scott open sets is a T_0 topology on P.*

(ii) *If $x, y \in P$ and there is some open set containing x but not containing y, then $x \not\leq y$.*

(iii) *The following are equivalent:*
   - (a) *The Scott topology is T_1.*
   - (b) *P has the discrete order.*
   - (c) *The Scott topology is T_2.*
   - (d) *The Scott topology is discrete.*

Proof. Let P be a partially ordered set. Clearly the union of upper sets from P is an upper set. And, if $D \subseteq P$ is directed and $\bigsqcup D \in \bigcup_i U_i$, with U_i open for each $i \in I$, then $\bigsqcup D \in U_i$ for some $i \in I$. Since U_i is Scott open, it follows that $D \cap U_i \neq \emptyset$, and so the same is true of $D \cap (\bigcup_i U_i)$.

If $x \not\sqsubseteq y \in P$, then the definition of Scott open implies that $\downarrow y = \{z \in P \mid z \sqsubseteq y\}$ is Scott closed. Since $x \notin \downarrow y$, we have $x \in P \setminus \downarrow y$, which is Scott open. Hence the Scott topology is T_0. this proves (i).

For (ii), if $x, y \in P$ and $x \in U$, then $x \leq y$ and $U = \uparrow U$ imply $y \in U$ as well. Thus $y \notin U$ must imply $x \not\leq y$.

Finally, (iii) follows easily from (ii). $\square$

In our example – $(\mathbb{N} \rightharpoonup \mathbb{N})$ – it is not hard to show that $\uparrow f$ is Scott open if $\mathrm{dom}(f)$ is finite (a directed union of functions extends a finite function if and only if one of the functions in the directed family extends the finite function), and, as it happens, this family of principal upper sets forms a basis for the Scott topology on $(\mathbb{N} \rightharpoonup \mathbb{N})$.

There are a number of important results that are true of the Scott topology. Below we summarize some of them; they all can be found for dcpo's (cpo's without least elements) as well as cpo's in, e.g., [3].

Proposition 2.7 *Let P be a cpo, and endow P with the Scott topology.*

(i) *If $D \subseteq P$ is directed, then $\bigsqcup D$ is a limit point of D, and it is the greatest limit point of D.*

(ii) *Let I and J be directed sets, and let $(i, j) \mapsto x_{ij} \colon I \times J \to P$ be monotone. Then*

(a) $\bigsqcup_{i\in I} \bigsqcup_{j\in J} x_{ij} = \bigsqcup_{j\in J} \bigsqcup_{i\in I} x_{ij}$.

(b) *If $I = J$, then $\bigsqcup_{i\in I} \bigsqcup_{j\in J} x_{ij} = \bigsqcup_{i\in I} x_{ii}$.*

(iii) *If Q also is a dcpo, then $f \colon P \to Q$ is continuous iff f is monotone and preserves sups of directed sets.* $\square$

As we shall see, the second part of this result is a very useful tool in proving results about continuous functions between dcpo's.

One of the most celebrated results about cpo's is the following. We attribute it to TARSKI, who first proved it for complete lattices [55]. However, a number of others – among them SCOTT and KNASTER – have contributed to this result.

Theorem 2.8 TARSKI [55]
If P is a cpo and $f \colon P \to P$ is monotone, then f has a least fixed point, $\mathit{fix}(f) = \bigsqcup_{\alpha\in\mathrm{Ord}} f^\alpha(\perp)$. If f is continuous, then $\mathit{fix}(f) = \bigsqcup_{n\in\mathbb{N}} f^n(\perp)$.

Proof. We confine ourselves to an outline. For the first part, the monotonicity of f and the fact that $\perp \sqsubseteq x$ $(\forall x \in P)$ implies $\perp \sqsubseteq f(\perp)$, and so $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain. Since P is a cpo, this chain has a least upper bound, which we use to define $f^\omega(\perp) = \bigsqcup_{n\in\mathbb{N}} f^n(\perp)$. A transfinite induction then shows that $f^\alpha(\perp)$ is well-defined for all ordinals α, and that $\alpha \leq \beta$ implies $f^\alpha(\perp) \sqsubseteq f^\beta(\perp)$. Since this holds for all ordinals, there must be one where the increasing chain stops growing, and the first place this happens is easily seen to be a fixed point of f. The fact that $\perp \sqsubseteq x$ $(\forall x \in P)$ implies $f^\alpha(\perp) \sqsubseteq f(y) = y$ for all fixed points y of f and all ordinals α, showing that the first ordinal α where $f(f^\alpha(\perp)) = f^\alpha(\perp)$ is the least fixed point of f.

If f is continuous, then it follows that

$$f(\bigsqcup_n f^n(\perp)) = \bigsqcup_n f(f^n(\perp)) = \bigsqcup_n f^{n+1}(\perp).$$

$\square$

Example 2.9 Returning to our example, we recall that the natural order on $(\mathbb{N} \rightharpoonup \mathbb{N})$ makes the empty function the least element; this corresponds to the pointwise order on the family $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$ of monotone selfmaps of $\mathbb{N}_\perp$, and the constant function with value $\perp$ is the least element.

Now, let $[(\mathbb{N}_\perp \to \mathbb{N}_\perp) \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)]$ be the family of Scott-continuous selfmaps of $(\mathbb{N}_\perp \to \mathbb{N}_\perp)$. Define

$$F \in [(\mathbb{N}_\perp \to \mathbb{N}_\perp) \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)] \text{ by } F(f)(n) = \begin{cases} \perp & \text{if } n = \perp, \\ 1 & \text{if } n = 0, \\ n \cdot f(n-1) & \text{otherwise,} \end{cases}$$

where we define $n \cdot \perp = \perp \cdot n = \perp$ for all $n \in \mathbb{N}$. Then

- F is Scott continuous: indeed, as the domain of the function f increases, then the domain of $F(f)$ also increases, which implies F is monotone. Continuity then follows from the fact that the supremum of a directed set of functions is just their union.
- If $\kappa_\perp$ denotes the function that has value $\perp$ at all points on $\mathbb{N}_\perp$, then $F^n(\kappa_\perp) = f_{n\perp}$ for $n > 0$: this is a routine induction argument.
- $F(\mathrm{FAC}) = \mathrm{FAC}$: indeed, FAC has maximal domain – $\mathbb{N}$ – and so it cannot be extended. And it is clear that $F(\mathrm{FAC})(n) = n! = \mathrm{FAC}(n)$ for all $n \in \mathbb{N}$.
- $\mathrm{FAC} = \mathrm{fix}\,(F)$: this follows from the second observation and the fact that $\mathrm{FAC} = \bigsqcup_n f_{n\perp}$.

Actually, this example shows an alternative approach to obtaining a recursively defined function from just one functional. We shall see a striking generalization of this result later.

We've already seen that continuous functions can be characterized purely order-theoretically. Another fact about continuous functions also is important to note.

Proposition 2.10 *Let P, Q and R be dcpo's, and let $f\colon P \times Q \to R$. Then f is (jointly) continuous wrt the Scott topology on $P \times Q$ if and only if f is separately continuous wrt the product of the Scott topologies on P and Q.*

Proof. If $D \subseteq P \times Q$ is directed, then it is routine to show that $\sqcup D = (\sqcup \pi_P(D), \sqcup \pi_Q(D))$. If $f\colon P \times Q \to R$ is separately continuous wrt the product of the Scott topologies on P and Q, then

$$\begin{aligned} f(\sqcup D) &= f((\sqcup \pi_P(D), \sqcup \pi_Q(D))) = f(\sqcup_{d \in D} \pi_P(d), \sqcup_{d' \in D} \pi_Q(d')) \\ &= \sqcup_{d \in D} f((\pi_P(d), \sqcup_{d' \in D} \pi_Q(d'))) = \sqcup_{d \in D} \sqcup_{d' \in D} f((\pi_P(d), \pi_Q(d'))) \\ &= \sqcup_{d \in D} f((\pi_P(d), \pi_Q(d))) = \sqcup f(D), \end{aligned}$$

so f also is continuous wrt the Scott topology on $P \times Q$.

Conversely, if $f\colon P \times Q \to R$ is continuous wrt the Scott topology of $P \times Q$, then f preserves suprema of directed sets in $P \times Q$, which clearly implies f preserves suprema of directed subsets of $P \times \{y\}$ and $\{x\} \times Q$, respectively, for all $x \in P$ and $y \in Q$. This characterizes separate continuity of f wrt the product of the Scott topologies. □

Remark 2.11 It should be noted that, for dcpos P and Q, the product of the Scott topologies of P and Q is in general weaker than the Scott topology of the product. However, these topologies do coincide for continuous dcpos (cf. Section 3).

Theorem 2.12 *For dcpo's P and Q, the family $[P \to Q]$ of Scott-continuous maps is a dcpo in the pointwise order.*

Proof. It's routine to show that the directed supremum of monotone functions is well-defined and monotone. The fact that order of computing the supremum of a product of directed sets can be reversed implies the supremum function itself also preserves directed suprema, hence is continuous. $\square$

Tarski's Theorem guarantees that the the operator $\mathrm{fix}_D\colon [D \to D] \to D$ is well-defined, and using part (ii) of Proposition 2.7, it is easy to show that this operator is continuous. The following discussion shows in what sense fix_D is unique.

A *fixed point operator* is a family of continuous maps $F_D\colon [D \to D] \to D$ for each cpo D, which satisfies $F_D(f) = f(F_D(f))$ for each $f \in [D \to D]$. Such a family is called *uniform* if $F_E(g) = h(F_D(f))$ for all continuous maps $f\colon D \to D$ and $g\colon E \to E$ and strict continuous maps $h\colon D \to E$ satisfying $h \circ f = g \circ h$:

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\curvearrowright"] & D \arrow[d, "h"] \\ E \arrow[r, "g"'] & E \end{tikzcd} \qquad \Longrightarrow \qquad F_E(g) = h(F_D(f))$$

Theorem 2.13 fix *is the unique uniform fixed point operator defined on the category* CPO. $\square$

It is clear that $\{\bot\}$ is a terminal object for the category CPO of cpo's and continuous maps, and that the product of cpo's is another such, so CPO is cartesian. It also is closed, as Theorem 2.12 shows. The fact that $[P \times Q \to R] \simeq [P \to [Q \to R]]$ also is clear from Proposition 2.10. Thus we have:

Theorem 2.14 *The category CPO of cpo's and Scott continuous maps is cartesian closed. The same also holds of the category DCPO of dcpo's and Scott continuous maps.* $\square$

Our aim in this section was to show how order theory together with topology provides a richer theory than topology alone. While we have not shown that topology alone cannot claim the results we have enumerated, it should be clear that results we have highlighted are available in a particularly simple way in the cpo setting, and that this theory offers some results (such as Tarski's Theorem and Theorem 2.13) that are not so easily available in other settings. We also shall see that these results are particularly useful in the area of programming language semantics, which is at the heart of theoretical computer science.

3 Domain Theory

While some aspects of our motivating example in the previous section clearly are close to computability, nothing in the general theory of the category CPO addresses this directly. Domain theory adds this aspect to the theory we have outlined.

3.1 Basic Results

We begin this development with some standard definitions.

Definition 3.1 Let P be a dcpo. An element $k \in P$ is *compact* iff $\uparrow k$ is Scott open. We let $K(P) = \{k \in P \mid K \text{ is compact}\}$, and for each $x \in P$, $K(x) = \downarrow x \cap K(P)$.

For example, if we consider $\mathbb{N}_\perp$ to be the flat natural numbers, then $K(\mathbb{N}_\perp) = \mathbb{N}_\perp$. We already noted that the partial functions with finite domain are compact in $(\mathbb{N} \rightharpoonup \mathbb{N})$, from which it follows that $K([\mathbb{N}_\perp \stackrel{!}{\to} \mathbb{N}_\perp]) \supseteq \{f \mid \mathrm{dom}(f) \text{ is finite}\}$, where $[\mathbb{N}_\perp \stackrel{!}{\to} \mathbb{N}_\perp]$ is the space of continuous selfmaps of $\mathbb{N}_\perp$ leaving $\perp$ fixed.

Definition 3.2 The dcpo P is *algebraic* if, for all $x \in P$

- $K(x)$ is directed, and
- $x = \bigsqcup K(x)$.

By a *domain*, we mean an algebraic cpo.

The following result is basic to the theory.

Theorem 3.3 *Let P be a dcpo and let $B \subseteq K(P)$ be a family of compact elements of P. If for all $x \in P$,*

(i) *$B(x) = \downarrow x \cap B$ is directed, and*

(ii) *$x = \bigsqcup B(x)$,*

then P is algebraic and $B = K(P)$.

Proof. If the conditions hold, then $x = \bigsqcup B(x)$ can be used to show that $K(x)$ is directed, so $x = \bigsqcup B(x) \sqsubseteq \bigsqcup K(x) \sqsubseteq x$, and then P is algebraic. If $k \in K(P)$, then $k = \bigsqcup B(k)$ implies $k \in B(k)$ by the definition of compactness. $\square$

We noted earlier that the functions $f \colon \mathbb{N}_\perp \to \mathbb{N}_\perp$ that leave $\perp$ fixed and that have a finite domain are compact elements in $[\mathbb{N}_\perp \to \mathbb{N}_\perp]$. A corollary of Theorem 3.3 is that $K([\mathbb{N}_\perp \stackrel{!}{\to} \mathbb{N}_\perp]) = \{f \mid \mathrm{dom}(f) \text{ is finite}\}$.

Definition 3.4 Let P be a partially ordered set. An *ideal* of P is a directed lower set of P. We let $\mathrm{Idl}(P)$ denote the family of ideals of P.

If P is a poset, then using Theorem 3.3 it is routine to show that $\mathrm{Idl}(P)$ is an algebraic dcpo whose compact elements are $K(\mathrm{Idl}(P)) = \{\downarrow x \mid x \in P\}$. The definition of algebraicity then implies that a cpo P is algebraic if and only if $P \simeq \mathrm{Idl}(P)$: indeed the mapping $x \mapsto K(x)$ has $I \mapsto \bigsqcup I$ as its inverse. Moreover, $\mathrm{Idl}(P)$ is a cpo if and only if P has a least element.

To elevate the above relationship to an equivalence of categories requires using relations between posets rather than functions. Since a continuous map $f: P \to Q$ between domains need not preserve compact elements, such a function f does not restrict to a function from $K(P)$ to $K(Q)$. But, for each $x \in P$, $f(x) = \bigsqcup K(f(x))$ is completely determined by the ideal $K(f(x))$ of compact elements of Q. This gives rise to the following notion.

Definition 3.5 Let P and Q be posets. An *approximable relation* $R \subseteq P \times Q$ is a relation satisfying:

- $(\forall x \in P)(\forall y, y' \in Q)\, xRy \sqsupseteq y' \;\Rightarrow\; xRy'$.
- $(\forall x \in P)(\forall M \subseteq Q \text{ finite})\, ((\forall y \in M)\, xRy \;\Rightarrow\; (\exists z \in Q)\, xRz \;\&\; M \subseteq \downarrow z)$.
- $(\forall x, x' \in P)(\forall y \in Q)\, x' \sqsupseteq xRy \;\Rightarrow\; x'Ry$.

These conditions insure that the set $\{y \in Q \mid xRy\}$ is an ideal of Q, and so the relation R is really a monotone function from P to $\mathrm{Idl}(Q)$; this then extends to a (unique!) continuous function from $\mathrm{Idl}(P)$ to $\mathrm{Idl}(Q)$. If $R \subseteq P \times P'$ and $S \subseteq P' \times P''$ are approximable relations, then so is $S \circ R \subseteq P \times P''$. Hence, there is a category POS_A of posets and approximable relations. The correspondence taking a continuous mapping $f: P \to Q$ between algebraic dcpo's to the approximable relation $R_f \subseteq K(P) \times K(Q)$ by $R_f = \bigcup\{\{k\} \times K(f(x)) \mid k \in K(P)\}$ then has as its inverse the assignment taking an approximable relation $R \subseteq P \times Q$ between posets to the continuous map $f_R: \mathrm{Idl}(P) \to \mathrm{Idl}(Q)$ by $f_R(I) = \bigsqcup(\bigcup R(I))$. Thus, we have an equivalence of categories POS_A and ADCPO between posets and approximable relations and algebraic dcpo's and continuous mappings. This equivalence cuts down to an equivalence between the full subcategories POS_{A0} of posets with least element and ALG of domains.

Theorem 3.6 *The category* ALG *of domains is equivalent to the category* POS_{A0} *of posets with least element and approximable relations.* □

If by a *locally compact space* we mean one in which each point has a neighborhood basis of compact sets, then the following is obvious from the definitions.

Proposition 3.7 *If P is a domain, then $\{\uparrow k \mid k \in K(P)\}$ is a basis for the Scott topology on P, and so P is locally compact.* □

We will see later that domains also are *sober*; for now we leave this issue and concentrate on bringing computability more to the fore. A detailed description would include an indication of how *enumerability* can be captured in this setting. The details are too many to go through here, so we confine ourselves to the following brief indicator.

We showed that the function $\mathrm{FAC}: \mathbb{N} \to \mathbb{N}$ could be realized as the least fixed point $\mathrm{fix}(F)$, where $F\colon[\mathbb{N}_\perp \to \mathbb{N}_\perp] \to [\mathbb{N}_\perp \to \mathbb{N}_\perp]$. This result has a striking generalization.

Theorem 3.8 Myhill–Sheperdson [50]
If $g\colon \mathbb{N} \rightharpoonup \mathbb{N}$ is partial recursive, then g can be realized as $g = \mathrm{fix}(G)$ for some continuous selfmap $G\colon[\mathbb{N}_\perp \to \mathbb{N}_\perp] \to [\mathbb{N}_\perp \to \mathbb{N}_\perp]$. □

This hints at the close relationship between notions of computability and domain theory. We could summarize this relationship with the following "slogans":

Algebraicity	captures	*Computability*
k compact	if and only if	*k is computable in finite time*
$f\colon P \to P$ *recursive*	implies	$(\exists F\colon[P \to P] \to [P \to P])\ f = \mathrm{fix}\,(F)$

3.2 Continuous Domains

Many of the basic results outlined in the previous section have an important generalization. In his seminal paper [49], Scott comments that the algebraic lattices he discovered as injective spaces are in some sense zero-dimensional, and to close up the class under quotients, one needs to consider positive-dimensional analogues. This was the impetus for the results in [20], where it is shown that continuous lattices form the class of objects so generated. At the more general level of cpo's, the corresponding objects are the *continuous cpo's*. Some of their theory was presented in the exercises in [20], but the nicest presentation we have seen is in [3]. Here's a brief outline of the basics of that theory.

Definition 3.9 Let P be a dcpo, and let $x, y \in P$.

- We write $x \ll y$ if for all $D \subseteq P$ directed sets, if $y \sqsubseteq \bigsqcup D$, then $D \cap \uparrow\! x \neq \emptyset$. If $x \ll y$, then we say x *is way-below* y. We let $\Downarrow\! y = \{x \in P \mid x \ll y\}$ for each $y \in P$.
- P is a *continuous dcpo* if for all $y \in P$,

(i) $\Downarrow\! y$ is directed, and
(ii) $y = \bigsqcup \Downarrow\! y$.

We let CON denote the category of continuous cpo's and Scott continuous maps.

Clearly $x \in P$ satisfies $x \ll x$ if and only if x is compact, and so each algebraic cpo is continuous. Given a continuous cpo P, in analogy with the poset $K(P)$, we can define the pre-ordered set $(P, \ll)$. In general, $\ll$ is not a partial order: $x \ll x$ iff x is compact. The proper generalization of $K(P)$ is given in the following definition.

Definition 3.10 An *abstract basis* is a set B together with a transitive relation $\prec$ which satisfies the *interpolation property:*

$$\mathrm{INT} \quad (\forall M \subseteq B\, \text{finite})(\forall x \in P)\, M \prec x \;\Rightarrow\; (\exists y \prec x)\, M \prec y \prec x.$$

Here, $M \prec x$ means $z \prec x$ for all $z \in M$.

Proposition 3.11 *Let P be a continuous dcpo. Then $\ll$ satisfies* INT. *Hence*

- $(P, \ll)$ *is an abstract basis.*
- $\Uparrow\!x = \{y \in P \mid x \ll y\}$ *is Scott open for each $x \in P$.*

Proof. Let P be continuous and let $x \in P$. Consider the set $A = \{z \in P \mid (\exists w \in P)\, z \ll w \ll x\}$. It is routine to show that A is directed, and clearly $\bigsqcup A \sqsubseteq x$. If $\bigsqcup A \neq x$, then $x = \bigsqcup \Downarrow\!x$ implies there is some $w \ll x$ with $w \not\sqsubseteq \bigsqcup A$, and then the same argument implies there is some $z \ll w$ with $z \not\sqsubseteq \bigsqcup A$. Then $z \in A$, so $z \sqsubseteq \bigsqcup A$, which is a contradiction. Hence $\bigsqcup A = x$.

Next, if $M \subseteq P$ is a finite set with $y \ll x$ for all $y \in M$, then, for each $y \in M$, there is some $y' \in A$ with $y \sqsubseteq y'$. Choosing $z \in A$ with $y' \sqsubseteq z$ for each $y \in M$ implies there is some $w \ll x$ with $z \ll w$. But, then $y \sqsubseteq y' \sqsubseteq z$ implies $y \ll w \ll x$ for all $y \in M$. Hence P satisfies INT.

The first part of the Proposition now follows. As for the second part, if $\bigsqcup D \in \Uparrow\! x$, then $x \ll \bigsqcup D$, and so INT implies there is some $y \in P$ with $x \ll y \ll \bigsqcup D$. Then $\exists d \in D$ with $y \sqsubseteq d$, and so $d \in \uparrow\! y \subseteq \Uparrow\! x$. $\Box$

Corollary 3.12 *If P is a continuous dcpo, then P is locally compact in the Scott topology.*

Proof. The fact that $\Uparrow\!x$ is Scott open implies $\uparrow\!x$ is a Scott-compact neighborhood of each point in $\Uparrow\!x$. $\Box$

Each abstract basis $(B, \prec)$ has an ideal completion $\mathrm{Idl}(B, \prec)$ – the set of $\prec$-directed lower sets of B, and this ideal completion is a continuous dcpo in which $x \prec y$ implies $\downarrow\!x \ll \downarrow\!y$ in $\mathrm{Idl}(B, \prec)$. Moreover, given a continuous dcpo P, the mapping $x \mapsto \Downarrow\!x : P \to \mathrm{Idl}(P, \ll)$ has as its inverse the mapping $I \mapsto \bigsqcup I : \mathrm{Idl}(P, \ll) \to P$.

In further analogy to the algebraic case, there is a notion of *approximable relations* between abstract bases, and the following theorem holds.

Theorem 3.13 Abramsky & Jung [3]
There is an equivalence between the categories ABAS *of abstract bases and approximable relations and* COND *of continuous dcpo's and continuous mappings. This equivalence restricts to an equivalence between the full subcategories* ABAS_0 *of abstract bases with minimum elements and* CON *continuous cpo's.* $\Box$

What this all says is that there is a uniform approach the algebraic and continuous cases in which the algebraic structure of continuous cpo's can be highlighted and used effectively to understand the structure of continuous cpo's. It has been known for some time that certain aspects of the theory of domains are more elegantly and simply presented in the continuous case (because of the closure under quotients), and the approach of abstract bases provides a method for developing that theory in a way that affords easy access to the results about algebraic cpo's that one might wish to highlight.

3.3 Categories of Domains

In the first section we noted that CPO and DCPO are cartesian closed categories. If P and Q are algebraic, then it is easy to show that $P \times Q$ also is algebraic and that $K(P \times Q) = K(P) \times K(Q)$. Since the terminal object also is algebraic, ALG – the category of domains and continuous maps – is cartesian. If we want to know whether ALG also is cartesian closed, the following result shows we don't have to look far for a potential internal hom:

Theorem 3.14 Smyth [51], Jung [30]
Let C be a full subcategory of ALG and let P, Q be objects of C.

(i) *If C has products, then $P \times Q$ is the product of P and Q in C.*

(ii) *If C has exponentials, then $[P \to Q]$ is the exponential of P and Q in C.* □

We attribute this theorem jointly to Smyth and Jung; Smyth [51] showed this for ω-algebraic domains (i.e., ones for which $K(P)$ is countable), and Jung [30] extended the result to the general case.

Unfortunately, ALG is *not* cartesian closed. Indeed, a simple example that hints at the problem is to show that $(\mathbb{N}, \leq)^{\mathrm{op}}$ – the natural numbers in the dual of the usual order – satisfies $[(\mathbb{N}, \leq)^{\mathrm{op}} \to (\mathbb{N}, \leq)^{\mathrm{op}}]$ is not algebraic. In fact, $K(f) = \emptyset$ for any function in this space. (This example is taken from [3].)

So, one might ask what cartesian closed categories exist within ALG. The first one we note is probably the best-known.

Definition 3.15 A domain P is a *Scott domain* if P is closed under the formation of non-empty infima.

Theorem 3.16 *The category SD of Scott domains and continuous maps is cartesian closed.* □

Clearly the product of Scott domains is another such, so the proof of this result requires only consideration of the function space. Here, a little work is required. The pointwise infimum of a family of continuous maps between Scott domains surely is well-defined, but it is not necessarily continuous. What one has to take for the infimum is the largest Scott continuous map which is pointwise below the pointwise infimum. Of course, even once it is shown that this map exists and that it is the infimum, it also needs to be shown that the family of continuous maps between Scott domains is algebraic. Here one explicitly shows that every continuous function is the supremum of "step functions" which clearly are compact elements in the function space. By taking finite sub-suprema of such step-functions, one sees that the compact elements of the function space form a basis.

A larger cartesian closed category of domains is obtained in the following way.

Definition 3.17 An *embedding–projection pair* $(e, p): P \to Q$ between domains P and Q is a pair of continuous maps $e: P \to Q$ and $p: Q \to P$ satisfying

- $p \circ e = 1_P$, and
- $e \circ p \sqsubseteq 1_Q$.

Actually, an e-p pair is a special case of a Galois adjunction between the domains P and Q: e is the lower adjoint and p the upper adjoint.

Definition 3.18 A domain P is *SFP* if there is a sequence of finite posets and e-p pairs, $\{(e_{nn+1}, p_{n+1n}): P_n \to P_{n+1}\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} P &\simeq \lim_n (P_n, (p_{nn-1} \circ \cdots \circ p_{m+1m})_{m\leq n\in\mathbb{N}}) \\ &\simeq \operatorname{colim}_n (P_n, (e_{n-1n} \circ \cdots \circ e_{mm+1})_{m\leq n\in\mathbb{N}}) \end{aligned}$$

This definition only makes sense once one shows that the indicated limit and colimit both exist and that they coincide. This was first demonstrated by Plotkin [43]. Plotkin constructed the category SFP of SFP-objects and continuous maps in order to have a cartesian closed category that was closed under all the operators he needed to create the sort of semantic models he had in mind. In particular, he needed a ccc that was closed under the Plotkin power domain construct, and this is something that is not true of Scott domains. Plotkin also conjectured the following result, which was proved by Smyth [51].

Theorem 3.19 Smyth [51]
The category SFP *of SFP-objects and continuous maps is the largest cartesian closed category of ω-algebraic domains.* □

In his celebrated thesis [30], Jung greatly extended our knowledge about maximal cartesian closed categories of domains. He first showed that the category of *bifinite domains* – those that are simultaneously the limit and colimit of a directed family of finite posets under e-p pairs – is cartesian closed, and in fact is maximal such among those ccc's of domains. He also defined the following class of domains.

Definition 3.20 An *L-domain* is a domain P in which $\downarrow\!x$ is a complete lattice for each $x \in P$.

Theorem 3.21 Jung [30]
There are two maximal cartesian closed full subcategories of domains:

- *The category* BIFIN *of bifinite domains and continuous maps, and*
- *The category* LDOM *of L-domains and continuous maps.* □

3.4 Categorical Generalizations

One of the basic aspects of the Scott topology is that directed sets converge to their suprema. Moreover, Tarski's Theorem guarantees that continuous selfmaps on cpo's have least fixed points that can be computed in a simple way – simply iterate the function starting at the least element. Smyth and Plotkin [53] were the first to elevate these ideas to the categorical level. In their approach, categories of cpo's and continuous maps were viewed as "large cpo's" in which colimits of what are called "expanding sequences" in [3] correspond to suprema in a cpo. Furthermore, domains satisfying desired properties can be viewed as "fixed points" of associated continuous endofunctors of the category, and these "fixed points" can be calculated in a way similar to the calculation of the least fixed point of a continuous selfmap of a cpo. We now outline this material along with the interesting phenomena that arise in related categories. All of this material is presented in detail in Section 5 of [3].

In order to mimic Tarski's Theorem at the level of a category A of cpo's, we first need to order A. This is accomplished by defining not a partial order on A, but rather a *pre-order* – a reflexive, transitive relation – on A.

Definition 3.22 Let D and E be cpo's. We write $D \sqsubseteq E$ if and only if there is an embedding-projection pair $(e,p)\colon D \to E$.

Lemma 3.23 $\sqsubseteq$ *is a pre-order on the class of cpo's.*

Proof. It is clear that the relation is reflexive, since the identity map forms an e-p pair on any cpo. Transitivity follows from the fact that $(e_2 \circ e_1, p_1 \circ p_2)\colon D_1 \to D_3$ is an e-p pair if $(e_1,p_1)\colon D_1 \to D_2$ and $(e_2,p_2)\colon D_2 \to D_3$ are e-p pairs. $\square$

Note, however, that it is unclear what it means for two dcpo's to be equivalent under $\sqsubseteq$.

Example 3.24 Let $I = [0,1]$ denote the unit interval, $E = I \times I$ the unit square in the product order, and

$$D = ([1/2, 1/2] \times [1/2, 1/2]) \cup \{(x,x) \mid 1/2 \leq x \leq 1\}.$$

Clearly D is a sub-cpo of E, and it is easy to see that there is a projection mapping $p\colon E \to D$ so that the embedding $i\colon D \to E$ together with p forms an embedding-projection pair. But, likewise, E can be embedded in D as the lower square, and this also has an associated projection $p'\colon D \to E$. Thus D and E are equivalent under $\sqsubseteq$, but they clearly are not isomorphic as cpo's.

Even though $\sqsubseteq$ is not a partial order, we can still use it as if it were one, and so our next goal is to show that increasing sequences on cpo's in this order have "least upper bounds".

Definition 3.25 Let $(e_n,p_n)\colon D_n \to D_{n+1}$ be a sequence of e-p pairs for each $n \in \mathbb{N}$. We define

$$D_\infty = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} D_n \mid p_n(x_{n+1}) = x_n\},$$

and we endow D_∞ with the order inherited from $\prod_{n\in\mathbb{N}} D_n$. It is not hard to show that D_∞ is a sub-cpo of $\prod_{n\in\mathbb{N}} D_n$, since the maps p_n all are continuous.

We also can define embedding-projection pairs $(E_n,P_n)\colon D_n \to D_\infty$ by $P_n((x_n)_{n\in\mathbb{N}}) = x_n$ and $E_n(x) = (f_{in}(x)_{i\in\mathbb{N}})$, where $f_{ij}\colon D_j \to D_i$ by

$$f_{ij} = \begin{cases} p_i \circ \cdots \circ p_{j-1} & \text{if } i < j, \\ 1_{D_i} & \text{if } i = j, \text{ and} \\ e_{i-1} \circ \cdots \circ e_j & \text{if } i > j. \end{cases}$$

Theorem 3.26 *If* $(e_n, p_n): D_n \to D_{n+1}$ *is a sequence of e-p pairs for each* $n \in \mathbb{N}$*, then* $(E_n, P_n): D_n \to D_\infty$ *as defined above is a sequence of e-p pairs satisfying* $E_n = E_{n+1} \circ e_n$ *and* $P_n = p_n \circ P_{n+1}$ *for each* $n \in \mathbb{N}$*.*

Moreover, if A *is a cpo and* $(E'_n, P'_n): D_n \to A$ *is a sequence of e-p pairs satisfying* $E'_n = E'_{n+1} \circ e_n$ *and* $P'_n = p_n \circ P'_{n+1}$ *for each* $n \in \mathbb{N}$*, then there is a unique e-p pair* $(E, P): D_\infty \to A$ *such that* $E \circ E_n = E'_n$ *and* $P_n \circ P = P'_n$ *for each* $n \in \mathbb{N}$*.*

Finally, if $(E'_n, P'_n): D_n \to A$ *is a co-cone over the sequence* $(e_n, p_n): D_n \to D_{n+1}$*, then the co-cone is co-limiting if and only if* $1_A = \bigsqcup(E'_n \circ P'_n)$*.* □

Let CPO_{ep} be the category of cpo's and e-p pairs; i.e., the objects of the category are cpo's, and morphisms are pairs of embedding-projection mappings between objects. The point of the previous result is that we can regard $(D_\infty, ((E_n, P_n): D_n \to D_\infty)_{n\in\mathbb{N}})$ as a co-cone over the diagram $(e_n, p_n): D_n \to D_{n+1})_{n\in\mathbb{N}}$ in CPO_{ep}, and this result asserts that it is co-limiting. Viewed as a colimit, D_∞ then is the "least" upper bound of the sequence $D_0 \sqsubseteq \cdots \sqsubseteq D_n \sqsubseteq D_{n+1} \cdots$, and so the category CPO of cpo's and continuous maps has least upper bounds relative to the order $\sqsubseteq$. The construction shows that this also holds for every full subcategory of CPO that is complete. Note also that CPO has a least cpo - the one-point cpo $\{\perp\}$, since there is an obvious e-p pair from $\{\perp\}$ to any cpo P.

The next point is to single out a family of continuous endofunctors for which we can prove an analogue of Tarski's Theorem. The obvious definition for continuity would be that a functor preserves least upper bounds, as defined in Theorem 3.26. But to make this precise, we first record a result that shows CPO is closed under limits and colimits.

Theorem 3.27 *If* $(P_i, \{(e_{ij}, p_{ij}): P_i \to P_j\}_{i \leq j \in I})$ *is a diagram in* CPO_{ep}*, then*

$$\lim(P_i, \{p_{ij}\}_{i \leq j \in I}) \simeq \mathrm{colim}(P_i, \{e_{ij}\}_{i \leq j \in I}). \qquad \square$$

So, if one has a diagram $(P_i, \{(e_{ij}, p_{ij}): P_i \to P_j\}_{i \leq j \in I})$ in CPO_{ep}, then the limit of the diagram $(P_i, \{p_{ij}: P_i \to P_j\}_{i \leq j \in I})$ and the colimit of the diagram $(P_i, \{e_{ij}: P_i \to P_j\}_{i \leq j \in I})$ both exist and they coincide. This limit can be regarded either as a colimit or a limit in the category CPO by taking the appropriate projection from CPO_{ep}. This result allows for a fine analysis of the limit of such a diagram, and this in turn is very useful in applying the techniques that are needed to construct domains to satisfying certain equations.

We already have seen that the colimit of a sequence $(P_i, \{(e_{ij}, p_{ij}): P_i \to P_j\}_{i \leq j \in I})$ in CPO can be regarded as the least upper bound of the sequence. Moreover, the order on CPO ensures that all functors between categories of cpo's are monotonic: if $(e, p): P \to Q$ is an e-p pair in a category A of cpo's and $F: \mathsf{A} \to \mathsf{B}$ is a functor, then $(F(e), F(p)): F(P) \to F(Q)$ also is an e-p pair. So what remains is to find the appropriate sense in which functors should be continuous.

Definition 3.28 Let A and B be co-complete categories of cpo's and continuous maps. The functor $F: \mathsf{A} \to \mathsf{B}$ is *continuous* if for every diagram

$(P_i, ((e_{ij}, p_{ij})\colon P_i \to P_j\colon)_{i\le j\in I})$ in A_{ep},

$$F(\mathrm{colim}(P_i, \{e_{ji}\}_{i\le j\in I})) \simeq \mathrm{colim}(F(P_i), \{F(e_{ji)}\}_{i\le j\in I}).$$

While this seems a reasonable definition for continuity (albeit somewhat opposite from the usual definition of a continuous functor), it can be a difficult property to prove. The following result shows that there is a simple test that makes it easy to show certain functors are continuous.

Definition 3.29 The functor $F\colon \mathsf{A} \to \mathsf{B}$ between full subcategories of CPO is *locally continuous* if for all objects P and Q of A,

$$F\colon [P \to Q] \to [F(P) \to F(Q)]$$

is continuous.

This definition does make sense: indeed, because operations on $[D \to E]$ are defined pointwise, even though $[D \to E]$ and $[F(D) \to F(E)]$ are not necessarily objects of A or B, respectively, nonetheless they are cpo's and $F\colon [D \to E] \to [F(D) \to F(E)]$ is a well-defined function, so it makes perfect sense that it might be continuous.

Theorem 3.30 Plotkin [44], Smyth & Plotkin [53]
If $F\colon \mathsf{A} \to \mathsf{B}$ is a locally continuous functor between full subcategories of CPO*, then $F\colon \mathsf{A} \to \mathsf{B}$ is continuous.* □

Now, let $F\colon \mathsf{CPO} \to \mathsf{CPO}$ be an endofunctor and let $(e, p)\colon \{\bot\} \to F(\{\bot\})$ be the natural e-p pair. If we let F^0 be the identity functor, then the following is the analogue to Tarski's Theorem we have been seeking:

Corollary 3.31 Tarski's Theorem for Categories of Cpo's
Let $F\colon \mathsf{A} \to \mathsf{A}$ be a continuous endofunctor on a full, complete subcategory of CPO*. Then $(F^n(e), F^n(p))\colon F^n(\{\bot\}) \to F^{n+1}(\{\bot\})$ is a sequence of e-p pairs and*

$$\begin{aligned}\mathbb{I} &= \{(x_n)_{n\in\mathbb{N}} \mid F^n(p)(x_{n+1}) = x_n\} \\ &\simeq \mathrm{colim}(F^n(\{\bot\}), (F^{m-1} \circ \cdots \circ F^n)(e)_{n<m\in\mathbb{N}})\end{aligned}$$

satisfies $F(\mathbb{I}) \simeq \mathbb{I}$. Moreover, $\mathbb{I}$ is the least such cpo, in the sense of Theorem 3.26. □

Since local continuity implies continuity, we can find a domain satisfying a desired isomorphism by starting with a continuous endofunctor $F\colon \mathsf{CPO} \to \mathsf{CPO}$ and seeking a cpo P satisfying $F(P) \simeq P$. The technique for finding such a cpo P is to apply Tarski's Theorem 3.31: iterate the functor F starting with the least domain, $\{\bot\}$, using the canonical e-p pair from $\{\bot\}$ to $F(\{\bot\})$. One should note the analogy to finding fixed points of continuous selfmaps of cpo's. We present perhaps the simplest example.

Example 3.32 Let $L\colon \mathsf{CPO} \to \mathsf{CPO}$ by $L(P) = P \cup \{\bot\}$, where $\bot \notin P$, and for $f\colon P \to Q$,

$$L(f)\colon L(P) \to L(Q) \text{ by } L(f)(x) = \begin{cases} f(x) & \text{if } x \in P, \\ \bot & \text{otherwise.} \end{cases}$$

Thus, L is the *lift functor* which adds a new bottom to the cpo P and which extends a continuous map between cpo's by sending the new bottom in the domain cpo to the new bottom in the range cpo. Clearly L is an endofunctor of CPO, and the local continuity of L should be obvious.

In seeking a cpo P satisfying $L(P) \simeq P$, we start with the cpo $\{\perp\}$ and the embedding-projection pair (e,p) between $\{\perp\}$ and the cpo $L(\{\perp\})$ which sends $\perp$ to the least element of $L(\{\perp\})$, and the projection which is the only map from $L(\{\perp\})$ to $\{\perp\}$. This leads to the following diagram in CPO_{ep}:

$$\{\perp\} \underset{e}{\overset{p}{\leftrightarrows}} L(\{\perp\}) \underset{L(e)}{\overset{L(p)}{\leftrightarrows}} L^2(\{\perp\}) \cdots L^n(\{\perp\}) \underset{L^n(e)}{\overset{L^n(p)}{\leftrightarrows}} L^{n+1}(\{\perp\}) \cdots$$

Theorem 3.27 then implies that

$$\begin{aligned}
&\lim_n \,(L^n(\{\perp\}), \{L^{n-1}(p) \circ \cdots \circ L^m(p)\}_{m<n\in\mathbb{N}}) \\
&\qquad \simeq \operatorname*{colim}_n \,(L^n(\{\perp\}), \{L^m(e) \circ \cdots \circ L^{n-1}(e)\}_{m<n\in\mathbb{N}}) \\
&\qquad \simeq (\mathbb{N}, \leq)^\top,
\end{aligned}$$

where $(\mathbb{N}, \leq)^\top$ is the natural numbers in the usual order with a top element added. It is important to note that the reason this is the colimit of the diagram $(L^n(\{\perp\}), \{L^m(e) \circ \cdots \circ L^{n-1}(e)\}_{m<n\in\mathbb{N}})$ is that the colimit is taken in CPO, where all objects must be directed complete. Hence, the colimit in POS – $(\mathbb{N}, \leq)$ – must have a largest element added to make it a cpo.

Now, since L is locally continuous, it is continuous. Hence

$$(\mathbb{N}, \leq)^\top \simeq L((\mathbb{N}, \leq)^\top),$$

and this provides a solution to the equation $L(P) \simeq P$.

In analogy to the situation with continuous selfmaps of cpo's, the solution $L(P) \simeq P$ we just found is least relative to the pre-order we have placed on CPO. There is another way to state this fact, which utilizes the notions of F-algebras.

Definition 3.33 Let $F\colon \mathsf{A} \to \mathsf{A}$ be an endofunctor on a category A. The object A of A is an *F-algebra* if there is a morphism $\pi_A\colon F(A) \to A$ in the category A. If A and B are F-algebras, then an *F-homomorphism* from A to B is an A-morphism $f\colon A \to B$ such the the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\pi_A"'] & F(B) \arrow[d, "\pi_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

If $F\colon \mathsf{A} \to \mathsf{A}$ is a locally continuous endofunctor on the full subcategory A of CPO, and if A contains the object $\{\perp\}$, then we can form the object

$$\mathbb{I} = \operatorname*{colim}_n((F^n(\{\perp\})_{n\in\mathbb{N}}), \{F^m(e) \circ \cdots \circ F^{n-1}(e)\}_{m\leq n\in\mathbb{N}}),$$

where $e: \{\bot\} \to F(\{\bot\})$ is the embedding sending $\bot$ to the least element of $F(\{\bot\})$, and this satisfies $F(\mathbb{I}) \simeq \mathbb{I}$. Recall that a map $f: P \to Q$ between cpo's is *strict* if it preserves the least element of P. The proof of the following result can be found in [3].

Theorem 3.34

(i) *If A is an F-algebra for which $\pi_A: F(A) \to A$ is an isomorphism, then there is a least F-homomorphism $f_{B,A}: A \to B$ for any F-algebra B.*

(ii) $\mathbb{I}$ *is a sub-cpo of every fixed point $B \simeq F(B)$ of F.*

(iii) $\mathbb{I}$ *is an initial F-algebra in the category $\mathsf{A}_!$ of A-objects and strict continuous maps from A.* □

So, if we take the case $F = L$, the lift functor, then this says that the lift algebra $(\mathbb{N}, \leq)^\top$ is a lift algebra in CPO that is the "least fixed point" in the category in the sense that there is a least homomorphism from it to any other lift algebra. Moreover, if we force the least element of $(\mathbb{N}, \leq)^\top$ to be mapped to the least element of a target lift algebra B, then there is a unique lift algebra homomorphism from $(\mathbb{N}, \leq)^\top$ to B.

Lastly, in Section 5.3 we will see how the assignment $P \mapsto [P \to P]$ can be made functorial, and how the techniques outlined here allow one to construct a non-degenerate fixed point for the associated functor. This result provides us with a model of the untyped lambda calculus of Church and Curry.

3.5 Further Results

The results we have outlined begin to make a case that domain theory has a number of interesting results to offer. From the start, there have been several attempts to duplicate the results we describe in other settings. For example, a number of authors have examined the possibility of developing analogous results in categories of metric spaces. Most notable among these is the seminal result of AMERICA AND RUTTEN [6] where it was shown that one of the most important techniques – solving "recursive domain equations" (like our lift algebra equation) – can be carried out in the metric setting. Analogous results also have been obtained by FLAGG AND KOPPERMAN [17] who use the different setting of quantales. Perhaps the most penetrating results so far have been obtained by WAGNER [59] who has shown that the domain-theoretic and metric space approaches can be understood as instances of a common theme. This theme is to regard the categories CPO and MET as enriched categories. For CPO, the enrichment is over the two-point lattice, while for MET, it is over the quantale $(\mathbb{R}^{\mathrm{op}}, +)$ of real numbers in the opposite order equipped with $+$ as the tensor product.

Another point that is worth making is that there are concerted attempts to understand just what portion of the properties of CPO are *fundamental* to a basis for theoretical computation. In this regard, we mention two research efforts:

(i) The work of FREYD [19] on *algebraically compact categories.* If T is an endofunctor of a category C, then T-*Inv* denotes the category of triples $\langle A, f, g\rangle$ where $\langle A, f\rangle$ is a T-algebra, $\langle A, g\rangle$ is a T-coalgebra, and $f \circ g$ and $g \circ f$ are the identity maps. T is *algebraically bounded* if T-*Inv* has an object that is both terminal and initial. If C is bi-complete (i.e., every covariant endofunctor of C or of C^{op} has an initial algebra), then C is *algebraically compact* if every endofunctor is algebraically bounded. For example, the category of countable sets is algebraically complete. These notions appear to characterize what is necessary for each endofunctor to have a "least fixed point" in the category.

(ii) The work of Plotkin, Fiore, *et al* on a "system of axioms for domain theory." The axiom system postulates a pair of categories in which there is a forgetful functor from one to the other whose left adjoint is "analogous to" the lift functor. The relation between CPO and $\mathsf{CPO}_!$ is the prime example: here $\mathsf{CPO}_!$ is the category of cpo's and strict continuous maps. The forgetful functor from $\mathsf{CPO}_!$ to CPO has lift as its left adjoint. The motivation is that $(\mathbb{N}, \leq)^\top$ is an initial lift algebra, and this is exactly what is needed to develop a theory of ω-cpo's and continuous maps – partial orders with least element where countable chains have least upper bounds, and in which maps preserve the suprema of such chains.

4 Domains as Topological Spaces

The "traditional approach" to domain theory emphasizes realizing domains as dcpo's P which are isomorphic to the family $\mathrm{Idl}(K(P))$ of order ideals of the set of compact elements. The results in [3] extend this approach to continuous domains by utilizing the notion of an abstract basis. In our opinion, this approach suffers from the drawback of having to deal with approximable relations, which we view as much less intuitive than continuous functions. In this section we outline an alternative approach that emphasizes topology, perhaps to the detriment of not highlighting the algebraic character of domains that the traditional approach offers. Nonetheless, we believe this approach has some intuitive advantages.

4.1 Order-theoretic Topology

To begin, we recall the well-worn connection between topology and algebra that has been extensively studied under the rubric "order-theoretic topology." A basic reference for this approach is the book [28]. However, we prefer to focus on the closed sets of a topological space, rather than the open sets.

Let TOP be the category of topological spaces and continuous maps.

Definition 4.1 If X is a topological space, then we define the family $\Gamma(X) = \{C \subseteq X \mid C = \overline{C}\}$ of closed subsets of X. If $f\colon X \to Y$ is continuous, we define $\Gamma(f)\colon \Gamma(Y) \to \Gamma(X)$ by $\Gamma(f)(C) = f^{-1}(C)$.

Definition 4.2 A *Brouwerian lattice* is a complete lattice L for which $x \vee (\bigwedge C) = \bigwedge_{y \in C}(x \vee y)$ for all $x \in L$ and all families $C \subseteq L$. A morphism of

Brouwerian lattices is a mapping $f\colon L \to M$ that preserves all infima and all finite suprema.

Theorem 4.3 *If* CBL *denotes the category of Brouwerian lattices and Brouwerian lattice maps, then* $\Gamma\colon \mathsf{TOP} \to \mathsf{CBL}^{\mathrm{op}}$ *is a contravariant functor.* $\square$.

To go back the other way, we first need some terminology.

Definition 4.4 For a complete lattice L, an element $p \in L$ is *co-prime* if for all $F \subseteq L$ finite, if $p \sqsubseteq \vee F$, then $F \cap \uparrow p \neq \emptyset$. We denote by $\mathrm{Spec}_{\vee}(L)$ the family of co-primes of L.

For example, given a topological space X, the set $\overline{\{x\}}$ is co-prime in $\Gamma(X)$ for each $x \in X$. Note that the least element of L cannot be co-prime, since $F = \emptyset$ is a possibility.

We want to topologize $\mathrm{Spec}_{\vee}(L)$, so we make the following definition.

Definition 4.5 If L is a lattice, we define $C \subseteq \mathrm{Spec}_{\vee}(L)$ to be *closed* if $C = \downarrow x \cap \mathrm{Spec}_{\vee}(L)$ for some $x \in L$. The *hull-kernel topology* on $\mathrm{Spec}_{\vee}(L)$ has these sets as its family of closed sets.

Of course, for this definition to make sense, it must be shown that the family of closed sets we have defined is closed under all intersections and all finite unions. The former is true since $\bigcap\{\downarrow x_i \cap \mathrm{Spec}_{\vee}(L) \mid i \in I\} = \downarrow (\bigwedge_i x_i) \cap \mathrm{Spec}_{\vee}(L)$, while the latter is an easy exercise using the fact that all elements of $\mathrm{Spec}_{\vee}(L)$ are co-prime.

Proposition 4.6 *Let* $\phi\colon L \to M$ *be a morphism of Brouwerian lattices. We define the* lower adjoint *of* ϕ *by* $\phi_*\colon M \to L$ *by* $\phi_*(x) = \bigwedge \phi^{-1}(\uparrow x)$. *Then:*

(i) $\phi \circ \phi_* \geq 1_M$ *and* $\phi_* \circ \phi \leq 1_L$*; .i.e.,* (ϕ, ϕ_*) *is a Galois adjunction between* L *and* M*.*

(ii) ϕ_* *preserves all suprema.*

(iii) $\phi_*(\mathrm{Spec}_{\vee}(M)) \subseteq \mathrm{Spec}_{\vee}(L)$.

(iv) $\phi_*|_{\mathrm{Spec}_{\vee}(M)}\colon \mathrm{Spec}_{\vee}(M) \to \mathrm{Spec}_{\vee}(L)$ *is hull-kernel continuous.*

Proof. It is clear that ϕ_* is well-defined since M is a complete lattice, and part (i) is then a routine exercise. Since ϕ preserves all infima, part (ii) follows from the general theory of adjunctions. Part (iii) follows from the fact that ϕ preserves finite suprema, and part (iv) again is easy. $\square$

Using this Proposition, we can prove the following result.

Corollary 4.7 *There is a functor* $\mathrm{Spec}\colon \mathsf{CBL}^{\mathrm{op}} \to \mathsf{TOP}$ *given by* $\mathrm{Spec}(L) = \mathrm{Spec}_{\vee}(L)$*, and for* $\phi\colon L \to M$*, Spec*$(\phi) = \phi_*|_{\mathrm{Spec}_{\vee}(L)}$. $\square$

Our aim is to use the functors Spec and Γ to establish as equivalence of categories, but this is not true in the generality we are in. For example, not every topological space is of the form $\mathrm{Spec}_{\vee}(L)$ for some complete Brouwerian lattice L. Indeed, the unit $\eta_X\colon X \to \mathrm{Spec}_{\vee}(\Gamma(X))$ satisfies $\eta_X(x) = \overline{\{x\}}$, and so this map is injective if and only if X is T_0. On the other side, the co-unit $\epsilon_L\colon L \to \Gamma(\mathrm{Spec}_{\vee}(L))$ given by $\epsilon_L(x) = \bigvee \downarrow X \cap \mathrm{Spec}_{\vee}(L)$ certainly is onto, but it is one-to-one if and only if every $x \in L$ is the supremum of the set $\downarrow x \cap \mathrm{Spec}_\vee(L)$. We make these special spaces and lattices the subject of our next definitions.

Definition 4.8 A closed subset $C \subseteq X$ of the topological space X is *irreducible* if C is a co-prime in $\Gamma(X)$; i.e., if C is not the union of two proper closed subsets. The space X is *sober* if every irreducible closed subset C satisfies $C = \overline{\{x\}}$ for a unique point $x \in X$. We let SOB denote the category of sober spaces and continuous maps.

The following proposition is routine.

Proposition 4.9

(i) *If L is a complete Brouwerian lattice, then $\mathit{Spec}_\vee(L)$ is a sober space in the hull-kernel topology.*

(ii) *If X is a topological space, then $\eta_X : X \to \mathrm{Spec}_\vee(\Gamma(X))$ is a continuous and open mapping onto its image.* □

Corollary 4.10 *The functor* $\mathrm{Spec} \circ \Gamma : \mathsf{TOP} \to \mathsf{SOB}$ *is left adjoint to the inclusion functor.* □

For a topological space X, the space $\mathrm{Spec}_\vee(\Gamma(X))$ is called the *sobrification* of X; it is the largest space having the same topology as X.

On the lattice side, we have the following.

Definition 4.11 A complete Brouwerian lattice L *has enough co-primes* if $x = \bigvee(\downarrow x \cap \mathrm{Spec}_\vee(L))$ for all $x \in L$. We let SCBL denote the category of such lattices and maps $\phi_* : L \to M$ that are upper adjoints to CBL-maps from M to L.

Proposition 4.12

(i) *If X is a topological space, then $\Gamma(X)$ has enough co-primes.*

(ii) *If L is a complete Brouwerian lattice, then the mapping $\epsilon_L : L \to \Gamma(\mathrm{Spec}_\vee(L))$ is a monomorphism of complete Brouwerian lattices.* □

Corollary 4.13 *The functor* $\Gamma \circ \mathrm{Spec} : \mathsf{CBL} \to \mathsf{SCBL}$ *is left adjoint to the inclusion functor.* □

A complete Brouwerian lattice also is called *spatial* if L has enough co-primes. All of this culminates in the following result.

Theorem 4.14 *The functors* $\Gamma|_{\mathsf{SOB}} : \mathsf{SOB} \to \mathsf{SCBL}^{\mathrm{op}}$ *and* $\mathrm{Spec}|_{\mathsf{SCBL}^{\mathrm{op}}} : \mathsf{SCBL}^{\mathrm{op}} \to \mathsf{SOB}$ *form a dual equivalence.* □

4.2 Continuous Posets

We know by now that we can endow each dcpo with its Scott topology, and obviously this would be a way to take advantage of the equivalence of categories we have just outlined. Unfortunately, in this generality, it is not clear whether every dcpo can be retrieved from its Scott topology. But we will be able to do this for continuous dcpo's, and it is convenient to generalize from the setting of dcpo's just a bit.

Definition 4.15 Let P be a poset. If $x, y \in P$, then we write $x \ll y$ if, for all directed sets $D \subseteq P$, if $\bigsqcup D$ exists in P and $y \sqsubseteq \bigsqcup D$, then $D \cap \uparrow x \neq \emptyset$. We say P is *continuous* if, for all $y \in P$,

- $\Downarrow y = \{x \in P \mid x \ll y\}$ is directed, and
- $y = \bigsqcup \Downarrow y$.

Likewise, $x \in P$ is *compact* iff $x \ll x$, and P is an *algebraic poset* if $K(x)$ is directed and satisfies $x = \bigsqcup K(x)$ for all $x \in P$. We let CPOS denote the category of continuous posets and Scott continuous maps, and APOS denote the full subcategory of algebraic posets.

The only difference between the definitions we just made and the earlier notions of continuity and algebraicity is that we no longer assume the underlying poset P is directed complete. Algebraic and continuous posets also have been studied in [36] and in [61], respectively. We shall see that the equivalence just outlined for sober spaces and spatial Brouwerian lattices yields a very satisfying theory for the categories CPOS and APOS. We begin our study with the following result, whose proof is the same as that for Proposition 3.11.

Lemma 4.16 *If P is a continuous poset, then $\ll$ satisfies the property*

$$\text{INT} \qquad x \ll y \;\Rightarrow\; (\exists z \in P)\; x \ll z \ll y$$

Hence, $\Uparrow x$ is Scott open for each $x \in P$. □

Lemma 4.17 *If P is a continuous poset and $C \in \mathrm{Spec}_\vee(\Gamma(P))$, then $\{x \in C \mid \Uparrow x \cap C \neq \emptyset\}$ is directed and $C = \bigsqcup \{\downarrow x \mid \Uparrow x \cap C \neq \emptyset\}$.*

Proof. Any closed set $C = \overline{\bigcup\{\downarrow x \mid x \in C\}}$ and each $x \in P$ satisfies $x = \bigsqcup \Downarrow x$ since P is continuous. Thus, $C = \overline{\bigcup\{\downarrow x \mid \Uparrow x \cap C \neq \emptyset\}}$. Suppose that $x, y \in C$ satisfy $\Uparrow x \cap C \neq \emptyset \neq \Uparrow y \cap C$. If $\Uparrow x \cap \Uparrow y \cap C = \emptyset$, then $C = (C \setminus \Uparrow x) \cup (C \setminus \Uparrow y)$ is the disjoint union of proper closed sets, which means $C \notin \mathrm{Spec}_\vee(\Gamma(P))$. This shows $\{x \in C \mid \Uparrow x \cap C \neq \emptyset\}$ is directed. And since $C = \overline{\bigcup\{\downarrow x \mid \Uparrow x \cap C \neq \emptyset\}}$, it follows that $C = \bigsqcup\{\downarrow x \mid \Uparrow x \cap C \neq \emptyset\}$. □

Proposition 4.18 *If P is a continuous poset, then*

(i) $\mathrm{Spec}_\vee(\Gamma(P))$ *is a dcpo.*

(ii) $C \ll D \in \mathrm{Spec}_\vee(\Gamma(P))$ *iff* $(\exists x \ll y \in P)\, C \subseteq \downarrow x \ll \downarrow z \subseteq D$.

(iii) $\mathrm{Spec}_\vee(\Gamma(P))$ *is continuous.*

(iv) *The mapping $\eta_P \colon P \to \mathrm{Spec}_\vee(\Gamma(P))$ is a homeomorphism onto its image, and the topology $\eta_P(P)$ inherits from the Scott topology on $\mathrm{Spec}_\vee(\Gamma(P))$ is the hull-kernel topology of $\eta_P(P)$.*

Proof. If L is a complete lattice and $D \subseteq L$ is a directed family of co-primes, then it is easy to show that $\bigsqcup D$ also is co-prime. This shows (i).

For (ii), suppose that $x \ll y \in P$ and that $\mathcal{D} \subseteq \mathrm{Spec}_\vee(\Gamma(P))$ is a directed family of closed sets whose supremum dominates $\downarrow y$. Then $y \in \overline{\bigcup \mathcal{D}}$. Since

$x \ll y$, it follows that $\Uparrow x$ is a Scott open set containing y, and so $\bigcup \mathcal{D} \cap \Uparrow x \neq \emptyset$. Since closed sets are lower sets, there is some set C in the family $\mathcal{D}$ with $x \in C$, and this means $\downarrow x \sqsubseteq C$. Thus $\downarrow x \ll \downarrow y$ in $\mathrm{Spec}_\vee(\Gamma(P))$. It then follows that $C \ll D$ for any sets C and D with $C \subseteq \downarrow x$ and $\downarrow y \subseteq D$.

Conversely, if $C \ll D$ in $\mathrm{Spec}_\vee(\Gamma(P))$, then the preceding lemma implies $D = \bigsqcup\{\downarrow x \mid \Uparrow x \cap D \neq \emptyset\}$, and this supremum is directed. Hence, $(\exists x \in D)$ $\Uparrow x \cap D \neq \emptyset$ & $C \subseteq \downarrow x$. Since $\Uparrow x \cap D \neq \emptyset$, Lemma 4.16 implies there is some $y \in D \cap \Uparrow x$ with $\Uparrow y \cap D \neq \emptyset$. Then the first part of the proof implies $\downarrow x \ll \downarrow y$, and so $C \subseteq \downarrow x \ll \downarrow y \subseteq D$, which proves part (ii).

Part (iii) follows from part (ii), Lemma 4.17 and the continuity of P. The first part of (iv) follows from the fact that P is T_0 in the Scott topology. Since directed sets in $\mathrm{Spec}_\vee(\Gamma(P))$ converge to the same point in the Scott topology of $\mathrm{Spec}_\vee(\Gamma(P))$ as they do in the hull-kernel topology (since $\mathrm{Spec}_\vee(\Gamma(P))$ is closed under directed suprema in $\Gamma(P)$), the identity map is continuous from the Scott topology to the hull-kernel topology. Conversely, if $C \in U \subseteq \mathrm{Spec}_\vee(\Gamma(P))$ and U is Scott open, then there is some $x \in P$ with $C \in \{D \mid \Uparrow x \cap D \neq \emptyset\} \subseteq U$. Then

$$\begin{aligned}\mathrm{Spec}_\vee(\Gamma(P)) \setminus \{D \mid \Uparrow x \cap D \neq \emptyset\} &= \{D \in \mathrm{Spec}_\vee(\Gamma(P)) \mid D \cap \Uparrow x = \emptyset\} \\ &= \{D \in \mathrm{Spec}_\vee(\Gamma(P)) \mid D \subseteq P \setminus \Uparrow x\},\end{aligned}$$

which clearly is hull-kernel closed, and so the topologies are the same. □

Since $\mathrm{Spec}_\vee(\Gamma(P))$ is sober for any continuous poset P, the following result is clear.

Corollary 4.19 *The functor* $\mathrm{Spec} \circ \Gamma \colon \mathsf{CPOS} \to \mathsf{CON}$ *is left adjoint to the forgetful functor. Hence, the continuous poset P is sober if and only if P is a dcpo.* □

Thus, the sobrification of a continuous poset is a continuous dcpo with "the same way-below relation." Of course, we can restrict our attention to the algebraic case to obtain the following.

Corollary 4.20 *If P is an algebraic poset, then* $\mathrm{Spec}_\vee(\Gamma(P))$ *is an algebraic dcpo with* $K(\mathrm{Spec}_\vee(\Gamma(P))) = \{\downarrow k \mid k \in P\}$. *Hence, P is a dcpo if and only P is sober in the Scott topology.* □

One might ask which algebraic posets P satisfy the property that $\mathrm{Idl}(P) \simeq \mathrm{Spec}_\vee(\Gamma(P))$ the answer is the following.

Proposition 4.21 *An algebraic poset P satisfies* $\mathrm{Idl}(P) \simeq \mathrm{Spec}_\vee(\Gamma(P))$ *if and only if $P = K(P)$.* □

All of the material we have presented has been for continuous posets, and the resulting directed complete partial orders are continuous dcpo's. Clearly a similar development can be made for continuous posets with least element, and then the resulting directed complete partial order would be cpo's.

Our stated motivation was to present a theory that avoided the use of approximable relations. This theory does that, but it does not have the "purely algebraic" flavor that using approximable relations affords. On the other hand, this theory provides a nice example of how the sobrification functor can yield pleasing results relating categories of incomplete partial orders to ones that are complete. This highlights that fact that sober spaces might best be thought of in terms of completeness, rather than separation.

Finally, one might ask whether the theory we have presented can be extended to a larger class of posets endowed with the Scott topology. While this may be true, such a theory cannot include all dcpo's as the target of the sobrification functor, as Johnstone's example [28] of a dcpo whose Scott topology is not sober shows.

4.3 Duality Theories

One of the appealing aspects of domain theory is the fact that rich duality theories can be devised for it. These theories rely on both aspects of domains: their intrinsic topological structure as represented by the Scott topology, and their intrinsic algebraic structure, represented by the role that compact elements play in the structure of domains. The basic theory relies on analyzing the use of spectral theory of the previous section somewhat more carefully.

In applying the sobrification functor, we "passed through" the family $\Gamma(P)$ for P a continuous poset. Since P and $\mathrm{Spec}_\vee(\Gamma(P))$ have the same closed sets, we can investigate the complete Brouwerian lattice $\Gamma(P)$ assuming that P is a continuous poset or a continuous dcpo.

Definition 4.22 A complete lattice L is *continuous* if L is a continuous cpo. Likewise, L is *algebraic* if L is algebraic as a cpo. The lattice L is *bicontinuous* (resp., *bialgebraic*) if both L and L^{op} are continuous (resp., algebraic).

An examination of the proof of part ii) of Proposition 4.18 shows that $x \ll y \in P$ implies $\downarrow\! x \ll \downarrow\! y$ in $\Gamma(P)$ for a continuous poset P. It then follows that, if $x_i \ll y_i$ for each $i = 1 \ldots, n$, then $\cup_i \downarrow\! x_i \ll \cup_i \downarrow\! y_i$ in $\Gamma(P)$. It then is routine to show that any closed subset C of a continuous poset P satisfies

$$C = \bigsqcup\{\downarrow\! F \mid F \subseteq C \text{ finite } \& \ \downarrow\! F \ll C\}.$$

That is, $\Gamma(P)$ is a continuous cpo.

Dually, it can be shown that $P\setminus \Uparrow\! y$ is way-below $P\setminus \Uparrow\! x$ in $(\Gamma(P), \supseteq)$ if $x \ll y \in P$, again for P a continuous poset. If $C \subseteq P$ is closed, then $C = \bigcap\{P\setminus \Uparrow\! F \mid F \subseteq P \setminus C \text{ finite}\}$, and $P\setminus \Uparrow\! F \ll C$ in $(\Gamma(P), \supseteq)$ for each such F. Since this family is easily seen to be directed under reverse inclusion, it follows that $(\Gamma(P), \supseteq)$ also is continuous. Hence $\Gamma(P)$ is bicontinuous if P is a continuous poset. Since $\Gamma(P)$ is Brouwerian, [20], Proposition VII-2.9 then implies that $\Gamma(P)$ also is completely distributive.

Theorem 4.23 *If P is a continuous poset, then $\Gamma(P)$ is a completely distributive bicontinuous lattice. Moreover, $C \ll D$ in $\Gamma(P)$ if and only if there are finite subsets $F, G \subseteq P$ such that $C \subseteq \downarrow\! F \ll \downarrow\! G \subseteq D$. Dually, $C \ll D$ in $(\Gamma(P), \supseteq)$ if and only if there are finite subsets $F, G \subseteq P$ such that $C \supseteq P\setminus \Uparrow\! F \ll P\setminus \Uparrow\! G \supseteq D$.*

Moreover, $\Gamma(P)$ is algebraic if and only if P is algebraic, in which case

$K(\Gamma(P)) = \{\uparrow F \mid F \subseteq K(P) \text{ finite}\}$, *and* $K(\Gamma(P)^{\mathrm{op}}) = \{P \setminus \uparrow F \mid F \subseteq K(P) \text{ finite}\}$. □

Lawson duality also asserts that the converse of this Theorem holds. Namely, if L is a completely distributive, bicontinuous lattice, then $\mathrm{Spec}_{\vee}(L)$ is a continuous dcpo and $L \simeq \Gamma(\mathrm{Spec}_{\vee}(L))$ (cf. [33]).

Theorem 4.23 can be raised to the level of a duality theory by applying the general equivalence between sober spaces and spatial complete Brouwerian lattices. Indeed, the continuous functions between continuous dcpo's are the Scott continuous maps, and these correspond precisely to the maps between the respective closed-set lattices that preserve all suprema and co-primes. To state this precisely, we let CDCPO denote the category of continuous dcpo's and Scott continuous maps, and BDL denote the category of bicontinuous completely distributive lattices and maps preserving all suprema and all co-primes.

Theorem 4.24 *The functors* $\Gamma|_{\mathsf{CDCPO}}\colon \mathsf{CDCOP} \to \mathsf{BDL}$ *and* $\mathrm{Spec}|_{\mathsf{BDL}}\mathsf{BDL} \to \mathsf{CDCOP}$ *form an equivalence.*

These functors further cut down to an equivalence between the full subcategories ALG *of algebraic dcpo's and* BAL *of bialgebraic completely distributive lattices.* □

4.4 Power Domains

One of the most important constructs for semantics is that of *power domains.* The idea is to have a model for nondeterminism. There are three traditional power domains, and these constructs can be defined purely algebraically – i.e., order-theoretically. We begin with the definitions, and then proceed to recast them topologically.

Nondeterministic choice is meant to be a binary operation which satisfies some simple algebraic rules: *associativity, commutativity* and *idempotency.* Thus, a model of nondeterministic choice is simply a semilattice. The traditional path to building a model for nondeterminism is to start with a model for sequential composition and perhaps some additional operations as well, and then to construct a model for nondeterminism "on top" of the existing model. Thus one usually begins with a *continuous algebra* relative to some signature Σ (i.e., a Σ-algebra whose underlying set is a continuous cpo such that the interpretation of all of the operations is continuous), and seeks to add a semilattice operation to the model. The following development modularizes this by first constructing free ordered semilattices over posets, and then extending them naturally to be continuous algebras.

Definition 4.25 Let P be a poset. We define the family

(a) $L_{\mathrm{fin}}(P) = \{\downarrow F \mid \emptyset \neq F \subseteq P \text{ finite}\}$ with

$$\downarrow F \sqsubseteq_L \downarrow G \text{ iff } \downarrow F \subseteq \downarrow G \quad \text{and} \quad \downarrow F + \downarrow G = \downarrow (F \cup G).$$

(b) $U_{\mathrm{fin}}(P) = \{\uparrow F \mid \emptyset \neq F \subseteq P \text{ finite}\}$ with

$$\uparrow F \sqsubseteq_U \uparrow G \text{ iff } \uparrow G \subseteq \uparrow F \quad \text{and} \quad \uparrow F + \uparrow G = \uparrow (F \cup G).$$

(c) $C_{\text{fin}}(P) = \{\langle F\rangle = \downarrow F \cap \uparrow F \mid \emptyset \neq F \subseteq P \text{ finite}\}$ with

$$\langle F\rangle \sqsubseteq_C \langle G\rangle \text{ iff } \downarrow F \sqsubseteq_L \downarrow G \;\&\; \uparrow F \sqsubseteq_U \uparrow G \quad \text{and} \quad \langle F\rangle + \langle \downarrow G\rangle = \langle F \cup G\rangle.$$

Proposition 4.26 *Let* POS *be the category of posets and monotone maps.*

(i) *If* SUP *is the category of sup-semilattices and sup-semilattice maps, then the functor* $L\colon \mathsf{POS} \to \mathsf{SUP}$ *by* $L(P) = L_{\text{fin}}(P)$ *and* $L(f)(\downarrow F) = \downarrow f(F)$ *is left adjoint to the forgetful functor from* SUP *to* POS*.*

(ii) *If* INF *is the category of inf-semilattices and inf-semilattice maps, then the functor* $U\colon \mathsf{POS} \to \mathsf{INF}$ *by* $U(P) = U_{\text{fin}}(P)$ *and* $U(f)(\uparrow F) = \uparrow f(F)$ *is left adjoint to the forgetful functor from* INF *to* POS*.*

(iii) *If* OSEM *is the category of ordered semilattices and ordered-semilattice maps, then the functor* $C\colon \mathsf{POS} \to \mathsf{OSEM}$ *by* $C(P) = C_{\text{fin}}(P)$ *and* $C(f)(\langle F\rangle) = \langle f(F)\rangle$ *is left adjoint to the forgetful functor from* OSEM *to* POS*.*

Proof. We outline the proof for (iii); the others are similar. Let P be a poset and S an ordered semilattice (i.e., a semilattice with a partial order relative to which the semilattice operation is monotone), and suppose $f\colon P \to S$ is a monotone map.

The family $C(P) = C_{\text{fin}}(P)$ is a semilattice under the operation $(\langle F\rangle, \langle G\rangle) \mapsto \langle F \cup G\rangle$. Moreover, if $\langle F_1\rangle \sqsubseteq_C \langle F_2\rangle$ and $\langle G_1\rangle \sqsubseteq_C \langle G_2\rangle$, then

$$\downarrow(\langle F_1 \cup G_1\rangle) = \downarrow(F_1 \cup G_1) = \downarrow F_1 \cup \downarrow G_1 \subseteq \downarrow F_2 \cup \downarrow G_2 = \downarrow(\langle F_2 \cup G_2\rangle),$$

and, similarly,

$$\uparrow(\langle F_2 \cup G_2\rangle) = \uparrow(F_2 \cup G_2) = \uparrow F_2 \cup \uparrow G_2 \subseteq \uparrow F_1 \cup \uparrow G_1 = \uparrow(\langle F_1 \cup G_1\rangle).$$

Thus the semilattice operation is monotone on $C_{\text{fin}}(P)$.

Now, define $C(f)\colon C_{\text{fin}}(P) \to S$ by $C(f)(\langle F\rangle) = x_1 * \cdots * x_n$, where $F = \{x_1, \ldots, x_n\}$ and $*$ is the semilattice operation on S. It is routine to show $C(f)$ is well-defined and that $C(f)$ is a semilattice map. Finally, $C(f)(\{x\}) = f(x)$ is clear, and this is the unique semilattice map from $C_{\text{fin}}(P)$ to S satisfying this property since $C_{\text{fin}}(P)$ is generated by the image of P under the map $x \mapsto \{x\} = \langle\{x\}\rangle$. □

In Section 3 we pointed out that the category of algebraic dcpo's and Scott continuous maps is equivalent to the category of posets and approximable relations. There is another relationship between the category of DCPO dcpo's and Scott continuous maps and the category POS of posets and monotone maps that is worth pointing out.

Proposition 4.27 *The functor* $\mathrm{Idl}\colon \mathsf{POS} \to \mathsf{DCPO}$ *defined by* $\mathrm{Idl}(P) = \{I \subseteq P \mid \emptyset \neq I = \downarrow I \text{ directed}\}$ *and* $\mathrm{Idl}(f)(I) = \downarrow f(I)$ *is left adjoint to the forgetful functor.*

Proof. Certainly $\mathrm{Idl}(P)$ is a dcpo for any poset P. And if $f\colon P \to Q$ is a monotone map from the poset P to the dcpo Q, then we can define the mapping $\hat{f}\colon \mathrm{Idl}(P) \to Q$ by $\hat{(f)}(I) = \bigsqcup f(I)$. Since f is monotone and I is an ideal, it follows that $f(I)$ is directed, so this supremum is well-defined. And if $C \subseteq Q$ is a Scott-closed set, then

$$\hat{f}^{-1}(C) = \{I \in \mathrm{Idl}(P) \mid \bigsqcup f(I) \in C\} = \{I \in \mathrm{Idl}(P) \mid f(I) \subseteq C\}.$$

It is clear that this is a lower set in $\mathrm{Idl}(P)$, and it is routine to show that this family is closed under directed suprema (which are just increasing unions).

Moreover, if we define $\eta_P : P \to \mathrm{Idl}(P)$ by $\eta_P(x) = \downarrow x$, then this map is continuous and $\hat{f} \circ \eta_P = f$, and this is the unique continuous map $g : \mathrm{Idl}(P) \to Q$ satisfying $g \circ \eta_P = f$ since $\eta_P(P)$ is Scott-dense in $\mathrm{Idl}(P)$. □

Next, we note the following result.

Proposition 4.28 *If S is an ordered semilattice, then $\mathrm{Idl}(S) = \{I \subseteq P \mid I = \downarrow I$ is directed$\}$ is an algebraic dcpo which also is a semilattice, and the semilattice operation on $\mathrm{Idl}(S)$ induced from that of S is continuous.*

Proof. Let $* : S \times S \to S$ be the semilattice operation on S. We define $* : \mathrm{Idl}(S) \times \mathrm{Idl}(S) \to \mathrm{Idl}(S)$ by $I * J = \downarrow \{x * y \mid x \in I \ \&\ y \in J\}$. Since I and J are directed and $* : S \times S \to S$ is monotone, it follows that $\{x * y \mid x \in I \ \&\ y \in J\}$ also is directed, and so $I * J$ is an ideal of S. Also, using Proposition 4.27 and the fact that $\mathrm{Idl}(P \times Q) = \mathrm{Idl}(P) \times \mathrm{Idl}(Q)$ for all posets P and Q, the fact that $* : S \times S \to S$ is monotone implies that $* : \mathrm{Idl}(S) \times \mathrm{Idl}(S) \to \mathrm{Idl}(S)$ is continuous. □

Note that a corollary of this last result is that, if S is a sup- (respectively, an inf-) semilattice, then so is $\mathrm{Idl}(S)$ under the operation induced from the semilattice operation from S.

Corollary 4.29 *The restriction of the functor Idl to each of the categories* SUP, INF *and* OSEM, *respectively, is a left adjoint to the inclusion functor from the associated category of continuous semilattices and continuous semilattice maps. The composition of this restriction with each of the left adjoints L, U and C gives a left adjoint to the inclusion of the associated category of continuous semilattices into* POS, *respectively.* □

This last result can be used along with the following purely categorical result to lift the free ordered semilattices just constructed to free algebraic semilattices, thus building of universal algebraic semilattices over algebraic cpo's.

Theorem 4.30 [37] *Let A, B and C be categories, and suppose $F_{\mathsf{AB}} : \mathsf{A} \to \mathsf{B}$ is left adjoint to $U_{\mathsf{AB}} : \mathsf{B} \to \mathsf{A}$, and that $F_{\mathsf{AC}} : A \to C$ is left adjoint to $U_{\mathsf{AC}} : \mathsf{C} \to \mathsf{A}$. Also, suppose there is a functor $U_{\mathsf{BC}} : C \to B$ satisfying $U_{\mathsf{AB}} \circ U_{\mathsf{BC}} \simeq$ [2] U_{AC}, and finally suppose that for each object b in B, there is an object Gb in A such that $F_{\mathsf{AB}} Gb \simeq b$. Then there is a left adjoint $F_{\mathsf{BC}} : B \to C$ to U_{BC} given by $F_{\mathsf{BC}} b = F_{\mathsf{AC}} Gb$ and $F_{\mathsf{BC}} \circ F_{\mathsf{AB}} \simeq F_{\mathsf{AC}}$.*

[2] By $\simeq$ we mean natural isomorphism.

$$\begin{tikzcd}
\mathsf{A} \arrow[r, shift left, "F_{\mathsf{AB}}"] \arrow[d, shift right, "F_{\mathsf{AC}}"'] & \mathsf{B} \arrow[l, shift left, "U_{\mathsf{AB}}"] \arrow[dl, dashed, shift right, "\exists F_{\mathsf{BC}}"'] \\
\mathsf{C} \arrow[u, shift right, "U_{\mathsf{AC}}"'] \arrow[ur, shift right, "U_{\mathsf{BC}}"'] &
\end{tikzcd}$$

□

Our particular application of this Theorem is to the case that $\mathsf{A} = \mathsf{POS}$ is the category of posets and monotone maps, $\mathsf{B} = \mathsf{CPO}$ is the category of algebraic cpo's, and $F_{AB} = \mathrm{Idl}$ is the ideal functor. The function G on objects of B associates to each algebraic cpo P the set $K(P)$ of compact elements of P. By choosing C to be an appropriate category of algebras over POS – in the case of power domains, these will be categories of continuous semilattices – we find that having a universal C-algebra over a poset $K(P)$ naturally leads to a universal C-algebra over the algebraic cpo P. We see that each of the power domains arises in exactly this fashion, so the construction of these objects has been broken down into two steps: first form the appropriate free ordered semilattice over the family of compact elements of an algebraic cpo, and then apply Corollary 4.29 to obtain the free continuous algebra over the cpo.

We let

ADCPO denote the category of algebraic dcpo's and continuous maps,

ASUP denote the category of algebraic dcpo's having a continuous sup-semilattice operation and continuous maps preserving finite suprema,

AINF denote the category of algebraic dcpo's which also have a continuous inf-semilattice operation and continuous maps preserving finite infima, and

ASEM denote the category of algebraic dcpo's having a continuous semilattice operation and continuous maps preserving finite products.

Theorem 4.31 Hennessy & Plotkin [23]

(i) *The functor* $\mathcal{P}_L\colon \mathsf{ADCPO} \to \mathsf{ASUP}$ *defined by* $\mathcal{P}_L(P) = \mathrm{Idl} \circ L_{\mathrm{fin}}(K(P))$ *and* $\mathcal{P}_L(f)\colon \mathcal{P}_L(P) \to \mathcal{P}_L(Q)$ *by*

$$\mathcal{P}_L(f)(I) = \bigsqcup\{\downarrow f(F) \mid \emptyset \neq F \subseteq K(P) \;\&\; \downarrow F \in I\}$$

is left adjoint to the forgetful functor from ASUP *to* ADCPO.

(ii) *The functor* $\mathcal{P}_U\colon \mathsf{ADCPO} \to \mathsf{AINF}$ *defined by* $\mathcal{P}_U(P) = \mathrm{Idl} \circ U_{\mathrm{fin}}(K(P))$ *and* $\mathcal{P}_U(f)\colon \mathcal{P}_U(P) \to \mathcal{P}_U(Q)$ *by*

$$\mathcal{P}_U(f)(I) = \bigsqcup\{\uparrow f(F) \mid \emptyset \neq F \subseteq K(P) \;\&\; \uparrow F \in I\}$$

is left adjoint to the forgetful functor from AINF *to* ADCPO.

(iii) *The functor* $\mathcal{P}_C\colon \mathsf{ADCPO} \to \mathsf{ASEM}$ *defined by* $\mathcal{P}_C(P) = \mathrm{Idl} \circ C_{\mathrm{fin}}(K(P))$ *and* $\mathcal{P}_C(f)\colon \mathcal{P}_C(P) \to \mathcal{P}_C(Q)$ *by*

$$\mathcal{P}_C(f)(I) = \bigsqcup\{\langle f(F)\rangle \mid \emptyset \neq F \subseteq K(P) \;\&\; \langle F\rangle \in I\}$$

is left adjoint to the forgetful functor from ASEM *to* ADCPO.

Proof. Again we confine ourselves to an outline of the last assertion, the others being similar. We know from Proposition 4.26 that the functor C from POS to OSEM is left adjoint to the forgetful functor, and Corollary 4.29 implies the restriction of the functor Idl is left adjoint to the forgetful functor from ASEM to OSEM. Corollary 4.29 further implies that $\mathrm{Idl}|_{\mathsf{OSEM}} \circ C\colon \mathsf{POS} \to \mathsf{ASEM}$ is left adjoint to the inclusion, and Theorem 4.30 then implies it induces $\mathcal{P}_C\colon \mathsf{ADCPO} \to \mathsf{ASEM}$ which is left adjoint to the inclusion. Given the definitions of Idl and of C, it is routine to show that $\mathcal{P}_C$ acts on objects and morphisms as indicated. $\square$

If we apply the construction for the free sup-semilattice to $K(P)$ for an algebraic poset P, and then take the ideal completion to obtain $\mathcal{P}_L(P)$, then we have the *lower power domain*, or the *Hoare power domain*, as it sometimes is called. Similarly, $\mathcal{P}_U(P)$ is the *upper power domain*, or *Smyth power domain* using the free inf-semilattice over $K(P)$, and $\mathcal{P}_C(P)$ is the *convex power domain* or *Plotkin power domain* over the algebraic dcpo P. It was first pointed out in [23] that each of these yields a left adjoint to the forgetful functor from a category of ordered semilattice dcpo's into ALG.

Theorem 4.31 serves to define the three traditional power domains for all algebraic dcpo's. The first of these also has a simple topological representation.

Proposition 4.32 *If P is an algebraic dcpo, then $\mathcal{P}_L(P) \simeq (\Gamma_0(P), \subseteq)$, where $\Gamma_0(P)$ denotes the family of non-empty closed subsets of P.*

Proof. For non-empty, finite subsets F, G of $K(P)$, $F \sqsubseteq_L G$ iff $F \subseteq \downarrow G$ iff $\downarrow F \subseteq \downarrow G$, and this implies the mapping $\downarrow F \mapsto \downarrow F\colon L_{\mathrm{fin}}(K(P)) \to \Gamma_0(P)$ is an isomorphism of $L_{\mathrm{fin}}(P)$ onto $K(\Gamma_0(P))$. Clearly this mapping preserves the semilattice operation, and since each of $\mathrm{Idl}(L_{\mathrm{fin}}(K(P)), \sqsubseteq_L)$ and $\Gamma_0(P)$ is an algebraic dcpo, the isomorphism extends to one of the structures themselves. $\square$

This leads to a definition of an analogue to the lower power domain for continuous dcpo's. We let SUPCON denote the category of continuous dcpo's endowed with a Scott continuous sup-semilattice operation and Scott continuous maps preserving the sup-semilattice operation.

Proposition 4.33 *The functor $\Gamma_0\colon \mathsf{CON} \to \mathsf{SUPCON}$ which sends each continuous dcpo P to $\Gamma_0(P)$ endowed with the union operation, and each mapping $f\colon P \to Q$ to the sup-semilattice mapping $\Gamma_0(f)(C) = \overline{f(C)}$ is left adjoint to the forgetful functor.*

Proof. The functor clearly is well-defined on objects. And the unit of the adjunction is the mapping $x \mapsto \downarrow x\colon P \to \Gamma_0(P)$. Suppose that S is a sup-semilattice continuous dcpo and that $f\colon P \to S$ is continuous. Then we can define $\Gamma_0(f)(\downarrow x) = f(x)$ for each $x \in P$. Given $C \in \Gamma_0(P)$, we have $C = \bigsqcup\{\downarrow F \mid \downarrow F \subseteq C, F \text{ finite}\}$, and this sup is directed. So we can extend $\Gamma_0(f)$ by defining $\Gamma_0(f)(C) = \bigsqcup(\vee_S f(F))$, This mapping is well-defined and continuous. $\square$

For the other power domains – the upper and convex – we require some more development and a restriction of the class of dcpo's considered for a topological analogue for them to be derived. We begin with the following definition.

Definition 4.34 A subset $S \subseteq X$ of the topological space X is *saturated* if it is the intersection of the open sets that contain it.

It is routine to show that a subset $A \subset X$ of a topological space X is compact if and only if $\mathrm{Sat}(A) = \bigcap\{U \mid A \subseteq U \text{ open}\}$ is saturated. Moreover, the saturated subsets of a partially ordered set endowed with the Scott topology are precisely the upper sets; this follows since $P\setminus \downarrow x$ is Scott open for all $x \in P$.

Theorem 4.35 Hofmann and Mislove [24]
Let X be a sober space, and let $\mathcal{F}$ be a filter basis of compact saturated subsets of X. Then

(i) *$\bigcap \mathcal{F}$ is compact, and*

(ii) *if $\bigcap \mathcal{F} \subseteq U$ with U open, then there is some $C \in \mathcal{F}$ such that $C \subseteq U$.* □

This theorem can be proved by first showing that the family of Scott compact saturated subsets of a sober space X is isomorphic to the semilattice of Scott-open filters in the lattice of Scott-open subsets of X. A simple analysis of this structure then yields the result. Alternatively, as in [31], a direct proof can be given.

It is important to note that the second part of the theorem implies that if the intersection of a filter basis of compact saturated sets is empty, then one of them is empty.

Finally, recall from Corollary 4.19 that continuous dcpo's are sober in the Scott topology, so the above result applies to them.

We now are ready to give a topological representation of the upper power domain. This result was first discovered by Smyth [52] for the case of domains. Given the tools at our disposal, however, we can extend the definitions to all continuous dcpo's. To begin, let INFCON denote the category of continuous dcpo's which also have a continuous inf-semilattice operation, and continuous, inf-preserving maps, and recall CON denotes the category of continuous dcpo's and continuous maps.

Proposition 4.36 *Let P be a continuous dcpo, and let $\mathcal{C}(P)$ denote the family of non-empty Scott compact, saturated subsets of P. Then:*

(i) *$(\mathcal{C}(P), \cup)$ is a continuous dcpo inf-semilattice.*

(ii) *The functor $\mathcal{C}\colon \mathsf{CON} \to \mathsf{INFCON}$ which associates $\mathcal{C}(P)$ to the continuous dcpo P, and to the continuous mapping $f\colon P \to Q$ the mapping $\mathcal{C}(f)\colon \mathcal{C}(P) \to \mathcal{C}(Q)$ by $\mathcal{C}(f)(C) = \uparrow f(C)$ is left adjoint to the forgetful functor.*

(iii) *If P is algebraic, then $(\mathcal{C}(P), \cup) \simeq (\mathcal{P}_U(P), \cup)$.*

Proof. Suppose P is continuous. Then each compact saturated subset C of P can be written as the filtered intersection of sets of the form $\uparrow F$, where $C \subseteq \uparrow F$ and $F \subseteq P$ is finite. Conversely, Theorem 4.35 implies that each filtered intersection of sets of the form $\uparrow F$ for $F \subseteq P$ finite is Scott compact, and so this accounts for all compact saturated subsets of P. Moreover, Theorem 4.35 also implies that $\mathcal{C}(P)$ is a dcpo under reverse containment.

Now, suppose C is saturated and compact, and $C \subseteq \Uparrow F$ for $F \subseteq P$ finite. If $\mathcal{F}$ is a filter basis of compact saturated subsets of P such that $C \supseteq \bigcap \mathcal{F}$, then since $\Uparrow F$ is Scott open and $C \subset \Uparrow F$, Theorem 4.35 again implies that $X \subseteq \Uparrow F$ for some $X \in \mathcal{F}$. It follows that $\uparrow F \ll C$ in $\mathcal{C}(P)$ (in the opposite order). The continuity of P then implies $(\mathcal{C}(P), \supseteq)$ is a continuous dcpo, and it is clear that union is a continuous inf-semilattice operation on $(\mathcal{C}(P), \supseteq)$. This proves (i).

For part (ii), it is clear that $\mathcal{C}\colon \mathsf{CON} \to \mathsf{INFCON}$ is well-defined. Suppose S is a continuous dcpo with a continuous inf-semilattice operation, and suppose $f\colon P \to S$ is continuous. If $C \in \mathcal{C}(P)$, then $C = \bigcap\{\uparrow F \mid C \subseteq \Uparrow F \;\&\; F \,\text{finite}\}$. Then $\hat{f}(\uparrow F) = *_S f(F)$ is well-defined,[3] and $\hat{f}(C) = \bigsqcup\{*_S f(F) \mid C \subseteq \Uparrow F \;\&\; F\,\text{finite}\}$ defines $\hat{f}(C)$. If $\mathcal{F} \subseteq \mathcal{C}(P)$ is directed, then $\bigsqcup \mathcal{F} = \bigcap \mathcal{F}$, and clearly $\{\hat{f}(C) \mid C \in \mathcal{F}\} \subseteq S$ is filtered. Now, $\bigsqcup \hat{f}(\mathcal{F}) \sqsubseteq_S \hat{f}(\bigcap \mathcal{F})$ holds just because $\hat{f}$ is monotone.

Conversely, let $\hat{f}(\bigcap \mathcal{F}) \in \Uparrow s$ for some $s \in S$. Then $\hat{f}(\bigcap \mathcal{F}) = \bigsqcup\{*_S f(F) \mid \bigcap \mathcal{F} \subseteq \Uparrow F \;\&\; F\,\text{finite}\}$, and so there is some finite subset $F \subseteq P$ with $\bigcap \mathcal{F} \subseteq \Uparrow F$ and $s \sqsubseteq_S *_S f(F)$. Since $\Uparrow F$ is a neighborhood of $\bigcap \mathcal{F}$, Theorem 4.35 implies $C \subseteq f^{-1}(U)$ for some $C \in \mathcal{F}$, and the monotonicity of $\hat{f}$ then implies $s \sqsubseteq_S *_S f(F) \sqsubseteq_S \hat{f}(C)$. Since $\hat{f}(\bigcap \mathcal{F}) = \bigsqcup_S \{s \in S \mid s \ll \hat{f}(\bigcap \mathcal{F})\}$, we conclude that $\bigsqcup_S \hat{f}(\mathcal{F}) = \hat{f}(\bigcap \mathcal{F})$, and this implies $\hat{f}\colon \mathcal{C}(P) \to S$ is continuous. This map clearly preserves finite infs, and it is completely determined by f. This proves (ii).

For part (iii), we note that the proof of the second part shows that $C = \bigsqcup\{\uparrow F \mid C \subseteq \Uparrow F \;\&\; F \text{ finite}\}$ characterizes the way-below relation in $(\mathcal{C}(P), \supseteq)$. If P is algebraic, then the finite sets $F \subseteq P$ such that $C \subseteq \Uparrow F$ can be taken to consist of compact elements, in which case $\Uparrow F = \uparrow F$. It then follows that $K(P_U(P)) = \{\uparrow F \mid \emptyset \neq F \subseteq K(P) \text{ finite}\}$ is a basis for $\mathcal{C}(P)$, and so $\mathcal{P}_U(P)$ and $\mathcal{C}(P)$ are isomorphic. $\square$

This affords the desired generalization of the upper power domain to continuous dcpo's. Obtaining a generalization of the convex power domain requires more work still. To derive the result we seek, we restrict ourselves to an interesting subclass of continuous dcpo's.

Definition 4.37 A domain P is *coherent* if it is Scott compact and the intersection of Scott-compact saturated subsets of P again is Scott compact.

Proposition 4.38 *A compact domain P is coherent if and only if*

$$(\forall F, G \subseteq K(P) \textit{ finite})(\exists H \subseteq K(P) \textit{ finite}) \;\; \uparrow F \cap \uparrow G = \uparrow H.$$

More generally, a continuous compact dcpo P is coherent iff

$$(\forall F, G \subseteq P \textit{ finite}) \uparrow F \cap \uparrow G \textit{ is compact.}$$

[3] By $*_S F$ we mean the product in S of the elements of the finite set F.

Proof. First, suppose P is a domain. If $F, G \subseteq K(P)$ are finite, then clearly $\uparrow F$ and $\uparrow G$ are Scott compact, and so coherence implies the same is true of $\uparrow F \cap \uparrow G$. But this set also is Scott open, and so it is the union of sets of the form $\uparrow k$, for $k \in K(P)$. Then compactness implies we can find finitely many such compact elements whose union is the set $\uparrow F \cap \uparrow G$, and this is the finite set H we seek.

To show the converse, first recall the sets of the form $\uparrow k$ for $k \in K(P)$ are a basis for the Scott topology, and any compact upper set A is the intersection of a filter basis of sets of the form $\uparrow F$, for $F \subseteq K(P)$ finite. So, given compact upper sets A and B, we conclude that

$$\begin{aligned} A \cap B &= (\bigcap\{\uparrow F \mid A \subseteq \uparrow F\}) \cap (\bigcap\{\uparrow G \mid B \subseteq \uparrow G\}) \\ &= \bigcap\{\uparrow F \cap \uparrow G \mid A \subseteq \uparrow F \ \&\ B \subseteq \uparrow G\} \end{aligned}$$

where the sets F and G all are finite subsets if $K(P)$ and the intersections are filtered. Theorem 4.35 then shows that $A \cap B$ is compact.

More generally, if P is a coherent dcpo and $F, G \subseteq P$ are finite, then $\uparrow F$ and $\uparrow G$ are compact, and so the same is true of $\uparrow F \cap \uparrow G$. Conversely, if this condition holds and $C, D \subseteq P$ are compact, then we can write each of these sets as a filtered intersection of sets of the form $\uparrow F$ where F is finite and the set in question is within $\Uparrow F$. The same argument as in the algebraic case then implies $C \cap D$ is compact. □

So far we have focused on the Scott topology on dcpo's. We now introduce a refinement of that topology.

Definition 4.39 Let P be a continuous dcpo. We define the *Lawson topology* on P to be the smallest topology for which $U \setminus \uparrow F$ is open for all $U \subseteq P$ Scott open and all $F \subseteq P$ finite.

It is routine to show that the collection $\{\Uparrow x \setminus \uparrow F \mid x \in P \ \&\ F \subseteq P \text{ finite}\}$ is a basis for the Lawson topology on a continuous dcpo P; in particular, if P is algebraic, then $\{x\} \cup F$ can be taken to be consist of compact elements.

Proposition 4.40 *Let P be a continuous dcpo. Then*

(i) *The Lawson topology on P is Hausdorff and the order $\sqsubseteq$ is closed in $P \times P$ in the product of the Lawson topologies.*

(ii) *The Lawson topology on an algebraic dcpo P is totally disconnected.*

(iii) *P is coherent if and only if the Lawson topology is compact.*

Proof. For (i), let $x, y \in P$ with $x \not\sqsubseteq y$. Since P is continuous, there is some $z \in P$ with $z \ll x$ and $z \not\sqsubseteq y$. Then $x \in \Uparrow z$, $y \in P \setminus \uparrow z$, and these are disjoint Lawson-open sets. This same argument shows that $(P \times P) \setminus \sqsubseteq$ is open in the product of the Lawson topologies.

For (ii), we simply note that the basis $\uparrow k \setminus \uparrow F$ where $k \in K(P)$ and $F \subseteq K(P)$ is finite consists of clopen subsets of P, since $\uparrow k$ is clopen in the Lawson topology if k is compact.

Finally, for (iii), we first assume that P is coherent, and we employ the

Alexander subbasis theorem to show P is Lawson compact. Note that a subbasis for the Lawson topology consists of sets of the form $\Uparrow x$ and $P\setminus \uparrow x$, for $x \in P$. Assume that

$$P = (\bigcup\{\Uparrow z \mid z \in A\}) \cup (\bigcup\{P\setminus \uparrow z \mid z \in B\}).$$

Since P is coherent, the sets $\cap\{\uparrow z \mid z \in F \subseteq B \text{ finite}\}$ are saturated and compact, and this family is filtered. Thus, Theorem 4.35 implies the intersection $\bigcap\{\uparrow z \mid z \in B\}$ is compact and saturated, and it does not intersect $P\setminus \uparrow z$ for any $z \in B$. It follows that $\bigcap\{\uparrow z \mid z \in B\} \subseteq \bigcup\{\Uparrow z \mid z \in A\}$, and so there is a finite subcover, $\bigcap\{\uparrow z \mid z \in B\} \subseteq \cup_{z\in F} \Uparrow z$, where F is a finite subset of A. Now, since $\cup_{z\in F} \Uparrow z$ is open and contains $\bigcap\{\uparrow z \mid z \in B\}$, Theorem 4.35 implies there is a finite subset $G \subseteq B$ with $\cap_{z\in G} \uparrow z \subseteq \cup_{z\in F} \Uparrow z$. It then follows that $\{\Uparrow z \mid z \in F\} \cup \{P\setminus \uparrow z \mid z \in G\}$ is a finite subcover of P, and so P is Lawson compact.

The converse – that P Lawson compact implies P coherent – follows from the fact that $\uparrow F$ is a Lawson-closed, hence compact, upper set for any finite set F, and the fact that Lawson compact upper sets are Scott compact in any continuous dcpo (which is easy to show). □

Note that the second part of this result gives substance to Scott's intuition that algebraic dcpo's are zero-dimensional. Also, note that the last part of the proof shows that the Scott-compact saturated sets – i.e., the members of the Smyth power domain $\mathcal{C}(P)$ – all are Lawson-closed sets in P.

Given a subset $X \subseteq P$ of a dcpo P, we define $\langle X\rangle = \downarrow X \cap \uparrow X$, the *convex hull of* X.

Proposition 4.41 *Let P be a coherent continuous dcpo. Then $\Gamma_\lambda(P) = \{X \subseteq P \mid X \text{ is } \lambda - \text{closed}\}$ is a continuous lattice with respect to* reverse *inclusion. In particular, $\Gamma_\lambda(P)$ is compact and Hausdorff in its Lawson topology.*

Proof. Since P is coherent, P is compact and Hausdorff in the Lawson topology. It is then routine to show that each compact subset $X \subseteq P$ is the filtered intersection of the family $\Downarrow X = \{Y \subseteq X \mid X \subseteq Y^{\circ\lambda}, Y\ \lambda-\text{closed}\}$. Moreover, if the intersection of a filter basis $\mathcal{F}$ of λ-closed sets satisfies $\bigcap \mathcal{F} \subseteq X$, then for any $Y \in \Downarrow X$, there is some $F \in \mathcal{F}$ with $F \subseteq Y$. This means $Y \ll X$ for each $Y \in \Downarrow X$, so this set deserves its name. □

Proposition 4.42 *Let P be a coherent continuous dcpo, and suppose $\mathcal{F} \subseteq \Gamma_\lambda(P)$ is a filter basis of λ-compact subsets of P. Then*

$$\langle\bigcap \mathcal{F}\rangle = \bigcap_{F\in\mathcal{F}} \langle F\rangle.$$

Thus the family of Lawson closed, order-convex subsets of P is a continuous lattice under reverse inclusion, and $X \ll Y$ for such sets if and only if $Y \subseteq X^{\circ\lambda}$.

Proof. If X is a closed subset of P, then X is compact and so $\uparrow X$ and $\downarrow X$ also are closed, hence compact, all because P is coherent. Thus, $X \mapsto$

$\langle X\rangle\colon \Gamma_\lambda(P) \to \Gamma_\lambda(P)$ is well-defined, and it clearly is monotone with respect to reverse inclusion. Thus $\langle\bigcap\mathcal{F}\rangle \subseteq \bigcap_{F\in\mathcal{F}}\langle F\rangle$.

For the converse, we suppose that $x \in \bigcap_{F\in\mathcal{F}}\langle F\rangle$. Then, for each $F \in \mathcal{F}$, there is a pair of elements $a_F, b_F \in F$ with $a_F \sqsubseteq x \sqsubseteq b_F$. Since P is compact in the Lawson topology, the nets $\{a_F\}_{F\in\mathcal{F}}$, $\{b_F\}_{F\in\mathcal{F}}$ must have subnets which converge to points $a, b \in P$, and, wlog, we assume that the nets $\{a_F\}_{F\in\mathcal{F}}$ and $\{b_F\}_{F\in\mathcal{F}}$ already converge to these points, respectively. Since each of the sets in $\mathcal{F}$ is closed and $\mathcal{F}$ is a filter basis, we must have $a, b \in F$ for each $F \in \mathcal{F}$, and so $a, b \in \bigcap\mathcal{F}$. And, since the order $\sqsubseteq$ on P is closed in the product Lawson topology and $a_F \sqsubseteq x \sqsubseteq b_F$ for all F, it follows that $a \sqsubseteq x \sqsubseteq b$. Hence $x \in \langle\bigcap\mathcal{F}\rangle$, so the two sets are equal.

Now, since the mapping $X \mapsto \langle X\rangle\colon \Gamma_\lambda(P) \to \Gamma_\lambda(P)$ preserves filtered intersections, it is a continuous kernel operator, and so its image, which is precisely the family of Lawson closed, order-convex subsets of P, also is a continuous lattice under reverse inclusion. The characterizing property of the way below relation in this lattice follows from the same property in $\Gamma_\lambda(P)$ (cf. [20], Corollary IV-1.7). $\square$

Definition 4.43 For a coherent continuous dcpo P, let

$$\mathcal{D}(P) = \{C \subseteq P \mid C = {\downarrow}C \cap {\uparrow}C,\ {\downarrow}C \in \Gamma_0(P)\ \&\ {\uparrow}C \in \mathcal{C}(P)\},$$

and define $C \sqsubseteq_D C'$ iff ${\downarrow}C \sqsubseteq_L {\downarrow}C'$ and ${\uparrow}C \sqsubseteq_U {\uparrow}C'$.

Proposition 4.44 *Let P be a coherent continuous dcpo. Then*

(i) $\mathcal{D}(P) = \{C \subseteq P \mid \emptyset \neq C = {\uparrow}C \cap {\downarrow}C = \overline{C}^\lambda\}$ *is the family of non-empty Lawson closed order-convex subsets of P.*

(ii) *If $C, D \in \mathcal{D}(P)$ and $D \subseteq C^{\circ_\lambda}$ (the interior in the Lawson topology), then there is a finite subset $F \subseteq{\uparrow}C$ such that $F \subseteq{\downarrow}D$ and $D \subseteq{\Uparrow}F$.*

Proof. The forward inclusion of part (i) follows from the fact that Scott-closed sets are Lawson closed, as are Scott-compact saturated sets. The reverse inclusion follows from the fact that coherent continuous dcpo's are Lawson compact.

For part (ii), since P is coherent, D and hence ${\uparrow}D$ are Lawson-compact. Then the continuity of P implies ${\uparrow}D = \bigcap\{{\Uparrow}F \mid F \text{ finite } \& \ F \subseteq{\downarrow}D\}$, and this intersection is filtered. Since $D \subseteq C^{\circ_\lambda}$, it follows that ${\uparrow}D \subseteq ({\uparrow}C)^{\circ_\sigma}$, and so Theorem 4.35 implies the result. $\square$

Note that Proposition 4.42 shows that $(\mathcal{D}(P), \supseteq)$ is a continuous cpo in which $X \ll Y$ if and only if $Y \subseteq X^{\circ_\lambda}$. We now investigate the other order on $\mathcal{D}(P)$.

Proposition 4.45 *If P is a coherent continuous dcpo, then $(\mathcal{D}(P), \sqsubseteq_D)$ is a continuous dcpo in which*

$$C \ll D \textit{ iff } (\exists F \subseteq P \textit{ finite})\ C \sqsubseteq_D \langle F\rangle \sqsubseteq_D D\ \&\ D \subseteq{\Uparrow}F,$$

and for which the operation

$$(C, D) \mapsto \langle C, D\rangle \equiv{\uparrow}(C \cup D) \cap {\downarrow}(C \cup D)\colon \mathcal{D}(P) \times \mathcal{D}(P) \to \mathcal{D}(P)$$

is continuous.

Proof. Assume that $\mathcal{F} \subseteq \mathcal{D}(P)$ is $\sqsubseteq_D$-directed and let $A = \overline{\bigcup\{\downarrow C \mid C \in \mathcal{F}\}}^{\sigma}$ be the Scott closure of the union of the lower sets of the members of $\mathcal{F}$. Then A is Scott closed, and hence also Lawson closed. Now, the sets $\{A \cap \uparrow C \mid C \in \mathcal{F}\}$ form a filtered intersection, and each is non-empty and Lawson compact since P is coherent, so their intersection also is non-empty and Lawson compact (e.g., by Theorem 4.35). Let B be that intersection. We claim that $B = \bigsqcup_{\mathcal{D}(P)} \mathcal{F}$. Indeed, it is obvious that $B \subseteq \uparrow C$ for all $C \in \mathcal{F}$. For the other direction – $C \subseteq \downarrow B$ for each $C \in \mathcal{F}$ – given $x \in C \in \mathcal{F}$, the fact that $\mathcal{F}$ is $\sqsubseteq_D$-directed implies that $\uparrow x\ \cap (A \cap \uparrow C')$ is non-empty and compact for each $C' \in \mathcal{F}$, so Theorem 4.35 shows the same is true of the $\uparrow x\ \cap B$. Thus B is an upper bound for $\mathcal{F}$, and a similar argument shows that B is the least upper bound of $\mathcal{F}$ in the order $\sqsubseteq_D$, so $(\mathcal{D}(P), \sqsubseteq_D)$ is a dcpo.

Next, if $C \in \mathcal{D}(P)$, then C is a convex Lawson-closed subset of P, and $\uparrow C$ is Lawson compact, hence also Scott compact. So, we can write $\uparrow C$ as the filtered intersection of sets $\uparrow F$ where $C \subseteq \Uparrow F$ and F is finite. Clearly we can arrange it so that $F \subseteq \downarrow C$ for each such F. Then $\langle F \rangle = \downarrow F \cap \uparrow F \in \mathcal{D}(P)$, and $\langle F \rangle \sqsubseteq_D C$. We now show that $\langle F \rangle \ll_D C$.

First, since $\uparrow C \subseteq \Uparrow F$, if $\mathcal{F} \subseteq \mathcal{D}(P)$ is directed and $C \sqsubseteq_D \bigsqcup \mathcal{F}$, then $\bigsqcup \mathcal{F} \subseteq \uparrow C \subseteq \Uparrow F$. The first part of the proof implies $\bigsqcup \mathcal{F} = \bigcap\{A \cap \uparrow C' \mid C' \in \mathcal{F}\}$, where $A = \overline{\bigcup\{\downarrow C' \mid C' \in \mathcal{F}\}}^{\sigma}$. Since this expresses $\bigsqcup \mathcal{F}$ as a filtered intersection, Theorem 4.35 implies there is some $C_0 \in \mathcal{F}$ with $A \cap \uparrow C_0 \subseteq \Uparrow F$. Thus, $C_0 \subseteq A \cap \uparrow C_0 \subseteq \Uparrow F \subseteq \uparrow \langle F \rangle$.

On the other hand, $C \sqsubseteq_D \bigsqcup \mathcal{F}$ also implies that $C \sqsubseteq \downarrow \bigsqcup \mathcal{F}$, and so $F \subseteq \downarrow \bigsqcup \mathcal{F}$. Using the facts that $\bigsqcup \mathcal{F} = \bigcap_{C' \in \mathcal{F}} (\overline{\downarrow \bigcup \mathcal{F}}^{\sigma} \cap \uparrow C')$, that $F \subseteq \downarrow C \subseteq \downarrow \bigsqcup \mathcal{F}$, and that F is finite, we conclude there is some $C_1 \in \mathcal{F}$ with $F \subseteq \downarrow C_1$. Since $\mathcal{F}$ is directed, we can choose a $C_2 \in \mathcal{F}$ such that $C_0, C_1 \sqsubseteq_D C_2$, and it then follows that $\langle F \rangle \sqsubseteq_D C_2$. This all goes to show that $\langle F \rangle \ll_D C$ in $\mathcal{D}(P)$, as claimed.

It is easy to show that the family $\{\langle F \rangle \mid F \text{ finite}, \langle F \rangle \sqsubseteq_D C \,\&\, C \subseteq \Uparrow F\}$ is $\sqsubseteq_D$-directed, so we conclude that $\mathcal{D}(P)$ is continuous and that $\{\langle F \rangle \mid$ finite, $\langle F \rangle \sqsubseteq_D C \,\&\, C \subseteq \Uparrow F\}$ is a basis for the way-below set of each C in $\mathcal{D}(P)$. Hence, $(\mathcal{D}(P), \sqsubseteq_D)$ is continuous and if $C' \ll_D C$ in $\mathcal{D}(P)$, then there is some $F \subseteq P$ finite with $C' \sqsubseteq_D \langle F \rangle \sqsubseteq_D C$ and $C \subseteq \Uparrow F$.

The proof that $(C, D) \mapsto \langle C, D \rangle \colon \mathcal{D}(P) \times \mathcal{D}(P) \to \mathcal{D}(P)$ is continuous is straightforward. □

Thus, $(\mathcal{D}(P), \supseteq)$ is a coherent continuous dcpo by Proposition 4.42, while $(\mathcal{D}(P), \sqsubseteq_D)$ continuous dcpo by Proposition 4.45. We now investigate the relationship between these distinct orders on $\mathcal{D}(P)$. We begin with a technical lemma.

Lemma 4.46 *Let $F \subseteq P$ be a finite subset of the continuous dcpo P. If F is an antichain, then*

$$\uparrow_D \langle F \rangle = \{X \in \mathcal{D}(P) \mid X \subseteq \uparrow F \ \&\ X \not\subseteq \uparrow (F \setminus \{x\})\ (\forall x \in F)\}$$

Proof. If $\langle F \rangle \sqsubseteq_D X$, then $X \subseteq \uparrow F$ and $F \subseteq \downarrow X$, and it is easy to show that

X is in the set on the right side of the claimed equality using the fact that F is an antichain. Conversely, if X is in the set on the right side of the claimed equality, then $X \subseteq \uparrow F$, and since $X \not\subseteq \uparrow(F \setminus \{x\})$ for any $x \in F$, it must be that $\uparrow x \cap X \neq \emptyset$ for each $x \in F$. But, then for each $x \in X$, $(\exists y \in X)\, x \sqsubseteq y$, which means $x \in \downarrow y \subseteq \downarrow X$. Hence $F \subseteq \downarrow X$, so $F \sqsubseteq_D X$. □

Proposition 4.47 *Let P be a coherent continuous dcpo. Then*

(i) *The identity mapping on $\mathcal{D}(P)$ is continuous from the λ-topology on $(\mathcal{D}(P), \supseteq)$ to the Scott topology on $(\mathcal{D}(P), \sqsubseteq_D)$.*

(ii) *The identity mapping on $\mathcal{D}(P)$ is continuous from the λ-topology on $(\mathcal{D}(P), \supseteq)$ to the lower topology on $(\mathcal{D}(P), \sqsubseteq_D)$.*

Proof. For (i), we note that given $C \in U \subseteq (\mathcal{D}(P), \sqsubseteq_D)$ with U Scott open, then there is a finite subset $F \subseteq P$ with $C \in \Uparrow_D \langle F\rangle \subseteq \uparrow_{\sqsubseteq_D}\langle F\rangle \subseteq U$. By substituting the minimal elements of F for F itself, we may assume that F is an antichain, and so $\langle F\rangle = F$. Now, if F is a singleton set, then $\Uparrow_D \langle F\rangle = \{D \in \mathcal{D}(P) \mid D \subseteq \Uparrow F\}$, which is Scott open in $(\mathcal{D}(P), \supseteq)$. Thus $C \in \Uparrow_D \langle F\rangle \subseteq U$.

In case F has more than one element, we also can assume that C has more than one minimal element (for otherwise we are back in the case of one element in F). Then we also can assume that $C \not\subseteq \uparrow x$ for any $x \in F$: if $x \in F$ with $C \subseteq \uparrow x$, then the fact that C is closed in the λ-compact space P implies that there is a finite set $G \subseteq P$ with $C \subseteq \Uparrow G$ and $x \notin \uparrow G$. We also can choose the elements of G so that $C \not\subseteq \uparrow y$ for any $y \in G$ (since C has more than one minimal element), and so we can substitute the finite set G for F in our argument. Now we consider the λ-open subset of $(\mathcal{D}(P), \supseteq)$ given by

$$\uparrow_{\supseteq}(\Uparrow \langle F\rangle) \setminus \uparrow_{\supseteq}(\cup_{x\in F} \uparrow(F \setminus \{x\})),$$

which consists of the sets in $\mathcal{D}(P)$ that are subsets of the Scott-open set $\Uparrow F$ but are not subsets of the set $\cup_{x\in F} \uparrow(F \setminus \{x\})$. Clearly C is in this set, which is λ-open since $\uparrow_{\supseteq}(\Uparrow \langle F\rangle)$ is the Scott interior in $(\mathcal{D}(P), \supseteq)$ of the set $\uparrow\langle F\rangle$. Moreover, any set X which lies in this set satisfies $\langle F\rangle \sqsubseteq_D X$ by the previous lemma. This shows part (i).

For part (ii), if $C \in \mathcal{D}(P) \setminus \uparrow_D D$ for some $D \in \mathcal{D}(P)$, then $C \not\subseteq \uparrow D$ or $D \not\subseteq \downarrow C$. In the first case, $\uparrow D \in \mathcal{D}(P)$ as P is coherent, and $C \in \mathcal{D}(P) \setminus \uparrow_{\supseteq} \uparrow D$. Moreover, if $X \in \mathcal{D}(P) \setminus \uparrow_{\supseteq} \uparrow D$ then $X \not\subseteq \uparrow D$, and so $X \in \mathcal{D}(P) \setminus \uparrow_D D$.

Finally, in the second case, $D \not\subseteq \downarrow C$ implies $x \notin \downarrow C$ for some $x \in D$. Then $\uparrow x \cap C = \emptyset$, and since C is compact, we can choose $y \ll x$ in P with $\uparrow y \cap C = \emptyset$. It follows that $P \setminus \Uparrow y \in \mathcal{D}(P)$ and $C \subseteq (P \setminus \Uparrow y)^{\circ\lambda}$. Thus $P \setminus \Uparrow y \ll_{(\mathcal{D}(P),\supseteq)} C$, and if $X \in \mathcal{D}(P)$ with $P \setminus \Uparrow y \ll_{(\mathcal{D}(P),\supseteq)} X$, then $X \cap \uparrow y = \emptyset$, and so $D \not\sqsubseteq_D X$. □

Corollary 4.48 *If P is a coherent continuous dcpo, then so is $(\mathcal{D}(P), \sqsubseteq_D)$.*

Proof. Since $(\mathcal{D}(P), \supseteq)$ is coherent, its λ-topology is compact, and the λ-topology of $(\mathcal{D}(P), \sqsubseteq_D)$ is Hausdorff. The previous Proposition then implies these topologies are the same, so the λ-topology of $(\mathcal{D}(P), \sqsubseteq_D)$ also is compact, making $(\mathcal{D}(P), \sqsubseteq_D)$ coherent. □

Lemma 4.49 *If S is a coherent continuous dcpo with a continuous semilattice operation $*: S \times S \to S$, then there is a continuous mapping $f_S: \mathcal{D}(S) \to S$ such that $f_S(\langle C \cup D \rangle) = f_S(C) * f_S(D)$ and $f_S(\{x\}) = x$ for all $x \in S$.*

Proof. If $x_1 * \cdots * x_m \sqsubseteq_S y_1 * \cdots * y_n$, if $F = \{x_1, \ldots, x_m\}$ and $G = \{y_1, \ldots, y_n\}$ are finite subsets of S with $\langle F \rangle \sqsubseteq_D \langle G \rangle$. This implies that $\langle \{x_1, \ldots, x_m\} \rangle \mapsto x_1 * \cdots * x_m$ is well-defined and monotone on sets of the form $\langle \{x_1, \ldots, x_m\} \rangle$. Since each set $C \in \mathcal{D}(S)$ is the directed supremum of sets of the form $\langle F \rangle$ for F finite, we have a well-defined mapping $f_S: \mathcal{D}(S) \to S$. To see that f_S is a continuous, assume $f_S(C) \in U \subseteq S$ is Scott open. Then there is some finite subset $F \subseteq S$ with $C \subseteq \Uparrow F \subseteq U$, and clearly we can assume $F \sqsubseteq_D C$. Hence, $C \in \Uparrow_{\mathcal{D}(S)} \langle F \rangle \subseteq f_S^{-1}(U)$, so $f_S^{-1}(U) \subseteq \mathcal{D}(S)$ is open. It also is routine to check that this mapping preserves the semilattice operation on finitely generated subsets of S, and so it must on all of $\mathcal{D}(S)$ by continuity. Finally, if $x \in S$, then clearly $f_S(\{x\}) = x$. $\square$

Theorem 4.50 *Let* SCCON *be the category of coherent continuous semilattice dcpo's and Scott continuous semilattice maps. Then:*

(i) *The functor $\mathcal{D}$:* CCON $\to$ SCCON *which assigns to each coherent continuous dcpo P the coherent continuous semilattice $\mathcal{D}(P)$, and to each morphism $f: P \to Q$ the mapping $\mathcal{D}(f)(C) = \langle f(C) \rangle$ is left adjoint to the forgetful functor from* SCCON *to* CCON.

(ii) *If P is algebraic, then $(\mathcal{D}(P) \sqsubseteq_D) \simeq (\mathcal{P}_C(P), \sqsubseteq_C)$.*

Proof. If P is a coherent continuous dcpo, then the previous result implies $\mathcal{D}(P)$ is a coherent continuous semilattice. Suppose S is a coherent continuous semilattice and $f: P \to S$ is continuous. Then the previous results about the lower and upper power domains imply there are continuous semilattice maps $\hat{f}_L: \Gamma_0(P) \to \Gamma_0(S)$ and $\hat{f}_U: \mathcal{P}_U(P) \to \mathcal{P}_U(S)$ that uniquely extend the map f. It is routine to show that the mappings $C \mapsto \downarrow C: \mathcal{D}(P) \to \Gamma_0(P)$ and $C \mapsto \uparrow C: \mathcal{D}(P) \to \mathcal{P}_U(P)$ are continuous semilattice maps, and then the mapping $C \mapsto \hat{f}_L(\downarrow C) \cap \hat{f}_U(\uparrow C): \mathcal{D}(P) \to \mathcal{D}(S)$ also is a continuous semilattice mapping. If we compose this map with $f_S: \mathcal{D}(S) \to S$, we have the desired map. This proves part (i).

Part (ii) follows from part (ii) of the previous result, where we showed that the sets of the form $\langle F \rangle$ with F finite form a a basis at C for each $C \in \mathcal{D}(P)$. If P is algebraic, then we can assume the finite sets F consist solely of compact elements, and so this basis is isomorphic the the basis $K(\mathcal{P}_C(P)) = \{\langle F \rangle \mid \emptyset \neq F \subseteq K(P) \text{ finite}\}$. $\square$

These results end our presentation of the power domain constructions. For each of the three "standard" power domains, we have constructed an analogous power domain over continuous coherent domains, and shown these topological constructions agree with the algebraic ones in the case the domain P on which they are built is algebraic.

4.5 Abramsky's Program

Domains have demonstrated utility in devising models for high-level programming languages. Part of this modeling process includes devising logics that allow one to reason about the programming language under study. One of the most impressive accomplishments in the area of domain theory has been the work of Abramsky [2] in which a tight connection between domain-theoretic models and program logics is detailed. We very briefly outline these results here.

Abramsky's starting point is the realization that, for domains, the compact elements completely determine the domain, and for logics, the Lindenbaum algebra provides a definitive model from which the logic can be recovered. So, if there were a canonical way to create a domain from the Lindenbaum algebra of a logic, and a canonical way to create a Lindenbaum algebra from the compact elements of a domain, then this would lead to a canonical method for associating domains to logics and vice-versa.

Now, a Lindenbaum algebra is a special sort of distributive lattice. If one wants a classical logic, then this algebra should be Boolean. On the other hand, if one seeks an intuitionistic logic, then the lattice should be a Heyting algebra. If one starts with a domain D, then $K(D)$ can be identified with certain Scott open sets – the basis $\{\uparrow k \mid k \in K(D)\}$. In fact, we can take the distributive lattice $K\Omega(D) = \{C \subseteq D \mid C \text{ is compact and Scott open}\}$.

Lemma 4.51 *For a domain D, $K\Omega(D) = \{\uparrow F \mid \emptyset \neq F \subseteq K(D) \text{ finite}\}$.* □

Now, if D also is coherent, then the family $K\Omega(D)$ is a lattice, and so we can view it as the Lindenbaum algebra of some logic. And this logic will be intuitionistic, since its Lindenbaum algebra is a Heyting algebra.

We can carry this a bit further. The Scott-open sets are ones that are "inaccessible by directed suprema." Viewing the compact elements as representing finite amounts of information, the set $\uparrow k$ then represents any information that supersedes that of k. But the point is that we can observe *in finite time* whether the supremum of a directed set gets in $\uparrow k$.

On the other hand, $\uparrow F$, for $F \subseteq K(D)$ finite, is Scott-compact. This means it is determined by a finite amount of finite information. Thus, the sets in $K\Omega(D)$ form the basis of the "topology of the finitely observable," as Abramsky likes to phrase it.

Going back the other way, given a Lindenbaum algebra L, what domain could it represent? If we look at the algebra $K\Omega(P)$ for P a coherent domain, we see it is the sets $\uparrow k$ for $k \in K(P)$ that we need to retrieve. In the algebra $K\Omega(P)$, these are distinguished by the fact that $\{\uparrow k \mid k \in K(P)\}$ is the set the co-primes. So, it is Stone Duality that we need to apply to retrieve the domain P from the lattice $K\Omega(P)$.

Of course, this all is rather fatuous, since it is not so easy to take *any* domain – even any coherent domain – P and figure out what logic has Lindenbaum algebra $K\Omega(P)$. So what is needed is a method for going back and forth between domains and logics in a way that allows one to identify precisely the logic that a given domain generates. This is where Abramsky's program focuses. He sets out a number of basic constructors of domain theory and shows how each corresponds to a constructor for the proof system in the Lindenbaum algebra of the desired logic. In the end, his theory can be applied to domains freely generated by these constructors.

Abramsky's theory applies to the category SFP of SFP-objects and continuous maps. This is reasonable, since the domains should be countably based to reflect computational reality, and some of the constructors are well-behaved only if the domains in question are coherent. The fact that SFP is the largest cartesian closed category of countably-based domains that are coherent makes it the right target for Abramsky's theory.

The domain constructors Abramsky incorporates are the following:

- *Lift*: $L(P) = P_\perp$,
- *Coalesced sum*: $P \oplus Q$ the disjoint union of P and Q with the least elements identified.
- *Products*: $P \times Q$,
- *Function space*: $[P \to Q]$,
- *Power domains*: $\Gamma_0(P), \mathcal{C}(P)$, and $\mathcal{D}(P)$, and
- *Recursion.*

Winskel [60] was the first to observe that each of the power domains can be characterized logically by suitable modal operators; the lower power domain corresponds to the *sometimes* operator, the upper power domain to the *eventually* operator, and the convex power domain to the combination of the two.

The inclusion of recursion in this list is fundamental. It is based on the notion of an *admissible predicate* on a cpo – a non-empty Scott-closed subset of the cpo – and the *Principle of Fixed Point Induction* (cf. [56]):

> *If $f\colon P \to P$ is a monotone selfmap of a cpo and if $f(x)$ satisfies a given admissible property whenever x does, then* $\mathrm{fix} f$ *also satisfies the property.*

An important outstanding question is how to extend Abramsky's theory to domains that are not freely generated. The problem is reflected by the fact that quotients of domains need not be algebraic – they can be continuous. But, even if the quotients in question again are domains, there still is no clear way to extend Abramsky's theory to handle them. More concretely, the model for the language CSP [10] is intrinsically a Scott domain, and it would be very nice to have an extension of Abramsky's theory that provided a logic for this model.

5 The Lambda Calculus

Topology is at the center of the only known approach to giving models of the *untyped lambda calculus.* This system originally was devised by Alonzo Church in the 1930's in an attempt to find a foundation for mathematics and logic that placed functions at the forefront. While Church's original formulation had inconsistencies, Curry put forth subsystems that are consistent. Because the theory focuses on the computational aspects of functions, in the spirit of, say, the First Recursion Theorem, it attracted the attention of computer scientists. But the lack of mathematical models for the theory made it little more than a "formal and unmotivated notation," in the words of Dana Scott (cf. the Foreword to [20]). Not long after he made this observation, however, Scott found the first of a whole family of models for the system, and thus began the present-day study of the calculus in earnest.

5.1 Syntax

We begin our discussion of the lambda calculus by describing the calculus and what it means to have a model for it. Let C denote a set of *constants,* V a set of *identifiers* or *variables*, and then consider the following set of BNF-like production rules:

(1) $M ::= c \mid x \mid MM \mid \lambda x.M,$

where $c \in C$ and $x \in V$. One way to think of the rules given above is as the signature of a (single-sorted) universal algebra. In this case, the algebra has nullary operators (i.e., constants) $c \in C$ and $x \in V$, a family of unary operators $\{\lambda x.- \mid x \in V\}$, and one binary operator, $(M, N) \mapsto MN$, which is given as the third clause of the set. This operator is called *application*, and it is meant to be an abstract model for the application of a function (the first M in the clause) to its argument (the second M in the clause). The operator in the last clause is called *abstraction*, and it is meant to model the way we can take, say, the polynomial x^2 and make it into a function $x \mapsto x^2 \colon \mathbb{R} \to \mathbb{R}$, here defined on the reals. So, in lambda notation, the function $f \colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ would be denoted $\lambda x : \mathbb{R} \,.\, x^2$.

The term $\lambda x : \mathbb{R} \,.\, x^2$ is from a *typed universe*, where functions are specified as having specific domains (and codomains). The type of this term is given by the syntax $x : \mathbb{R}$, which denotes that the variable x is restricted to the set $\mathbb{R}$ of real numbers. But the calculus whose syntax is given in equation (1) has no such restriction; all terms are assumed to have the *same* domain - the family of all objects defined by the syntax. In this sense, the untyped lambda calculus could also be viewed as a *unityped system*, where there is just one type. Since terms either can be arguments for functions (the second M in the application clause of the BNF), or functions (the first M in the clause), the abstraction clause provides a way of taking a term and making it into a function of the variable x; the term $\lambda x.M$ makes the term M into a function of the variable x.

With no further rules, then, the lambda calculus whose abstract syntax is given above is simply the initial universal algebra with the signature just described. This algebra consists of all terms we can create by repeatedly applying the operators, starting with the constant terms $C \cup X$. Such algebras often are called *term algebras*.

But the lambda calculus is not meant to be just an abstract anomic universal algebra. Indeed, it is *supposed* to be an abstract model of functions and how they operate. This is why Church chose to focus on the two operations we have included: *application* and *abstraction*. In order to model functions more accurately, we impose three *conversion rules*; these simply are equations we want to hold in the algebra, and so the calculus is really the universal algebra we get as above, modulo the algebra congruence the following rules generate:

(α) $\lambda x.M = \lambda y.M[y/x]$, for y neither free nor bound in M.

(β) $(\lambda x.M)N = M[N/x]$.

(η) $\lambda x.Mx = M$, for x not free in M.

We let $\Lambda(C)$ denote the set of terms of the lambda calculus with constants from S, modulo the equations $(\alpha), (\beta)$ and (η). These rules presuppose the notions of when a variable is *bound* in a term (i.e., within the scope of a λ-abstraction), and when it is free (not within such a scope). In essence, the α-rule says that being a function of a variable has nothing to do with the name of the particular variable that is used. The β-rule provides the vital link between the two operators application and abstraction via the usual notion of substitution. And, finally, the η-rule says that all terms can be regard as functions. Certainly all functions we encounter satisfy these laws, and so it seems reasonable to include them in the calculus.

5.2 The Notion of a Lambda Model

One question to consider is what sort of *models* the calculus might have. Since the calculus is meant to be an abstract version of functions, we expect to find some *mathematical model* D in which

- the constants $c \in C$ are interpreted as constants in $[\![c]\!]$[4] $\in D$,
- the variables $x \in X$ also are constants in $[\![x]\!] \in D$,
- the term $\lambda x.M$ is interpreted as a function $[\![\lambda x.M]\!]\colon D \to D$.
- application MN is interpreted as the application of the term M interpreted as a function $[\![M]\!]\colon D \to D$ to the term $[\![N]\!] \in D$.

In all, this says we expect to find a mathematical object D which is a universal algebra with the same signature as the calculus, and a homomorphism of universal algebras $[\![-]\!]\colon \Lambda(C) \to D$ from the terms of the calculus to D. Of course, the calculus and the identity map form one such model, but it is difficult to think of any other model.

This difficulty is reflected in certain aspects of the calculus. For example, the idea that objects can be functions and arguments for functions at the same time seems a bit odd. The fact that all terms "live at the same level" seems unintuitive. This clash with intuition is brought home when we consider the term

$$\lambda x.xx.$$

[4] It is traditional to denote the semantic meaning of a term x in a model by $[\![x]\!]$.

This certainly is a valid term of our calculus. And for any term M of the calculus, it expresses the fact that we can apply M to itself; $(\lambda x.xx)M = MM$ by the β-rule. Since $(\lambda x.xx)$ can be applied to any term of the calculus, each selfmap of the domain arising as the interpretation of an element of the calculus must also be an element of the domain. Transporting this to our hoped-for model D, we see that there must some way to interpret each element of D as a selfmap of D. This means, that, at least, there must be maps

(2) $p\colon D \to [D \to D]$ and $e\colon [D \to D] \to D$ such that $p \circ e = 1_{[D\to D]}$.

Here, $[D \to D]$ denotes the family of selfmaps of D. In general then, we seek a mathematical object D satisfying the equations (2) in whatever universe D resides in. What seems to be required minimally is a cartesian closed category A and an object D from A for which $\mathsf{A}(D, D)$ is a retract of D in A. Such objects are called *reflexive*, so we seek a reflexive object is some cartesian closed category as a model for the calculus.

We note that $\Lambda(C)$ also has an operation of *composition* intrinsically defined on it:

Proposition 5.1 *The operation* $(M, N) \mapsto M \circ N \equiv \lambda x.MNx\colon \Lambda(C) \times \Lambda(C) \to \Lambda(C)$ *defines a monoid structure on* $\Lambda(C)$ *whose identity is the term* $I = \lambda x.xx$.

Proof. This is a routine verification using the conversion rules $(\alpha) - (\eta)$. $\square$

Lemma 5.2 *If* X *is an object of a ccc* A, *then* $[X \to X]$ *has a monoid structure.*

Proof. This also is a standard result about cartesian closed categories. A proof can be found in the appendix of [25]. $\square$

Using these observations, and denoting by MON the category of monoids and monoid homomorphisms, we now can formulate precisely what we mean by a *lambda model*.

Definition 5.3 An object X in a ccc A is a *lambda model* if there is a mapping $p\colon X \twoheadrightarrow [X \to X]$ and maps

- $\psi\colon \Lambda(C) \twoheadrightarrow X$ in SET, and
- $\phi\colon (\Lambda(C).\circ) \to ([X \to X], \circ)$ in MON

such that $\phi = p \circ \psi$ and $\phi(M)(\psi(N)) = \psi(MN)$.

$$\begin{tikzcd}
{[X \to X] \times X} \arrow[r, "\mathrm{app}_{\mathrm{A}}"] & X \\
\Lambda(C) \times \Lambda(C) \arrow[u, "\phi\times\psi"] \arrow[ur, "\psi\circ\mathrm{app}_{\Lambda(C)}"'] &
\end{tikzcd}$$

From this definition, it is clear that we are interested only in models that are SET-based. The following result clarifies the situation further; a more complete presentation about this connection can be found in Chapter 9 of [7]:

Theorem 5.4 *There is a one-to-one correspondence between models of the untyped lambda calculus and reflexive objects in ccc's.* □

We remark that there are other notions of what a *lambda model* should be. Of particular note are the results of [34] which elaborate the relationships between a number of approaches to defining this concept, as well as the detailed discussion in [7]. These involve concepts such as combinatory algebras and combinatory models.

For us, the question is how to find *an* example of a non-degenerate reflexive object in a ccc. The place to start a search for a lambda model is the category SET of sets and functions. But, there Cantor's Lemma says no non-degenerate set can admit its space of selfmaps as a quotient, let alone a retract. It turns out that, with one exception, it is relatively difficult to find cartesian closed categories that have any non-degenerate reflexive objects. We'll say more about the search for reflexive objects in other categories in a moment, but first we want to present the construction of one such model.

5.3 Finding a Lambda Model

While the first mathematical model of the untyped lambda calculus was found by Scott in the category of algebraic lattices and Scott-continuous functions [49], there is an underlying construction technique which is applicable much more broadly, and which highlights the nature of the model much better than simply reiterating Scott's construction *verbatim*.

We begin with the observation that what makes SET fail to have a non-degenerate model is that there are too many functions in SET. Namely, the set of selfmaps of a non-degenerate set has larger cardinality than the set itself, and this is what is getting in the way. But, if we restrict our attention to topological spaces and continuous maps, then this no longer is a problem. For example, the space of continuous selfmaps of a space X is of the same cardinality as X, providing X has a dense subspace of cardinality smaller than the cardinality of X. For example, $|[\mathbb{R} \to \mathbb{R}]| = |\mathbb{R}|$ for precisely this reason. But, it turns out that finding a model in Hausdorff spaces is not a simple matter – in fact, it remains an unsolved problem. So, we turn our attention to spaces not satisfying such a strong separation condition.

One place to find such spaces is within the area of partially ordered sets. As we have seen, the categories DCPO and CPO are cartesian closed, and the Scott topology is T_0 but rarely T_1. Moreover, we already have encountered a technique for finding domains with desired properties – solving *recursive domain equations*. Our example of the domain $(\mathbb{N}^{\mathrm{op}}, \leq) \simeq L(\mathbb{N}^{\mathrm{op}}, \leq)$ is a case in point. So, one way to find a cpo satisfying our needs would be to seek a solution of a similar domain equation.

The problem is to find a non-degenerate cpo that is isomorphic to its cpo of continuous selfmaps. We cannot expect an actual *set-inclusion* $[P \to P] \subseteq P$, since this is forbidden within Zermelo-Frankel set theory by the Foundation

Axiom,[5] so the the nearest we can come is an isomorphism. In this approach, the likely equation is $P \simeq [P \to P]$, and this gives rise to the operator $F(P) = [P \to P]$ on cpo's. However, this operator has P in two places, and it is not at all clear how to turn F into a functor. We now recall a definition that allows us to overcome this problem.

If P and Q are dcpo's, then the pair of Scott-continuous maps $e\colon P \to Q$ and $p\colon Q \to P$ is an *embedding-projection pair* if $p \circ e = 1_P$ and $p \circ e \leq 1_Q$. Given an embedding-projection pair $(e, p)\colon P \to Q$, we can define related functions

$$F(e)\colon [P \to P] \to [Q \to Q] \quad \text{by} \quad F(e)(f) = e \circ f \circ p,$$

and

$$F(p)\colon [Q \to Q] \to [P \to P] \quad \text{by} \quad F(p)(f) = p \circ f \circ e.$$

Moreover, these mappings $F(e)$ and $F(p)$ are Scott-continuous, since they are defined via composition. The following result captures an important fact about embedding-projection pairs.

Lemma 5.5 *If $(e, p)\colon P \to Q$ is an embedding-projection pair, then so is the pair of mappings $(F(e), F(p))\colon [P \to P] \to [Q \to Q]$.*

Proof. If $f\colon P \to P$, then $F(e)(f) = e \circ f \circ p\colon Q \to Q$ and so

$$(F(p) \circ F(e))(f) = p \circ (e \circ f \circ p) \circ e = f,$$

since $p \circ e = 1_P$. Similarly, for $g \in [Q \to Q]$,

$$(F(e) \circ F(p))(g) = e \circ (p \circ g \circ e) \circ p \leq g,$$

since $e \circ p \leq 1_Q$. □

Both DCPO and CPO are complete categories: the limit of diagram $\Phi\colon G \to \mathsf{DCPO}$ defined on the directed graph $G = (N, E)$ is the usual family

$$\underset{\longleftarrow}{\lim}\, (\Phi(n), \Phi(e))_{(n,e) \in (N,E)} = \{(x_n) \in \prod_{n \in \mathbb{N}} \Phi(n) \mid \Phi(e)(x_{n_i}) = x_{n_j}, e = n_i \to n_j\}.$$

Both categories also are co-complete: the colimit of the diagram $\Phi\colon G \to \mathsf{DCPO}$ is the *ideal completion* under directed suprema of the colimit in POS. Theorem 3.27 shows that for embedding-projection pairs, the limit and colimit that arise naturally are in fact the same: If $\Phi\colon G \to \mathsf{DCPO}_{ep}$ is a diagram in the category of dcpo's and embedding-projection pairs. For each edge e in G, let $\Phi(e) = (e_{ij}, p_{ij})$, where $e = n_i \to n_j$. Then

$$\underset{\longleftarrow}{\lim}\, \{(\Phi(n), p_{ij}) \mid n \in \mathbb{N}, e = n_i \to n_j\} \simeq \underset{\longrightarrow}{\lim}\, \{(\Phi(n), e_{ij}) \mid n \in \mathbb{N}, e = n_i \to n_j\}.$$

If we start with the canonical embedding-projection pair

$$(e.p)\colon \{\bot, \top\} \to F(\{\bot, \top\}) \quad \text{where} \quad e(x)(y) = x \text{ and } p(f) = f(\bot),$$

then we can consider the object

$$\mathbb{I}' = \operatorname*{colim}_n (F^n(\{\bot, \top\}), \{F^n(e) \circ \cdots \circ F^{m+1}(e)\}_{m \leq n \in \mathbb{N}}).$$

[5] This follows by a simple argument using the von Neumann definition of ordered pair $(x, y) = \{\{x\}, \{x, y\}\}$ and the identification of functions as sets of ordered pairs.

We would like to claim that $F(\mathbb{I}') \simeq \mathbb{I}'$, but to do this, we need to know that F is continuous. Since F is not the same sort of functor as before, we need the following definitions and result.

Definition 5.6 Let $F\colon \mathsf{A}^{\mathrm{op}} \times \mathsf{B} \to \mathsf{C}$ be a functor defined on categories of dcpo's which is contravariant in its first argument and covariant in its second (such a functor is said to be of *mixed variance*). We say F is *continuous* if for all diagrams $\Phi\colon G \to \mathsf{A}_{ep}$ and all diagrams $\Phi'\colon G' \to \mathsf{B}_{ep}$, we have

$$\begin{aligned} &F(\lim_{\mathsf{A}}(\Phi(n), \Phi(e))_{(n,e)\in G} \times \operatorname{colim}_{\mathsf{B}}(\Phi'(n'), \Phi'(e'))_{(n',e')\in G'}) \\ \simeq\, &\operatorname{colim}_{\mathsf{C}}(F(\Phi(n) \times \Phi'(n')), F(\Phi(e) \times \Phi'(e'))_{((n,e),(n',e'))\in G\times G'}). \end{aligned}$$

Also, G is *locally continuous* if for all objects P_1, P_2 of A and Q_1, Q_2 of B,

$$G\colon (\mathsf{A}^{\mathrm{op}} \times \mathsf{B})((P_1, Q_1), (P_2, Q_2)) \to \mathsf{C}(G(P_1, Q_1), G(P_2, Q_2))$$

is continuous.

In analogy to Theorem 3.30, we have the following:

Proposition 5.7 *If $G\colon \mathsf{A}^{\mathrm{op}} \times \mathsf{B} \to \mathsf{C}$ is mixed variance, locally continuous functor, then G is a continuous functor.* □

Applying this to our functor $F\colon \mathsf{CPO}^{\mathrm{op}} \times \mathsf{CPO} \to \mathsf{CPO}$, we see that local continuity of F is sufficient to prove the desired isomorphism $\mathbb{I}' \simeq F(\mathbb{I}')$. And, indeed, the proof that F is locally continuous is straightforward. The object thus defined, $\mathbb{I}' \simeq F(\mathbb{I}')$ is the D_∞-model first discovered by Scott [50]. This rather general treatment of how to construct such a model is taken from Section 5 of [3], where more details of proofs can be found, and where further results about the canonicity of solutions to domain equations for mixed variance, locally continuous functors also can be found.

There is one subtle point we have skipped over here. Namely, in generating the fixed point $\mathbb{I}'$, we started with the domain $\{\bot, \top\}$ instead of the least domain, $\{\bot\}$. Of course, the reason is that were we to apply the functor $F(P) = [P \to P]$ to $\{\bot\}$, we would never get anywhere, and our solution to $P \simeq [P \to P]$ would be $\{\bot\}$. But there is a way to bring this construction into the realm of generality considered in Section 3.4. Namely, we redefine the functor F. Instead of using $F(P) = [P \to P]$, we can take instead the functor $F(P) = A_\bot \oplus [P \to P]$, where $A_\bot$ is the flat domain defined on the set A of constants we wish to include in the syntax of the calculus, and $\oplus$ denotes coalesced sum. Since this set A can be assumed to include the variables, it is non-empty, and so the resulting domain $P \simeq F(P)$ must be non-degenerate. For example, taking A to be a singleton yields the original D_∞-model of Scott.

5.4 The Search for Other Models

We now have constructed a non-degenerate model of the untyped lambda calculus in CPO. An interesting question is whether models exist in other categories. We already commented that this is not possible in SET. The "next place" one might look is the category POS of posets and monotone maps.

5.4.1 Models in POS

It is well-known that POS is cartesian closed: the terminal object is the one-point poset, products are ordered in the product order, and the internal hom is the space of monotone maps.

Definition 5.8 If P is a poset, we let $L(P) = \{Y \subseteq P \mid Y = \downarrow Y\}$ denote the family of lower sets of P.

The following result of Gleason and Dilworth is crucial to our development.

Theorem 5.9 Gleason & Dilworth [21]
If P is a poset, then there is no monotone surjection of a subposet of P onto $L(P)$. $\square$

We also need the following result, whose proof is straightforward.

Lemma 5.10 *If $f\colon P^{\mathrm{op}} \hookrightarrow Q$ is an injection, then $x \mapsto \chi_{Q\setminus\downarrow f(x)}\colon P \to [Q \to 2]$ is also an injection, where 2 denotes the two-point lattice.* $\square$

Theorem 5.11 POS *has no non-degenerate lambda models.*

Proof. Abramsky
Suppose that P is a poset and that $[P \to P]$ is a retract of P in POS. If P is an antichain (i.e., if P is a set with the discrete order), then $[P \to P] = P^P$, and the result follows from Cantor's Lemma.

So we assume P is not an antichain. Then there are $a < b \in P$, and so $\chi_{P\setminus\downarrow a}\colon P \to 2$ retracts P onto 2. Then $[P \to 2]$ is a retract of $[P \to P]$, and hence also of P. Applying the same reasoning again, we see that $[[P \to P] \to 2]$ also is a retract of P. But, the mapping

$$I \mapsto \chi_I\colon L(P)^{\mathrm{op}} \to [P \to 2]$$

is an order-isomorphism, and this implies that $L(P)$ is isomorphic to a subposet of $[[P \to P] \to 2]$. Since the latter is a retract of P, there is some subposet $Q \subseteq P$ which maps under the retraction onto $L(P)$, and this contradicts the Gleason–Dilworth Theorem. $\square$

We should note that an alternative proof of this is contained in [12].

5.4.2 Models in Complete Ultrametric Spaces

Another well-known ccc is the category CU of complete ultrametric spaces and non-expansive mappings. In this category, the terminal object is the one-point space, products are given the product metric, and the internal hom is the space of non-expansive mappings between the spaces. We proceed to show that CU has no non-degenerate lambda models.

Definition 5.12 A metric $\mathrm{d}\colon X \times X \to \mathbb{R}^+$ is an *ultrametric* if $(\forall x, y, z \in X)\ \mathrm{d}(x, z) = \max(\mathrm{d}(x, y), \mathrm{d}(y, z))$.

Lemma 5.13 *If (X, d) is an ultrametric space and $x \in X$ and $\epsilon > 0$, then $B(x, \epsilon)$ is closed as well as open.*

Proof. Suppose that $y \notin B(x,\epsilon)$ Then $\mathrm{d}(x,y) \geq \epsilon$. Now consider $B(y,\epsilon)$. We claim $B(x,\epsilon) \cap B(y.\epsilon) = \emptyset$. Indeed, if $z \in B(x,\epsilon) \cap B(y.\epsilon)$, then $\mathrm{d}(x,z) < \epsilon$ and $\mathrm{d}(y,z) < \epsilon$. Hence, $\epsilon = \mathrm{d}(x,y) = \max(\mathrm{d}(x,z), \mathrm{d}(y,z)) < \max(\epsilon,\epsilon) = \epsilon$, which is a contradiction. □

Finally, we recall the *paradoxical combinator* Y$\in \Lambda(C)$ defined by

$$\mathrm{Y}M = \lambda f.\lambda x.f(xx)f(xx).$$

A routine calculations show that, for all terms M in $\lambda(C)$, $M(\mathrm{Y}M) = \mathrm{Y}M$; i.e., Y$M$ is a fixed point of M. This means that in any lambda model, each selfmap must have a fixed point.

Theorem 5.14 CU *has no non-degenerate lambda models.*

Proof. Plotkin
If X is a non-degenerate complete ultrametric space, then there are distinct point $a, b \in X$. If $\mathrm{d}(a,b) = \epsilon$, then $B(a,\epsilon)$ is a clopen ball in X which does not contain b. The mapping $f\colon X \to X$ by

$$f(x) = \begin{cases} a & \text{if } x \notin B(a,\epsilon), \text{ and} \\ b & \text{otherwise,} \end{cases}$$

clearly has no fixed points. Moreover, it is non-expansive. If $x, y \in B(a,\epsilon)$ or $x, y \in X \setminus B(a,\epsilon)$, then $f(x) = f(y)$, so this is clear. On the other hand, if $x \in B(a,\epsilon)$ and $y \in X \setminus B(a,\epsilon)$, then the argument in the proof of the lemma shows that $d(x,y) \geq \epsilon$, and so $d(f(x), f(y)) = \epsilon \leq d(x,y)$. Thus X cannot be a lambda model. □

It is interesting that this proof is quite different from the ones for SET and POS, both of which relied on a cardinality principle. Of course, knowing that all selfmaps in a lambda model must have fixed points already says topological spaces that are lambda models must be connected.

5.4.3 Hausdorff Lambda Models

A last category we consider in our search for lambda models is a ccc of Hausdorff spaces. If one starts with the category of compact spaces and continuous maps, then the natural ccc one comes to is the category K of k-spaces and continuous maps. A *K-space* is a topological space in which a subset is open if and only if its intersection with each compact subset of the space is open in the subspace. If we are given a topological space X, we can "k-ify" it by taking the topology generated by the sets satisfying this property. The basic results about the category K are contained in [54]. They include the fact that the terminal object is the one-point space, that the product is obtained by "k-ifying" the product topology, and the internal hom is the space of continuous maps endowed with the "k-ification" of the compact-open topology.

In the last subsection, we introduced the combinator Y which assigns to each term of the lambda calculus a "canonical" fixed point. Along with Y there is another combinator, K. This combinator is defined by

$$\mathrm{K} = \lambda MN.M$$

and it gives a way of recognizing constant functions in the lambda calculus. In fact, it is not hard to show that $\mathrm{K}(M)N = M$ using β-reduction. Moreover, $\mathrm{Y} \circ \mathrm{K} = I$ is the identity operator.

Our first result about non-degenerate lambda models in K eliminates compact spaces. To obtain this result, we first derive a simple result from the theory of compact semigroups. This result is not new; it can be found, e.g., in [27].

Proposition 5.15 *Let $(S, \cdot)$ be a compact monoid. If $x, y \in S$ satisfy $x \cdot y = 1_S$, then $y \cdot x = 1_S$.*

Proof. First, we show that any compact semigroup T has a smallest closed ideal M_T satisfying $M_T \subseteq I$ for any (closed) ideal $I \subseteq T$. Indeed, the semigroup T itself is an ideal, and if I and J are closed ideals, then so is $I \cdot J$, the set-product of I with J. Moreover, $I \cdot J \subseteq I, J$, so $I \cap J \neq \emptyset$. Thus, the family of non-empty, closed ideals is filtered, and so it has a non-empty intersection. This intersection also is an ideal, and so it must be the minimal ideal we seek.

Next, we note that, if T is commutative, then M_T is a group. Indeed, if $x \in M_T$, then $x \cdot M_T \subseteq M_T$ is compact (being a translate of a compact set), and $T \cdot (x \cdot M_T) = T \cdot (M_T \cdot x) = (T \cdot M_T) \cdot x \subseteq M_T \cdot x = x \cdot M_T$. Dually, $(x \cdot M_T) \cdot T = x \cdot (M_T \cdot T) \subseteq x \cdot M_T$. Thus, $x \cdot M_T$ is an ideal, and so it must be equal to M_T as M_T is minimal. Similarly, $M_T = M_T \cdot x$, so M_T is indeed a group.

Finally, let $x, y \in S$ with S a compact monoid, and suppose $x \cdot y = 1_S$. Let $S_x = \overline{\{x^n \mid n \in \mathbb{N}\}}$ be the closed subsemigroup of S that x generates. Since the semigroup of powers of x is commutative and multiplication is continuous on S, it follows that S_x is a commutative, and so this is a compact, commutative semigroup. Its minimal ideal M_{S_x} then is a group, and we let $e = e^2$ be the identity of this group. Furthermore, since M_{S_x} is a group, $x \cdot e \in M_{S_x}$ has an inverse x^{-1} in this group.

We claim that $e = 1_S$. Indeed, since $e \in S_x$, there is a net $\{x^{n_\alpha}\} \subseteq \{x^n \mid n \in \mathbb{N}\}$ such that $e = \lim_\alpha x^{n_\alpha}$. Now, S is compact, and so the net $\{y^{n_\alpha}\}$ has a cluster point, and by picking subnets if necessary, we can assume $\lim y^{n_\alpha} = z \in S$. Then, $x^n \cdot y^n = 1_S$ for all $n \in \mathbb{N}$, and so

$$e \cdot z = \lim_\alpha x^{n_\alpha} \cdot y^{n_\alpha} = 1_S,$$

and this means

$$1_S = e \cdot z = (e \cdot e) \cdot z = e \cdot (e \cdot z) = e \cdot 1_S = e.$$

Now, $x \cdot e = x \cdot 1_S = x$, and so x is a member of the group of units of S, and x^{-1} is the inverse of x in S. But since $x \cdot y = 1_S$, it follows that $y = x^{-1}$, and so $y \cdot x = 1_S$ as well. $\square$

Theorem 5.16 HOFMANN & MISLOVE [25] *There are no non-degenerate, compact Hausdorff lambda models in K.*

Proof. Suppose that X is a compact Hausdorff space that is a reflexive object in K. Then $[X \to X]$ is a retract of X, and so it too is compact and Hausdorff. $[X \to X]$ also is a topological semigroup under the operation of composition, and the identity map is the identity of this semigroup. Now, the combinators Y and K can be recognized in $[X \to X]$, and so there are functions $Y, K \in [X \to X]$ satisfying the property that $Y \circ K = 1_X$. It is routine to verify that the combinator K gives rise to the "constant picker" $K(x): X \to X$ by $K(x)(y) = x$. Hence, the image of X under K consists of the constant maps.

Now, the previous Proposition shows that in the compact monoid $[X \to X]$, if $s \cdot t = 1_X$, then $t \cdot s = 1_X$ as well. It follows that $K \circ Y = 1_X$ in $[X \to X]$, and this in turn implies that $1_X \in K(X)$ is a constant map. Hence, X is degenerate. □

Of course, this result doesn't eliminate non-degenerate lambda models from K; it merely says they cannot be compact. The class of possible models can be shrunk further by the following result. It is taken from [26].

Proposition 5.17 *Let X be a lambda model in a ccc A, and suppose that Z is a retract of X in A. Then there is a morphism Y_Z in $A([Z \to Z], Z)$ such that $f(Y_Z(f)) = Y_Z(f)$ for all $f \in [Z \to Z]$.*

Proof. Since X is a lambda model, $[X \to X]$ is a retract of X in A, and the interpretation Y_X of the paradoxical combinator Y in $[X \to X]$ implies each morphism $f \in [X \to X]$ satisfies $f(Y_X(f)) = Y_X(f)$. Now, if $\iota: Z \leftrightarrow X: \pi$ expresses Z as a retract of X in A, then it is routine to show that the mappings

$$f \mapsto \pi \circ f \circ \iota: [Z \to Z] \to [X \to X] \text{ and } g \mapsto \iota \circ g \circ \pi: [X \to X] \to [Z \to Z]$$

express $[Z \to Z]$ as a retract of $[X \to X]$. Another calculation shows that $f \mapsto \pi(Y_X(\pi \circ f \circ \iota)): [Z \to Z] \to Z$ produces a fixed point combinator for $[Z \to Z]$, and clearly $Y_Z = \pi \circ Y_X \circ \pi \circ - \circ \iota: Z \to Z$ is the desired morphism. □

Recall that the unit interval $[0, 1]$ in the usual topology is an absolute neighborhood retract.

Theorem 5.18 HOFMANN & MISLOVE [26]
If X is a normal space that contains a homeomorphic image of $[0, 1]$, then X cannot be a lambda model.

Proof. Indeed, if X is normal and contains a copy of $[0, 1]$, then there is a retraction of X onto $[0, 1]$. Then the previous result implies $[0, 1]$ has a continuous fixed-point picker. But this is not true. Indeed, consider the mapping

$$H: [0, 1] \to [[0, 1] \to [0, 1]] \text{ by } H(t)(x) = \begin{cases} 2tx + 1 - 2t & \text{if } 0 \le t \le 1/2, \\ (1 - 2t)x & \text{if } 1/2 < t \le 1. \end{cases}$$

The fixed points of the mappings $H(t)$ are

$$\mathrm{Fix}(H(t)) = \begin{cases} \{1\} & \text{if } 0 \le t < 1/2 \\ [0, 1] & \text{if } t = 1/2, \\ \{0\} & \text{if } 1/2 < t \le 1. \end{cases}$$

Clearly Fix cannot be made to be a continuous function on the arc of functions $\{H(t) \mid t \in [0, 1]\}$. □

RICE [46] has an alternative approach to proving these results. His methods rely more on the syntactical structure of the lambda calculus, and less on the structure of the objects of K.

5.5 Models of the λI-calculus

The calculus we have focused on has included the constant combinator K $= \lambda xy.x$. This calculus is sometimes called the "call-by-name" calculus, since it does not require evaluating terms before other terms can be applied to them. Another calculus is the λI-calculus, which corresponds to the "call-by-value" calculus. We begin by giving its syntax.

The λI-calculus includes:

- the constants $c \in C$ and the variables $x \in X$,
- the application MN of any term M in λI to any term N in λI, and
- the abstraction $\lambda x.M$ *only if* $x \in \mathrm{FV}(M)$, the set of free variables of M.

Clearly, this eliminates the constant functions from the calculus. In fact, any term of the λ-calculus can be defined from terms in the λI-calculus and the combinator $\mathrm{K} = \lambda xy.x$ (cf. [8]). But our argument showing compact Hausdorff models of the λ-calculus are degenerate uses the combinator K crucially. To prove a similar result for the λI-calculus, we first define three combinators:

- $\mathrm{B} = \lambda xyz.x(yz)$,
- $\mathrm{C}^* = \lambda xy.yx$, and
- $\langle \mathrm{I} \rangle = \lambda x.x\mathrm{I} = \lambda x.x(\lambda y.y)$, the so-called *list of* I.

B is the combinator that instantiates composition in the calculus. The following result is a routine computation.

Proposition 5.19 *In the λI-calculus, we have*

(i) $BC^*\langle \mathrm{I} \rangle = \lambda zt.t(z\mathrm{I})$, *and*

(ii) $B\langle \mathrm{I} \rangle \mathrm{C}* = \mathrm{I}$. □

Corollary 5.20 *There are no non-degenerate compact Hausdorff models of the λI-calculus.*

Proof. According to Proposition 5.15, if there is a compact Hausdorff model, then $\lambda x.x = \mathrm{I} = \lambda zt.t(z\mathrm{I})$. If we take any element a in the model, then this implies $a = (\lambda x.x)a = (\lambda zt.t(z\mathrm{I})a = \lambda t.t(a\mathrm{I})$. So, for any b in the model, $ab = (\lambda t.t(a\mathrm{I}))b = b(a\mathrm{I})$. Taking $a = \mathrm{I}$, we see that $b = \mathrm{I}b = b(\mathrm{II}) = b\mathrm{I}$ for all b in the model, and so $ab = b(a\mathrm{I}) = ba$.

Now, by a *basis* for the calculus we mean a family $\mathcal{K}$ of terms such that every term in the calculus can be realized as the application of terms in $\mathcal{K}$; i.e., every term of the calculus is in the set of terms that $\mathcal{K}$ generates under the operation of application. A result of Rosser (cf. [47] and Proposition 9.3.7 of [8]) states that the terms

- $\mathrm{I} = \lambda x.x$, and

- $\mathrm{J} = \lambda wxyz.wx(wzy)$

form a basis for the λI-calculus. But, according to our result above, J = IJ = JI, and so the only terms that I and J generate are powers of J. But, further,

$$\mathrm{J} = \mathrm{IJ} = \mathrm{JI} = \lambda xyz.x(zy) = \mathrm{J}',$$

and so further,

$$\mathrm{J} = \mathrm{J}' = \mathrm{IJ}' = \mathrm{J}'\mathrm{I} = \lambda yz.zy = \mathrm{J}'',$$

and finally,

$$\mathrm{J} = \mathrm{J}'' = \mathrm{IJ}'' = \mathrm{J}''\mathrm{I} = \lambda z.z = \mathrm{I},$$

Thus, the model is degenerate. □

Furio Honsell is to be thanked for pointing out to the author that the combinators B, C* and ⟨I⟩ can be used as described above to show that compact Hausdorff models of the λI-calculus also are degenerate in the same way as K and Y can be used to show compact Hausdorff models of the λ-calculus are degenerate.

6 Programming Languages and Other Applications

Domain theory began in an attempt by Dana Scott to avoid the untyped lambda calculus and find a more intuitive, mathematical structure for providing models for programming languages. As it happened, the search also produced the first mathematical models of the untyped lambda calculus. But we have not said much about the methods used to build programming language models themselves. We close this paper with some comments along this line; they are more hints at places to look for details than they are precise descriptions of such models themselves.

The "classical" languages which domain theory has proven most useful for modeling are functional languages; this is understandable, since the lambda calculus itself is a prototypical such language. An excellent introduction here is the book by Gunter [22]. A great deal of research has gone into this area, and Gunter's book also is a good resource for finding further applications along this line.

An area which the author has been motivated by is *process algebra* and languages supporting concurrent computation. Among the most prominent of these are the languages CSP studied by Hoare, Brookes, Roscoe, Reed —citebhr84,rr88 and others at Oxford, and CCS, invented by Milner[35] and studied by many people. In either case, the approach is to focus on the communication events that take place between machines running in a concurrent environment, rather than on the actual computations each machine makes. Domain theory has proved a fruitful tool in this area. For example, the seminal paper of Hennessy and Plotkin [23] showed how power domains correspond to forms of nondeterminism. The paper [37] carries this theme further by showing for a simple parallel programming language how each of the power domains corresponds to a distinct form of nondeterminism.

Process algebra also provides an interesting contrast to two opposing approaches for giving denotational models. As we have tried to demonstrate from the outset, the role of domain theory is to give meanings to recursive constructs, and the property of domains that facilitates this is the least fixed point property that continuous selfmaps on domains enjoy. Indeed, these fixed points not only exist, they are canonical. The alternative approach here is to use complete metric spaces and the Banach Fixed Point Theorem. This approach has been championed by DE BAKKER and his colleagues [13]. This approach has a shortcoming, however, in that it requires a restriction to so-called *guarded terms* in order to guarantee the associated selfmaps are contractive.

A synthesis between the domain-theoretic and metric-space approaches also has begun to emerge. A seminal result is the paper of AMERICA AND RUTTEN [6] in which it is shown how to solve recursive domain equations within the metric space world. Further work along this line has been done by FLAGG AND KOPPERMAN [18], as well as ALESSI, BALDAN, BELLÉ AND RUTTEN [5]. But the most extensive results along the lines of synthesizing domain theory and metric spaces are due to WAGNER [59].

Our discussion of power domains focused on presenting them first in their original algebraic formulation, and then from a topological view. There are topological analogues to these constructs, which have evolved from the original Vietoris hyperspaces. These constructs have received new interest because of the work of EDALAT [14–16]. In these works Edalat has found new and exciting applications of domain theory to the areas of fractals, neural networks and perhaps most notably to the theory of integration. Indeed, Edalat has used domain theory to provide a very simple derivation of the Riemann integral and, at the same time, he has found solutions to problems that do not seem available from the more traditional methods.

Lastly, we mention set theory as an area of application. One of the motivations for Church in devising the lambda calculus was to provide a new foundation for mathematics. Through work on set theory and process algebra, ACZEL [4] devised a new formulation for set theory in which he replaces the traditional Foundation Axiom by a more general axiom that allows a much wider family of sets - including ones that contain themselves as members. This set theory has special appeal for theoretical computation, since it provides simple models for processes that want to "call themselves." An obvious topic is then the relation between this new set theory and the more "traditional" domain theory. One aspect of this relation is presented in [38], where it is shown how to present a domain-theoretic model for the hereditarily finite portion of Aczel's theory. A direct application of Aczel's theory to providing programming models also can be found in [48].

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