Variations on an Interval Domain Theme

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Outline

- The interval domain
- Probability and Domain Theory
- Majorization and entropy
- The Bayesian Order
- Generalizations

The Interval Domain

$$\mathbb{IR} \;=\; (\{[a,b] \mid a \leq b \in \mathbb{R}\}, \supseteq)$$

- Directed complete
  - $D \subseteq P$ *directed* if $(\forall F \subseteq D$ finite$)$ $(\exists d \in D)$ $F \subseteq \downarrow d$
    $D \subseteq \mathbb{IR}$ directed iff D is a filterbase
  - Each directed subset of P has a least upper bound in P.
    $D \subseteq \mathbb{IR}$ directed $\implies \sqcup D = \bigcap D$

- $\mathbb{IR}$ also *continuous*:
  - $x \ll y \in P$ iff $(\forall D$ directed$)$ $y \sqsubseteq \sqcup D \;\Rightarrow \uparrow x \cap D \neq \emptyset$

    $[a,b] \ll [c,d]$ iff $[c,d] \subseteq (a,b)$
  - $\Downarrow y = \{x \mid x \ll y\}$ directed & $y = \sqcup \Downarrow y$

    $[c,d] = \bigcap\{[a,b] \mid [c,d] \subseteq (a,b)\}$
- Originally proposed by Dana Scott as model for functional programming language with abstract data type 'real'.
- $[a,b] \sqsubseteq [c,d]$ means $[c,d]$ has more information than $[a,b]$.

  Maximal – ideal – elements are points: $[r,r]$, $r \in \mathbb{R}$.

Notice $\mathbb{IR}$ has additional structure:

- $[a, b] \wedge [c, d] = [a \wedge c, b \vee d]$

Works for all non-empty families of intervals – if we add $\mathbb{R}$ to $\mathbb{IR}$.
These are called *continuous Scott domains*
CSD – Continuous Scott domains and Scott continuous maps
CSD is a cartesian closed category:

- terminal object
- finite products
- internal hom: $P^Q = \mathsf{CSD}[Q, P]$

Escardó took up Scott's proposal to develop Real PCF

Real PCF

- Lazy functional language based on simply typed λ-calculus with recursion
- has real numbers abstract data type realized via interval domain

Operational semantics for Real PCF associates to expressions of type real a shrinking sequence of rational intervals representing the real number.

Model built over $\mathbb{IR}$ is *computationally adequate*: Intersection of intervals agrees with the $\mathbb{IR}$-based *denotational* semantics of real expression

But, Real PCF requires parallel evaluation ("dovetailing") for its operational semantics:

$$\mathsf{pif}\colon bool \times real \times real \to real$$

$$\mathsf{pif}(true, x, y) = x; \mathsf{pif}(false, x, y) = y; \mathsf{pif}(\bot, x, y) = x \sqcap y$$

Requires evaluating b, x, y in parallel and outputting partial results.
Used to ensure Real PCF is Turing universal

Recently, Escardó, M. Hofmann & T. Streicher showed pif is intrinsic:
Under mild conditions, *any* language that is computationally adequate wrt $\mathbb{IR}$ must allow a "weak parallel or:"

$$\mathsf{wpor}\colon bool \times bool \to \{\bot, \top\}$$

$$\mathsf{wpor}(x, y) = \top \text{ iff } x = \top \text{ or } y = \top$$

The Scott Topology

$U \subseteq P$ is Scott open if:

- $U = \uparrow U = \{x \in P \mid (\exists u \in U)\ u \sqsubseteq x\}$
- $(\forall D \text{ directed})\ \sqcup D \in U \implies D \cap U \neq \emptyset$

Note: P continuous & $x \ll y \in P) \implies (\exists z)\ x \ll z \ll y$
This implies $\Uparrow x = \{y \mid x \ll y\}$ is Scott open.

$f \colon P \to Q$ is *Scott continuous* iff

- f is monotone
- $f(\sqcup D) = \sqcup f(D)$ for D directed.

Domain Models

Notice that $x \mapsto [x, x] \colon \mathbb{R} \to \mathrm{I}\mathbb{R}$ is a homeomorphism onto its image in the relative Scott topology.

$\mathrm{I}\mathbb{R}$ is a *domain model* for $\mathbb{R}$.

P is a *domain model* for X if

$\exists \phi \colon X \to (\mathrm{Max}(P), \sigma(P)|_{\mathrm{Max}(P)})$ homeomorphism

P is ω–continuous if $(\exists B \subseteq P)$ countable with $\Downarrow x \cap B$ directed and $x = \sqcup(\Downarrow x \cap B)$ $(\forall x \in P)$

Theorem (Lawson): X is Polish iff X has a domain model into an ω–continuous domain whose Scott and Lawson topologies agree on $\mathrm{Max}(P)$.

$$\lambda(P) = \sigma(P) \vee \omega(P), \quad \omega(P) = \langle \{P \setminus \uparrow F \mid F \subseteq P \text{ finite}\} \rangle$$

$\mu\colon P \to [0,\infty)^{op}$ *measurement* if μ Scott continuous and

$$x \in \mathrm{Max}(P) \ \& \ x \in U \in \sigma(P) \implies (\exists \epsilon > 0)\ \mu_\epsilon(x) \subseteq U$$

$$\mu_\epsilon(x) = \{y \sqsubseteq x \mid \mu(y) \in [0,\epsilon)\}$$

$$\ker(\mu) = \{x \in P \mid \mu(x) = 0\}$$

$\mu\colon \mathrm{I\!R} \to [0,\infty)^{op}$ by $\mu([a,b]) = b - a$

Theorem (Martin & Reed):

- X is developable & T_1 iff $X = \ker(\mu)$ for μ defined on a continuous poset.
- Each Cech-complete, developable space is the kernel of a measurement on a domain.
- But, there is a domain whose maximal elements are a G_δ but they are not kernel of any measurement.

These results utilize Mike's Moore space construction.

Probability Theory & Domains

$\mu\colon \sigma(P) \to [0,1]$ *valuation* if:

- $\mu(\emptyset) = 0$
- $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
- $U \subseteq V \implies \mu(U) \leq \mu(V)$

Theorem (Lawson, Edalat, Saheb-Djarhomi, Alvarez-Manilla) Scott continuous valuations on a domain correspond to subprobability measures.

Probabilistic Power Domain

$\mathbb{V}P = \{\mu \mid \mu \text{ Scott continuous valuation on } P\}$

$\mu \sqsubseteq \nu$ iff $\mu(U) \leq \nu(U)$ $(\forall U \in \sigma(P))$

$(\mathbb{V}P, \sqsubseteq)$ a domain if P is one.

DCPO, Coh both closed under $\mathbb{V}$

No known ccc of continuous domains closed under this construct.

For P finite, $\mathbb{V}P = \{\Sigma_{x\in P} r_x \delta_x \mid \Sigma_x r_x \leq 1, 0 \leq r_x \leq 1\}$

P *flat* if $x \sqsubseteq y \implies x = y$

In this case,

$$\Sigma r_x \delta_x \sqsubseteq \Sigma s_x \delta_x \text{ iff } r_x \leq s_x \ (\forall x \in P)$$

Theorem For P flat & $|P| = n > 1$, $\mathbb{V}P \simeq \mathrm{I}([0,1]^{n-1})$

Proof (n=2) $\mathrm{Max}(\mathbb{V}P) = \{r\delta_x + (1-r)\delta_y \mid 0 \leq r \leq 1\}$

$$r\delta_x + (1-r)\delta_y \mapsto [r,r] \colon \mathrm{Max}(\mathbb{V}P) \to \mathrm{Max}(\mathrm{I}([0,1]))$$

homeomorphism. Extends because

$$(r\delta_x + (1-r)\delta_y) \wedge (s\delta_x + (1-s)\delta_y = (r \wedge s)\delta_x + (1-(r \vee s))\delta_y$$

$(P, \{+_r \mid 0 \le r \le 1\}, *)$ *uniform choice algebra* if

1. $x +_p y = y +_{1-p} x$
2. $x +_1 y = x$
3. $(x +_r y) +_s z = x +_{rs} (y +_{\frac{s(1-r)}{1-rs}} z), \quad r < 1$
4. $x +_r x = x$

and

5. $x * y = y * x$
6. $x * (y * z) = (x * y) * z$
7. $x * x = x$.

$\mathbb{V}P$ is a uniform choice algebra with

$$\Sigma r_i \delta_{x_i} +_r \Sigma s_i \delta_{y_i} = \Sigma r r_i \delta_{x_i} + \Sigma (1-r) s_i \delta_{y_i}$$

and $* = \wedge$

Discrete Random Variables

$f\colon (X,\mu) \to (Y,\Omega)$ random variable

$f\mu(A) = \mu(f^{-1}(A)) \quad (\forall A \in \Omega)$

X countable $\implies$ f is discrete. Then

$$f\mu = \Sigma_{x\in X}\, r_x \delta_{f(x)}, \quad \text{where } \Sigma_x r_x = 1$$

Theorem

- If D is a domain, then so is $(\overline{\mathbb{R}_+} \times D)^n/S(n)$.
- If D is RB or FS, so is $(\overline{\mathbb{R}_+} \times D)^n/S(n)$.

$\mathcal{B}^{\mathbb{R}}(D) = \oplus_{n \geq 0}(\overline{\mathbb{R}_+} \times D)^n / S(n) \quad - \quad$ separated sum

$\mathbb{RV}(D) = \{[r_i, d_i]_n \in \mathcal{B}^{\mathbb{R}}(D) \mid \Sigma\, r_i \leq 1\}$

satisfies laws:

1. $\langle\ \rangle \oplus [r_i, d_i]_n = [r_i, d_i]_n$
2. $[r_i, d_i]_m +_r [s_j, e_j]_n = [s_j, e_j]_n +_{1-r} [r_i, d_i]_m$
3. $([r_i, d_i]_m +_r [s_j, e_j]_n) +_s [t_k, f_k]_p$
   $= [r_i, d_i] +_{rs} ([s_j, e_j]_n +_{\frac{s(1-r)}{1-rs}} [t_k, f_k]_p), \quad r < 1$

If D has semilattice operation, then
$\mathbb{RV}(D)$ is a uniform choice algebra.

$\mathbb{RV}$ leaves RB and FS invariant.

Majorization

$(r_i), (s_i) \in [0,1]^n$ with $\Sigma r_i = \Sigma s_i = 1$

$$(r_i) \preceq (s_i) \quad \text{iff} \quad \Sigma_{i=1}^{k} r_{[i]} \leq \Sigma_{i=1}^{k} s_{[i]} \quad (k \leq n, r_{[i]} = i^{th} \text{ largest } r_j)$$

- Discovered by Muirhead in 1903
- Arises in optimization problems
  – economics, algorithms, quantum computing...
- Studied by Hardy, Littlewood and Pólya
  and by Warshall and Oilkin

Theorem $(r_i) \preceq (s_i)$ iff $(r_i) = M(s_i)$
for some doubly stochastic M.

$\preceq$ is a preorder – not antisymmetric

$\Lambda^n = (\{(r_i) \in [0,1]^n \mid \Sigma r_i = 1,\ r_1 \geq r_2 \geq \cdots \geq r_n\}, \preceq)$

For example

$\perp = (1/n, \ldots, 1/n)$ and $(1, 0, \ldots, 0)$ maximal

Theorem (Martin & M) $(\Lambda^n, \preceq)$ is a continuous lattice.

$(r_i) \wedge (s_i) = (t_i)$ where $t_k = (\Sigma_{i \leq k} r_i) \wedge (\Sigma_{i \leq k} s_i) - t_{k-1}$

Moreover, entropy is a canonical measurement on $(\Lambda^n, \preceq)$

For $(r_i) \in [0,1]^n$ define $o(r_i) = \{i_1, i_2, \ldots, i_n\}$ where

$$i_1 = \min\{i \mid r_i = \max\{r_j \mid j = 1 \ldots, n\}\}$$

$$i_2 = \min\{i \mid r_i = \max\{r_j \mid j = 1 \ldots, n, j \neq i_1\}\}$$

$$\vdots$$

For $(r_i), (s_i) \in [0,1]^n$ define

$$(r_i) \sqsubseteq_M (s_i) \quad \text{iff} \quad o(r_i) = o(s_i) \;\&\; (r_i) \preceq (s_i)$$

Then $([0,1]^n, \sqsubseteq_M)$ is a domain.

On $([0,1] \times D)^n$ define

$$(r_i, d_i) \sqsubseteq_m (s_i, e_i) \quad \text{iff} \quad (r_i) \sqsubseteq_M (s_i) \;\&\; d_i \sqsubseteq e_i \; (\forall i)$$

Then $(([0,1] \times D)^n, \sqsubseteq_m)$ is a domain if D is one.

So, $\mathrm{Maj}(D)_n = (([0,1] \times D)^n / S(n), \sqsubseteq_m / \equiv_m)$

is a domain if D is one.

Inside we find

$$\Lambda(D)_n = \{(r_i, d_i) \in [0,1] \times D)^n \mid \Sigma r_i = 1\} / S(n)$$

also is a domain if D is one.

Extend to $\bigcup_n \Lambda(D)_n$ by $[r_i, d_i]_m \sqsubseteq_w m(s_j, e_j)_n$ iff

$$\Sigma_{i \leq k} r_{[i]} \leq \Sigma_{i \leq k} s_{[i]} \ \& \ d_k \sqsubseteq e_k \ (\forall k \leq m \leq n)$$

$(\Lambda(D), \sqsubseteq_{wm})$ is a continuous poset (ie., not directed complete). Unclear if its completion lies in any ccc of domains.

Bayesian Order

$\Delta^n = \{(r_i) \in [0,1]^n \mid \Sigma_i r_i 1\}$

$p_i : \Delta^{n+1} \rightharpoonup \Delta^n$ by $p_i(r_j) = \frac{1}{(1-r_i)}(r_1, \ldots, \widehat{r_i}, , \ldots, r_n)$

$n \geq 2$ & $x, y \in \Delta^{n+1}$

$x \sqsubseteq_B y \quad \text{iff} \quad (\forall i)(x, y \in \text{dom}(p_i) \;\Rightarrow\; p_i(x) \sqsubseteq_B p_i(y))$

$x, y \in \Delta^2$

$x \sqsubseteq_B y \quad \text{iff} \quad (y_1 \leq x_1 \leq 1/2) \text{ or } (1/2 \leq x_1 \leq y_1)$

$x \sqsubseteq_B y \quad$ iff

$(\exists \sigma \in S(n))(\forall i)(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} \leq (x \cdot \sigma)_{i+1} (y \cdot \sigma)_i$

$(\Delta^n, \sqsubseteq_B)$ directed complete partial order.

$(\Lambda^n, \sqsubseteq_B)$ domain.

Believe same approach as for $(\Lambda^n(D), \sqsubseteq_{wm})$ will apply here.

Further Work

- What is structure of $(\Lambda(D), \sqsubseteq_{wm})$?
- Is $(\Lambda(D), \sqsubseteq_{wm})$ image of $\mathrm{Maj}(D)$ under closure operator?
- Extend to quantum case?
  - Replace $S(n)$ by unitary group...