A(nother) Random Variable Monad

Preliminary Report

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A Disclaimer

- This is *NOT* a talk about stochastic processes or Skorohod’s Theorem.
  
  Apologies – a funny thing happened on the way to the talk...

- Instead, I’ll talk about some preliminary results on computational models for probabilistic choice.

  Active theme over last several years
  Other talks at the meeting: Plotkin, Barker, Jung
Some History

- The *probabilistic powerdomain* is the traditional domain model for probabilistic choice.
  - Realized on DCPO as $\forall(D)$ – the space of Scott-continuous valuations of $\mathcal{O}(D)$
  - Defines a monad on DCPO
  - Leaves Coh invariant, but *no CCC of domains is known to be invariant*
  - No distributive law wrt power domains

- Varacca (2003) devised a monad that overcomes these problems
  - Weakens the probabilistic law: $p + r p \neq p$.

- Based on Varacca’s work, M. (2007) devised a monad of *finite random variables* that leaves RB and FS invariant, and satisfies a distributive law.
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- Based on Varacca’s work, M. (2007) devised a monad of *finite random variables* that leaves RB and FS invariant, and satisfies a distributive law.
- Goubault-Larrecq and Varacca (2011) proposed a monad of *continuous random variables*, but the Kleisli lift failed to be Scott continuous.
- Their work inspired this work – and that by Tyler Barker you’ll hear after this talk.
Some Background

- **Random variables versus Prob**
  - Order on $\text{Prob}(D)$ is more complicated than that on $D$:
    \[ \mu \sqsubseteq \nu \text{ iff } \mu(U) \leq \nu(U) \ (\forall U \text{ open}) \]
    \[ d \mapsto \delta_d : D \to \text{Prob}(D) \text{ unit of monad} \]
  - **Idea of Random Variables:**
    Choose fixed domain $C$ for which $\text{Prob}(C)$ is well-behaved, and use
    \[ f : C \to D \text{ to define outcomes of probabilistic choices in } C \]
Some Background

- Random variables versus Prob
  - Which domain $C$ to choose?
    Use the Cantor tree – the full binary tree together with the Cantor set at the top:

    $\cdots \cdots \cdots \cdots \cdots \cdots C$
    
    | 0 | 1 | 0 | 1 |
    |----|----|----|----|
    |øj |0ö |01 |10 |

    - So, the model looks like $Prob(C) \times [C \to D]$ for a domain $D$
Sanity Check

\[ \text{RanV}(D) \equiv \text{Prob}(C) \times [C \rightarrow D] \text{ defines a functor:} \]

\[ g : D \rightarrow E \implies \text{RanV}(g)(\mu, f) = \langle \mu, g \circ f \rangle. \]
**Sanity Check**

But to be a monad, we need a Keisli lift:

\[ \text{Prob}(C) \times [C \to D] \]

\[ \eta_D \]

\[ D \]

\[ h \]

\[ \to \]

\[ \text{Prob}(C) \times [C \to E] \]

\[ h^\dagger \]

where \( \eta_D(d) = \langle \delta_\epsilon, \text{const}_d \rangle \).

If \( h^\dagger(\mu, f) = \langle \nu, g \rangle \),

\( g \) is easy: \( f : C \to D \) & \( \pi_2 \circ h : D \to E \) \( \Rightarrow \) \( g = \pi_2 \circ h \circ f : C \to E \).
Sanity Check
But to be a monad, we need a Keisli lift:

$$
\Prob(C) \times [C \to D]
\xrightarrow{\eta_D}
\xrightarrow{h^\dagger}
D
\xrightarrow{h}
\Prob(C) \times [C \to E]
$$

where $\eta_D(d) = \langle \delta_\epsilon, \text{const}_d \rangle$.

If $h^\dagger(\mu, f) = \langle \nu, g \rangle$,

For $\nu$, we should combine $\mu$ and the family $\{\pi_1 \circ h \circ f(d)\}_{d \in \text{supp} \mu}$.

Suppose $\mu = \sum_{d \in F} r_d \delta_d$ is simple. Then we expect:

$$
\begin{align*}
    h^\dagger \left( \sum_{d \in F} r_d \delta_d, f \right) &= \left\langle \sum_{d \in F} r_d (\delta_d \otimes \pi_1 \circ h \circ f(d)), \pi_2 \circ h \circ f \right\rangle
\end{align*}
$$

where $\otimes$ is a composition operation on measures.

$\otimes$ also should also be monotone!
More Background

- **Prob** forms a monad on:
  - **Comp** – category of compact $T_2$ spaces and continuous maps
    Algebras are compact affine spaces
  - **CompMon** – category of compact monoids and continuous monoid morphisms
    Algebras are compact affine monoids
  - **CompGrp** – category of compact groups and continuous group maps
    Algebras again compact affine monoids – right adjoint changes to $M \mapsto H(1_M)$

- On **CompMon**, the monoid operation on $\text{Prob}(S, \cdot, e)$ is *convolution*:
  \[
  \ast : \text{Prob}(S) \times \text{Prob}(S) \to \text{Prob}(S) \text{ by }
  \]

\[
\mu \ast \nu(A) = \cdot (\mu \times \nu)(A) = \mu \times \nu(\cdot^{-1}(A)) = \mu \times \nu(\{\langle x, y \rangle \mid x \cdot y \in A\})
\]
More Background

- An ordered monoid is a monoid \((S, \cdot, e)\) which has a partial order relative to which \(\cdot : S \times S \rightarrow S\) is monotone.
  
  - \(C\) is a monoid under concatenation, but it is not an ordered monoid.
  
  - \(C_\sqrt{} \equiv \{0, 1\}^\infty \cup \{0, 1\}^* \sqrt{}\) with the prefix order, and let
    
    \[
    s \cdot t = \begin{cases} 
    s't & \text{if } s = s' \\
    s & \text{otherwise}
    \end{cases}
    \]
  
  - \((C_\sqrt{}, \cdot, \sqrt{})\) is a compact ordered monoid.

- Prob is a monad on CompOrdMon – category of compact, ordered monoid and continuous, monotone monoid morphisms.

  The convolution operation on \(\text{Prob}(S, \cdot, e, \leq)\) is monotone:
  
  \[
  \mu \sqsubseteq \mu', \nu \sqsubseteq \nu' \implies \mu \ast \nu \sqsubseteq \mu' \ast \nu'.
  \]
  
  - This applies to any bounded complete domain in the Lawson topology – in particular to \(C_\sqrt{}\).
Towards a Monad

The monad should be: \( \text{Ran} V(D) = \text{Prob}(C_\sqrt{\varepsilon}) \times [C_\sqrt{\varepsilon} \to D] \) where

\[
\text{Prob}(C_\sqrt{\varepsilon}) \times [C_\sqrt{\varepsilon} \to D]
\]

where:

\[
\eta_D = \langle \delta_\sqrt{\varepsilon}, \text{const}_d \rangle \quad \text{and for} \quad \mu = \sum_{d \in F} r_d \delta_d
\]

\[
h^\dagger(\mu, f) = \left\langle \sum_{d \in F} r_d (\delta_d \ast (\pi_1 \circ h \circ f(d)), \pi_2 \circ h \circ f) \right\rangle
\]
Towards a Monad

\[
\text{Prob}(C_\sqrt{\cdot}) \times [C_\sqrt{\cdot} \to D]
\]

where:

\[
\eta_D = \langle \delta_\sqrt{\cdot}, \text{const}_d \rangle \quad \text{and for} \quad \mu = \sum_{d \in F} r_d \delta_d
\]

\[
h^\dagger(\mu, f) = \left( \sum_{d \in F} r_d (\delta_d \ast (\pi_1 \circ h \circ f(d)), \pi_2 \circ h \circ f) \right)
\]

Todo's:
- Validate monad structure – May require restricting \(\mu\) to be “thin”
- Work out equational laws
Towards a Monad

\[ \text{Prob}(C\sqrt{\ )} \times [C\sqrt{ \to D} \]

\[ \eta_D \]

\[ D \xrightarrow{h} \text{Prob}(C\sqrt{\ )} \times [C\sqrt{ \to E} \]

where:

\[ \eta_D = \langle \delta_{\sqrt{\ )}, \text{const}} \rangle \] and for \( \mu = \sum_{d \in F} r_d \delta_d \)

\[ h^\dagger(\mu, f) = \left\langle \sum_{d \in F} r_d (\delta_d \ast (\pi_1 \circ h \circ f(d)), \pi_2 \circ h \circ f) \right\rangle \]

ToDo's:

- Devise closed form for probability component with \( \mu \) non-discrete

\[ \mu = \lim_n \sum_{x \in F_n} r_x \delta_x \implies \mu \ast \nu = \lim_n \sum_{x \in F_n} r_x (\delta_x \ast (\pi_1 \circ h \circ f(x))) \]