Random Variable Models of Computation

Michael W. Mislove

Tulane University
New Orleans, LA

Samson@60 Fest
Oxford, May 28 –30, 2013

Joint work with Tyler Barker

Work sponsored by AFOSR & NSF

The Real Samson - MFPS XXIV

2007 – 08 UEFA Champions League

Manchester United	Chelsea
1	1

Manchester United won 6–5 on penalties

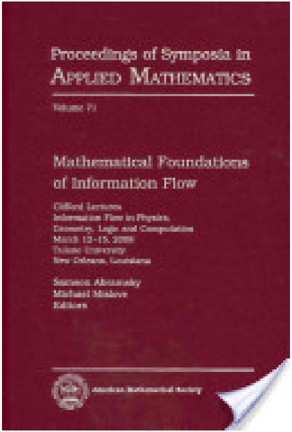
Date 21 May 2008

Venue Luzhniki Stadium, Moscow

UEFA Man of the Match:
Edwin van der Sar
(Manchester United)

Fans' Man of the Match:
Cristiano Ronaldo
(Manchester United)

The Real Samson - MFPS XXIV



Continuous Random Variables

- $f\colon (X,\mu) \to (Y,\Omega)$ *random variable*
  - (X,μ) probability space,
  - (Y,Ω) measure space
  - f is *measurable:* $f^{-1}(A)$ measurable $(\forall A \in \Omega)$
  - *Continuous* if X and Y topological spaces, f continuous and X, Y endowed with Borel σ-algebras.

Continuous Random Variables

- $f\colon (X,\mu) \to (Y,\Omega)$ *random variable*
- Assume X, Y *domains* endowed with Scott topology.

  $CRV(X,Y) = \{(\mu, f) \mid \mu \in Prob(X), f\colon \operatorname{supp}\mu \to Y\}$

  $\operatorname{supp}\mu = \bigcap\{X \mid \mu(X) = 1 \ \&\ X \text{ closed}\}$.

Continuous Random Variables

- $f\colon (X,\mu) \to (Y,\Omega)$ *random variable*
- Assume X, Y *domains* endowed with Scott topology.

  $$CRV(X,Y) = \{(\mu,f) \mid \mu \in Prob(X), f\colon \operatorname{supp}\mu \to Y\}$$

- *Goubault-Larrecq & Varacca, LICS 2011:*
  BCD closed under

$$\begin{array}{rcl} \Theta RV(\mathcal{C},P) & = & \{(\mu,f) \in CRV(\mathcal{C},P) \mid \mu \text{ thin}\} \\ (\mu,f) \le (\nu,g) & \text{iff} & \pi_{\operatorname{supp}\mu}(\nu) = \mu \;\&\; f \circ \pi_{\operatorname{supp}\mu} \le g \end{array}$$

$\mathcal{C}$ - Cantor tree

Continuous Random Variables

- $f\colon (X,\mu) \to (Y,\Omega)$ *random variable*
- Assume X, Y *domains* endowed with Scott topology.

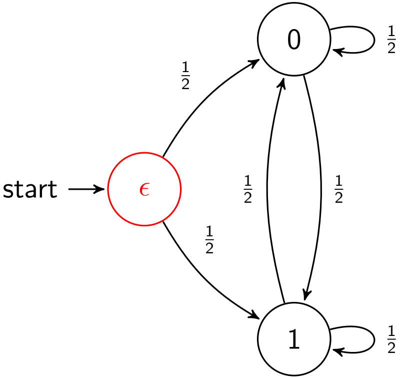
  $$CRV(X,Y) = \{(\mu, f) \mid \mu \in Prob(X), f\colon \operatorname{supp}\mu \to Y\}$$
- *Goubault-Larrecq & Varacca, LICS 2011:*
  BCD closed under

  $$\begin{array}{rcl} \Theta RV(\mathcal{C}, P) & = & \{(\mu, f) \in CRV(\mathcal{C}, P) \mid \mu \textit{ thin}\} \\ (\mu, f) \leq (\nu, g) & \text{iff} & \pi_{\operatorname{supp}\mu}(\nu) = \mu \;\&\; f \circ \pi_{\operatorname{supp}\mu} \leq g \end{array}$$

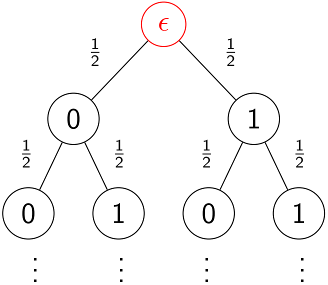
  $\mathcal{C}$ - Cantor tree
- *Goal:* Understand $\Theta RV(\mathcal{C}, P)$ construction for $P \in \mathrm{BCD}$

Motivating the Order - Automata

- A *(generative) probabilistic automaton* A has a finite set S of states, a start state $s_0 \in S$, a finite set of actions, Act, and a transition relation $\longrightarrow \subseteq S \times \mathrm{Prob}(Act \times S)$.
- Here's a simple example with one action, *flip*:



Unfolding the automaton:



Motivating the Order - Trace Distributions

- Typically, such automata are modeled by their *trace distributions:*

  $\mu_0 = \delta_\epsilon$

  $\mu_1 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$

  $\mu_2 = \frac{1}{4}\delta_{00} + \frac{1}{4}\delta_{01} + \frac{1}{4}\delta_{10} + \frac{1}{4}\delta_{11}$

  $\vdots$

- Stripping away the probabilities, we have the following sets on which these measures are *concentrated:*

  $X_0 = \{\epsilon\}$
  $X_1 = \{0, 1\}$
  $X_2 = \{00, 01, 10, 11\}$
  $\vdots$

Motivating the Order - Trace Distributions

- Stripping away the probabilities, we have the following sets on which these measures are *concentrated:*

$$X_0 = \{\epsilon\}$$
$$X_1 = \{0, 1\}$$
$$X_2 = \{00, 01, 10, 11\}$$
$$\vdots$$

- Notice that the X_ns are *antichains*, and

$X_0 \sqsubseteq_C X_1 \sqsubseteq_C X_2 \sqsubseteq_C \cdots$, where

$$\begin{aligned} X \sqsubseteq_C Y \quad &\Leftrightarrow \quad X \subseteq \downarrow Y \ \&\ Y \subseteq \uparrow X \\ &\Leftrightarrow \quad \pi_X(Y) = X \end{aligned}$$

The Underlying Structure - Domains and Trees

- $A^\infty = A^* \cup A^\omega$ is a *domain* under the prefix order.
  $KA^\infty = A^*$ – the finite words

  If A is finite, then A^∞ is *coherent*

  Compact in the *Lawson topology*

  Open sets: $U = \uparrow k \setminus \uparrow F, \quad k \in A^*, F \subseteq A^*$ finite

The Underlying Structure - Domains and Trees

- $A^\infty = A^* \cup A^\omega$ is a *domain* under the prefix order.
- $AC(A^\infty) = (\{X \mid \text{Lawson-compact antichain}\}, \sqsubseteq_C)$

$$\begin{aligned} X \sqsubseteq_C Y \quad &\Leftrightarrow \quad X \subseteq {\downarrow}Y \;\&\; Y \subseteq {\uparrow}X \\ &\Leftrightarrow \quad \pi_X(Y) = X \end{aligned}$$

Subdomain of $\mathcal{P}_C(A^\infty)$.

The Underlying Structure - Domains and Trees

- $A^\infty = A^* \cup A^\omega$ is a *domain* under the prefix order.
- $AC(A^\infty) = (\{X \mid \text{Lawson-compact antichain}\}, \sqsubseteq_C)$
- **Theorem:** $AC(A^\infty)$ is a bounded complete domain: all nonempty subsets have infima.

  $$(\emptyset \neq \mathcal{F} \subseteq AC(A^\infty)) \;\Rightarrow\; \inf \mathcal{F} = \mathrm{Max}(\textstyle\bigcap_{X \in \mathcal{F}} {\downarrow} X)$$

The Underlying Structure - Domains and Trees

- $A^\infty = A^* \cup A^\omega$ is a *domain* under the prefix order.
- $AC(A^\infty) = (\{X \mid \text{Lawson-compact antichain}\}, \sqsubseteq_C)$
- **Theorem:** $AC(A^\infty)$ is a bounded complete domain: all nonempty subsets have infima.
  Moreover, given $\{X_n\}_{n\in\mathbb{N}} \subseteq AC(A^\infty)$ directed and $X \in AC(A^\infty)$, TAE:

  (i) $X = \sup_n X_n$

  (ii) $X = \lim_n X_n$ in the Vietoris topology on $\Gamma(A^\infty)$.
- In particular, *any* $X \in AC(A^\infty)$ satisfies

  $X = \sup_n \pi_n(X) = \lim_n \pi_n(X)$, where

  $\pi_n \colon A^\infty \to A^{\leq n}$ is the canonical retraction.

Thin Probability Measures

- $\mu \in \mathit{Prob}(A^\infty)$ is *thin* if $\mathsf{supp}_\Lambda \mu \in AC(A^\infty)$.

  Note: $\mathsf{supp}_\Lambda \mu$ is in the *Lawson* topology.

- Define $\mu \leq \nu$ iff $\pi_{\mathsf{supp}_\Lambda \mu}(\nu) = \mu$

  Agrees with usual domain order (*qua* valuations) / functional analysis order via cones.

  $\Theta \mathit{Prob}(A^\infty) = (\{\mu \in \mathit{Prob}(A^\infty) \mid \mu \text{ thin}\}, \leq)$.

Thin Probability Measures

- $\mu \in \mathit{Prob}(A^\infty)$ is *thin* if $\mathsf{supp}_\Lambda\, \mu \in AC(A^\infty)$.
- **Proposition:** $(\Theta \mathit{Prob}(A^\infty), \leq)$ is a bounded complete domain: all nonempty subsets have infima.

$$(\emptyset \neq \mathcal{M} \subseteq \Theta \mathit{Prob}(A^\infty) \;\Rightarrow\; \forall \nu \in \mathcal{M},$$

$$\inf \mathcal{M} = \pi_M(\nu), \quad M = \inf_{\mu \in \mathcal{M}} \mathsf{supp}_\Lambda\, \mu)$$

Thin Probability Measures

- $\mu \in \mathit{Prob}(A^\infty)$ is *thin* if $\mathrm{supp}_\Lambda \, \mu \in AC(A^\infty)$.
- **Proposition:** $(\Theta \mathit{Prob}(A^\infty), \leq)$ is a bounded complete domain: all nonempty subsets have infima.

  Moreover, given $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \Theta \mathit{Prob}(A^\infty)$ directed and $\mu \in \Theta \mathit{Prob}(A^\infty)$, TAE:

  (i) $\mu = \sup_n \mu_n$

  (ii) $\mu = \lim_n \mu_n$ in the weak *-topology on $\Theta \mathit{Prob}(A^\infty)$.
- In particular, *any* $\mu \in \Theta \mathit{Prob}(A^\infty)$ satisfies

  $\mu = \sup_n \pi_n(\mu) = \lim_n \pi_n(\mu)$, where

  $\pi_n \colon A^\infty \to A^{\leq n}$ is the canonical retraction.

Adding Function Spaces

- $X \sqsubseteq_C Y \in AC(A^\infty), P \in \mathrm{BCD} \implies$
  $f \mapsto f \circ \pi_X \colon [X \longrightarrow P] \hookrightarrow [Y \longrightarrow P]$ &
  $g \mapsto \widehat{g} \colon [Y \longrightarrow P] \twoheadrightarrow [X \longrightarrow P]$ by $\widehat{g}(x) = \inf g(\pi_X^{-1}(x))$.

Adding Function Spaces

- $X \sqsubseteq_C Y \in AC(A^\infty), P \in \mathrm{BCD} \implies$
  $f \mapsto f \circ \pi_X \colon [X \longrightarrow P] \hookrightarrow [Y \longrightarrow P]$ &
  $g \mapsto \widehat{g} \colon [Y \longrightarrow P] \twoheadrightarrow [X \longrightarrow P]$ by $\widehat{g}(x) = \inf g(\pi_X^{-1}(x))$.
- $X \in AC(A^\infty), P \in \mathrm{BCD} \implies [X \longrightarrow P] \in \mathrm{BCD}$:

$$[X \longrightarrow P] \simeq \lim_n [\pi_n(X) \longrightarrow P] \simeq \lim_n P^{\pi_n(X)}.$$

Adding Function Spaces

- $X \in AC(A^\infty), P \in \mathrm{BCD} \implies [X \longrightarrow P] \in \mathrm{BCD}$:

  $$[X \longrightarrow P] \simeq \lim_n [\pi_n(X) \longrightarrow P] \simeq \lim_n P^{\pi_n(X)}.$$

- $(\bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P], \leq_R) \in \mathrm{BCD}$:

  $$f \leq_R g \quad \text{iff} \quad \mathrm{dom}\, f \sqsubseteq_C \mathrm{dom}\, g \;\&\; f \circ \pi_{\mathrm{dom}\, f} \leq g.$$

Adding Function Spaces

- $X \in AC(A^\infty), P \in \mathrm{BCD} \implies [X \longrightarrow P] \in \mathrm{BCD}$:

  $$[X \longrightarrow P] \simeq \lim_n [\pi_n(X) \longrightarrow P] \simeq \lim_n P^{\pi_n(X)}.$$

- $(\bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P], \leq_R) \in \mathrm{BCD}$:

  $$f \leq_R g \quad \text{iff} \quad \mathrm{dom}\, f \sqsubseteq_C \mathrm{dom}\, g \;\&\; f \circ \pi_{\mathrm{dom}\, f} \leq g.$$

Defining the Model

- $\Theta Prob(A^\infty) \times \bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P] \in \mathrm{BCD}$ if $P \in \mathrm{BCD}$.

Adding Function Spaces

- $X \in AC(A^\infty), P \in \mathrm{BCD} \implies [X \longrightarrow P] \in \mathrm{BCD}$:

  $$[X \longrightarrow P] \simeq \lim_n [\pi_n(X) \longrightarrow P] \simeq \lim_n P^{\pi_n(X)}.$$

- $(\bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P], \leq_R) \in \mathrm{BCD}$:

  $$f \leq_R g \quad \text{iff} \quad \mathrm{dom}\, f \sqsubseteq_C \mathrm{dom}\, g \ \&\ f \circ \pi_{\mathrm{dom}\, f} \leq g.$$

Defining the Model

- $\Theta Prob(A^\infty) \times \bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P] \in \mathrm{BCD}$ if $P \in \mathrm{BCD}$.
- For $P \in \mathrm{BCD}$

  $\Theta RV(A^\infty, P) = \{(\mu, f) \mid \mu \in \Theta Prob(A^\infty), f \colon \mathrm{supp}_\Lambda\, \mu \longrightarrow P\}$

  is a retract of $\Theta Prob(A^\infty) \times \bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P]$:

  $(\mu, f) \mapsto (\pi_Y(\mu), f \circ \pi_Y)$ is the projection

  $Y = \mathrm{supp}_\Lambda\, \mu \wedge \mathrm{dom}\, f.$

The Monad

Given $P \in \mathrm{BCD}$, define

- $\eta_P \colon P \to \Theta RV(A^\infty, P)$ by $\eta_P(x) = (\delta_\epsilon, \mathrm{const}_x)$.
- Given $h \colon P \longrightarrow \Theta RV(A^\infty, Q)$ and $(\sum_{x\in F} r_x\delta_x, f) \in \Theta RV(A^\infty, P)$ define $h^\dagger \colon \Theta RV(A^\infty, P) \longrightarrow \Theta RV(A^\infty, Q)$ by $h^\dagger(\sum_{x\in F} r_x\delta_x, f) = (\sum_{x\in F} r_x\delta_x * (\pi_1 \circ h \circ f)(x), g)$, where $g \colon (\bigcup_{x\in F} x \cdot \mathrm{supp}_\Lambda(\pi_1 \circ h \circ f)(x)) \to Q$ by

$$g(y) = (\pi_2 \circ h \circ f)(y)(y).$$

The Monad

Given $P \in \mathrm{BCD}$, define

- $\eta_P \colon P \to \Theta RV(A^\infty, P)$ by $\eta_P(x) = (\delta_\epsilon, \mathrm{const}_x)$.
- Given $h \colon P \longrightarrow \Theta RV(A^\infty, Q)$ and $(\sum_{x \in F} r_x \delta_x, f) \in \Theta RV(A^\infty, P)$ define $h^\dagger \colon \Theta RV(A^\infty, P) \longrightarrow \Theta RV(A^\infty, Q)$ by $h^\dagger(\sum_{x \in F} r_x \delta_x, f) = (\sum_{x \in F} r_x \delta_x * (\pi_1 \circ h \circ f)(x), g)$, where $g \colon (\bigcup_{x \in F} x \cdot \mathrm{supp}_\Lambda(\pi_1 \circ h \circ f)(x)) \to Q$ by
$$g(y) = (\pi_2 \circ h \circ f)(y)(y).$$
- These constructs satisfy the monad laws

The Monad

Given $P \in \text{BCD}$, define

- $\eta_P \colon P \to \Theta RV(A^\infty, P)$ by $\eta_P(x) = (\delta_\epsilon, \text{const}_x)$.
- Given $h \colon P \longrightarrow \Theta RV(A^\infty, Q)$ and

  $(\sum_{x \in F} r_x \delta_x, f) \in \Theta RV(A^\infty, P)$ define

  $h^\dagger \colon \Theta RV(A^\infty, P) \longrightarrow \Theta RV(A^\infty, Q)$ by

  $h^\dagger(\sum_{x \in F} r_x \delta_x, f) = (\sum_{x \in F} r_x \delta_x * (\pi_1 \circ h \circ f)(x), g)$, where

  $g \colon (\bigcup_{x \in F} x \cdot \text{supp}_\Lambda(\pi_1 \circ h \circ f)(x)) \to Q$ by

  $$g(y) = (\pi_2 \circ h \circ f)(y)(y).$$
- These constructs satisfy the monad laws
- BUT $h^\dagger$ is *not* monotone: $h(x) = (\delta_b, \text{const}_y)$

  $(\delta_\epsilon, f) \leq (\delta_a, f)$, but $(\delta_b, g_1) \not\leq (\delta_{ab}, g_2)$, if $a \neq b \in A$.

The Monad - Remedies

Extract a subdomain

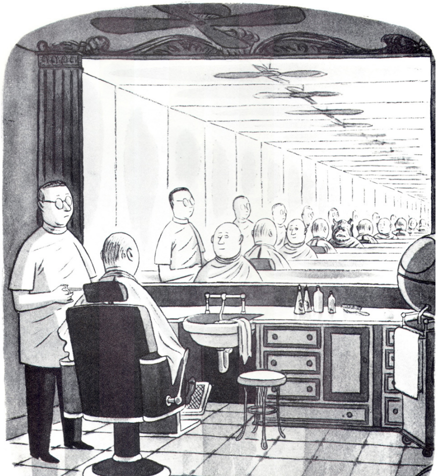
$$\Theta RV(A^\omega, P) \equiv \{(\mu, f) \in \Theta RV(A^\infty, P) \mid \mu = \delta_\epsilon \vee \mu \in Prob(A^\omega)\}.$$

Same as $\bigoplus_{\mu \in Prob(A^\omega)}(\{\mu\} \times [A^\infty \longrightarrow P])$.

The Monad - Remedies

Extract a subdomain

$$\Theta RV(A^{\omega}, P) \equiv \{(\mu, f) \in \Theta RV(A^{\infty}, P) \mid \mu = \delta_{\epsilon} \vee \mu \in Prob(A^{\omega})\}.$$



The Monad - Remedies

Extract a subdomain

$$\Theta RV(A^{\omega}, P) \equiv \{(\mu, f) \in \Theta RV(A^{\infty}, P) \mid \mu = \delta_{\epsilon} \vee \mu \in Prob(A^{\omega})\}.$$

Goubault-Larrecq Redefine the order on $\Theta Prob(A^{\infty})$

The Monad - Remedies

Extract a subdomain

$$\Theta RV(A^\omega, P) \equiv \{(\mu, f) \in \Theta RV(A^\infty, P) \mid \mu = \delta_\epsilon \vee \mu \in Prob(A^\omega)\}.$$

Goubault-Larrecq Redefine the order on $\Theta Prob(A^\infty)$

Tyler Barker Redefine $h^\dagger$

The Monad - Remedies

Extract a subdomain

$$\Theta RV(A^{\omega}, P) \equiv \{(\mu, f) \in \Theta RV(A^{\infty}, P) \mid \mu = \delta_{\epsilon} \vee \mu \in Prob(A^{\omega})\}.$$

Goubault-Larrecq Redefine the order on $\Theta Prob(A^{\infty})$

Tyler Barker Redefine $h^{\dagger}$

Use existing model Believe monad exists in kTop

The Monad - Remedies

Happy Birthday SAMSON!!