Random Variables over Domains

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Nondeterminism vs. Probabilistic Choice

• *Nondeterminism*: Represents the environment making choices
  • Like riders selecting floors on an elevator

• *Probabilistic choice*: Represents random events affecting the system
  • Like random stops the elevator makes
Standard Models for Nondeterminism and Probabilistic Choice

- **$S$ - finite set of states**

- **Nondeterministic choice:** $(\mathcal{P}(S), \cup)$

- **Probabilistic choice:** $(\mathcal{V}(S), \{r+ \mid 0 \leq r \leq 1\})$

$$\mathcal{V}(S) = \left\{ \sum_{i=1}^{n} r_i \delta_{x_i} \mid 0 \leq r_i; \sum_i r_i \leq 1; \ x_i \in S \right\}$$

$$\sum_{i=1}^{m} r_i \delta_{x_i} r + \sum_{j=1}^{n} s_j \delta_{y_j} = \sum_{i=1}^{m} r \cdot r_i \delta_{x_i} + \sum_{j=1}^{n} (1 - r) \cdot s_j \delta_{y_j}$$

$$\sum_{i=1}^{m} r_i \delta_{x_i} \subseteq \sum_{j=1}^{n} s_j \delta_{y_j} \quad \text{iff} \quad \sum_{x_i \in X} r_i \leq \sum_{y_j \in X} s_j \quad \forall X \in \mathcal{P}(S)$$
Nondeterminism & Probability

- Each defines an algebraic theory
- Nondeterminism: theory of semilattices
  \[ x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z; \quad x \sqcap y = y \sqcap x; \quad x \sqcap x = x \]
- Probabilistic choice: theory of probabilistic algebras
  \[ x_{r+y} = y_{1-r+x} \quad x_{1+y} = x \quad x_{r+x} = x \]
  \[ (x_{r+y})_{s+z} = x_{r \cdot s + \left( y_{s \cdot (1-r)} + z \right)} \text{ if } r < 1 \]
Combining Theories

- \( \mathcal{P}(\mathcal{V}(S)) \) - Sets of valuations
- \( X \oplus Y = \{ x \oplus y \mid x \in X, y \in Y \} \)
- Nondeterministic choice and probabilistic choice are entangled:
  \[
  (\{ \delta_x \} \cup \{ \delta_y \}) \frac{1}{2} + (\{ \delta_x \} \cup \{ \delta_y \}) = \{ \delta_x, \delta_y, \delta_x \frac{1}{2} + \delta_y \}
  \]
- \( X \cap Y = X \cup Y \cup (\bigcup_r X \oplus Y) \)
- Theorem (Tix 1999/ M- 2000)
  The power set induces an endofunctor on probabilistic algebras.
Two Theorems

• **Theorem 1 (Beck)**
  If \( \langle S, \eta_S, \mu_S \rangle, \langle T, \eta_T, \mu_T \rangle : A \to A \) are monads, then the following are equivalent:
  
  - There is a distributive law \( d : ST \to TS \)
  - \( S \) lifts to a monad of \( T \)-algebras

• **Theorem 2 (Plotkin & Varacca)**
  There is no distributive law of the power set over the probabilistic power domain, or *vice versa.*
**Alternative approach**

- **Weaken (eliminate) one of the laws:**
  \[x^r + y = y^{1-r} + x \quad x^1 + y = x \]
  \[(x^r + y)^s + z = x^r s + (y \frac{s(1-r)}{1-r s} + z) \text{ if } r < 1\]

- **New structures - (finite) indexed valuations:**
  \[IV(X) = \left[ \bigcup_{n>0} (\mathbb{R}_+ \times X)^n / \equiv \right] \cup \{0\}\]
  \[(r_i, x_i) \equiv (s_i, y_i) \iff (\exists \phi \in S(n))\]
  \[(r_{\phi^{-1}(i)}, x_{\phi^{-1}(i)}) = (s_i, y_i) \quad (i = 1, \ldots, n)\]
Understanding Indexed Valuations

- \([r_i, x_i]_m \simeq \sum_{i=1}^{m} r_i \delta_{x_i} \); \([1, x] \neq [(\frac{1}{2}, x), (\frac{1}{2}, x)]\)

- \(r \cdot [r_i, x_i]_m \mapsto [r \cdot r_i, x_i]_m : \mathbb{R}_+ \times IV(X) \rightarrow IV(X)\)

- \([r_i, s_i]_m \oplus [s_j, y_j]_n = [t_k, z_k]_{m+n}\)

- \((t_k, z_k) = \begin{cases} (r_i, x_i) & \text{if } k \leq m \\ (s_j, y_j) & \text{if } m < k \leq n \end{cases}\)

- \([r_i, x_i]_m r + [s_j, y_j]_n = r \cdot [r_i, x_i]_m \oplus (1 - r) \cdot [s_j, y_j]_n\)
Universal Properties

• Indexed valuations are real quasi-cones:

\[ A + (B + C) = (A + B) + C \quad A + B = B + A \]
\[ r(A + B) = rA + rB \quad r(sA) = (rs)A \]
\[ 0A = 0, \quad 1A = A \quad 0 + A = A \]
\[ (r + s)A = rA + sA \quad r, s \in \mathbb{R}_+ \quad A, B, C \in IV(X) \]

• Theorem (Varacca)

\( IV : \text{Set} \to \text{Set} \) defines a monad of real quasi-cones that enjoys a distributive law over \( \mathcal{P} \). So, \( (\mathcal{P} \circ IV)(X) \) is a real quasi-cone that also is a semilattice.
Justifying Indexed Valuations

\[ [r_i, x_i]_m \simeq \sum_{i=1}^{m} r_i \delta_{x_i}; \quad [1, x] \neq [(\frac{1}{2}, x), (\frac{1}{2}, x)] \]

\( f: (P, \mu) \to (X, \Omega) \) a random variable.

\( f: (P, \mu) \to (X, \Omega) \) induces \( (f \cdot \mu)(U) = \mu(f^{-1}(U)) \)

– too coarse

\( Flat: IV(X) \to \mathbb{P}(X) \) by \( Flat([r_i, x_i]_m) = \sum_i r_i \delta_{x_i} \)

– morphism of real quasi-cones.
Generalizing to Domains

Domain: Partial order in which directed sets have suprema

- A ⊆ D directed if each finite subset has an upper bound in A
- Continuous: x << y iff y ≤ sup A ⇒ x ≤ a ∈ A

\[ y = \sup \{ x \mid x \ll y \} \] - directed

Example: \(([0,1], \leq)\) \(x \ll y\) iff \(x = 0\) or \(x < y\)
Categories of Domains

\[ f : D \to E \text{ continuous if } f \text{ preserves the order} \]
\[ \text{and } f \text{ preserves sups of directed sets} \]

\[ \text{Dom} – \text{domains and continuous functions} \]
\[ – \text{not cartesian closed} \]

\[ \text{BCD} – \text{bounded complete domains and continuous functions} \]
\[ \cap – \text{is cartesian closed} \]

\[ \text{RB} – \text{retracts of bifinite domains and continuous functions} \]
\[ \cap – \text{is cartesian closed} \]

\[ \text{FS} – \text{FS-domains and continuous functions} \]
\[ – \text{maximal cartesian closed} \]
Constructing Bag Domains

\[ E \simeq D \times E = \bigcup_n D^n \] – domain of lists over \( D \)
- leaves RB, FS invariant
- free domain monoid over \( D \)

\[ D^n/ \equiv S(n) \] – domain of \( n \)-bags over \( D \)
- leaves RB, FS invariant

\[ E_C = \bigcup_n (D^n/ \equiv S(n)) \] – bag domain over \( D \)
- leaves RB, FS invariant
- free commutative domain monoid over \( D \)
Applying the Construction

\[ IV(D) = \bigcup_n ((\mathbb{R}_+ \times D)^n / \equiv_{S(n)}) \cup \{0\} \]

- leaves \( \text{RB}, \text{FS} \) invariant
- free commutative domain monoid
  over \( \mathbb{R}_+ \times D \) with \( \bot = 0 \)

\[ \mathbb{R}V(D) = \{[r_i, x_i]_m \in IV(D) \mid \sum_i r_i \leq 1\} \cup \{0\} \]
- discrete random variables over \( D \)

**Theorem:** \( \mathbb{R}V : \text{RB} \to \text{RB} \) is a continuous endofunctor; the same is true for \( \text{FS} \). \( \text{Flat} : \mathbb{R}V(D) \to \mathcal{V}(D) \) is an epimorphism.
Summary and Future Work

• Daniele Varacca first defined indexed valuations
  • Used *abstract bases*
  • No categorical results
• Our work first to introduce random variables
• Categorical results also new
• Bag domain results also new
• Possible further work:
  • Generalize to nondiscrete random variables
  • Applications to quantum computing - *entropy & majorization*