The Search for Random Variable Monads

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The Cantor Tree

$$\mathcal{C} \simeq \Sigma^{\infty} = \Sigma^{*} \cup \Sigma^{\omega}, \quad \Sigma = \{0, 1\}$$

Build Two Models Based on $\mathcal{C}$:

1. First from random coin flips
2. Second from “fixing” 1)

Domain Properties of $\mathcal{C}$

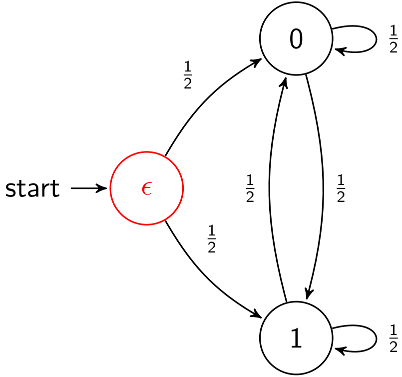
- $\mathcal{C} \simeq \{0,1\}^\infty = \{0,1\}^* \cup \{0,1\}^\omega$ is Scott domain
- $\mathcal{C}$ is a rooted tree
- $X \subseteq \mathcal{C}$ Scott closed $\Rightarrow \exists \pi_X \colon \mathcal{C} \to X$
- $X \subseteq \mathcal{C}$ Scott closed iff $X = {\downarrow} \mathit{Max}\, X$ &

  $\mathit{Max}\, X$ is Lawson compact antichain.

Flipping Coins

- *Random oracles* in computation:
  - Execute some steps of computation...
    - Come to a branch point – need to make a *random choice*
    - Consult an oracle – or, simply flip a coin
    - Make choice based on outcome of coin flip
  - REPEAT
- Examine first how to model the coin flips
  - Utilize structure of *trace distributions*
- Incorporate into *random variable models*

Model of coin flipping



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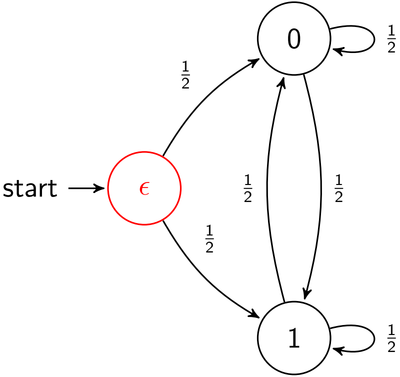
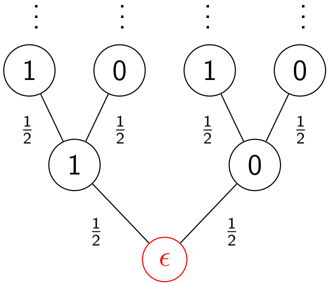


Unfolding the automaton:



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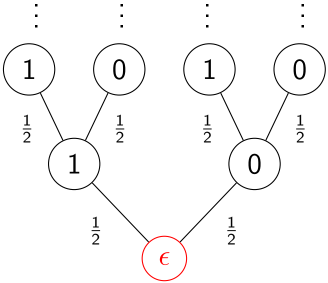

Model such automata be their *trace distributions:*

$$\vdots$$

$$\mu_2 = \frac{1}{4}\delta_{00} + \frac{1}{4}\delta_{01} + \frac{1}{4}\delta_{10} + \frac{1}{4}\delta_{11}$$

$$\mu_1 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$

$$\mu_0 = \delta_\epsilon$$

Model of coin flipping

Stripping away the probabilities, we have the following sets on which these measures are *concentrated*:

$$\begin{array}{ll} \vdots & \vdots \\ \mu_2 = \frac{1}{4}\delta_{00} + \frac{1}{4}\delta_{01} + \frac{1}{4}\delta_{10} + \frac{1}{4}\delta_{11} & X_2 = \{00, 01, 10, 11\} \\ \mu_1 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 & X_1 = \{0, 1\} \\ \mu_0 = \delta_\epsilon & X_0 = \{\epsilon\} \end{array}$$

Notice that the X_ns are *antichains*, and

$X_0 \sqsubseteq_C X_1 \sqsubseteq_C X_2 \sqsubseteq_C \cdots$, where

$$X \sqsubseteq_C Y \;\Leftrightarrow\; X \subseteq {\downarrow}Y \;\&\; Y \subseteq {\uparrow}X.$$

These are *Lawson-compact antichains* in the *Egli-Milner order.*

The Underlying Structure - Domains and Trees

- $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ is a *domain* under the prefix order.
  $K\Sigma^\infty = \Sigma^*$ – the finite words

  Scott topology: $\{\uparrow k \mid k \in K\Sigma^\infty\}$ as basis.

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  If Σ is finite, then Σ^∞ is *coherent*

  Compact in the *Lawson topology*

  Open sets: $U = \uparrow k \setminus \uparrow F, \quad k \in \Sigma^*, F \subseteq \Sigma^*$ finite

The Underlying Structure - Domains and Trees

- $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ is a *domain* under the prefix order.
- $AC(\Sigma^\infty) = (\{X \mid \text{Lawson-compact antichain}\}, \sqsubseteq_C)$

$$\begin{aligned} X \sqsubseteq_C Y \quad &\Leftrightarrow \quad X \subseteq {\downarrow}Y \ \&\ Y \subseteq {\uparrow}X \\ &\Leftrightarrow \quad \pi_X(Y) = X \end{aligned}$$

Subdomain of $\mathcal{P}_C(\Sigma^\infty)$ – convex power domain over Σ^∞.

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- **Theorem:** $AC(\Sigma^\infty)$ is a Scott domain: all nonempty subsets have infima. Moreover, given $\{X_n\}_{n\in\mathbb{N}} \subseteq AC(\Sigma^\infty)$ directed and $X \in AC(\Sigma^\infty)$, TAE:

  (i) $X = \sup_n X_n$

  (ii) $X = \lim_n X_n$ in the Vietoris topology on $\Gamma(\Sigma^\infty)$.

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- In particular, *any* $X \in AC(\Sigma^\infty)$ satisfies

  $X = \sup_n \pi_n(X) = \lim_n \pi_n(X)$, where

  $\pi_n \colon \Sigma^\infty \to \Sigma^{\leq n} = \{\sigma \mid |\sigma| \leq n\}$ is the canonical retraction.

Thin Probability Measures

- $\mu \in \mathit{Prob}(\Sigma^\infty)$ is *thin* if $\mathrm{supp}_\Lambda\, \mu \in \mathit{AC}(\Sigma^\infty)$.

  Note: $\mathrm{supp}_\Lambda\, \mu$ is in the *Lawson* topology.

Thin Probability Measures

- $\mu \in \mathit{Prob}(\Sigma^\infty)$ is *thin* if $\mathsf{supp}_\Lambda\, \mu \in \mathit{AC}(\Sigma^\infty)$.
- Define $\mu \leq \nu$ iff $\pi_{\mathsf{supp}_\Lambda\, \mu}(\nu) = \mu$

  Agrees with usual domain order (*qua* valuations) / functional analysis order via cones.

Thin Probability Measures

- $\mu \in \mathit{Prob}(\Sigma^\infty)$ is *thin* if $\mathrm{supp}_\Lambda\, \mu \in AC(\Sigma^\infty)$.
- **Proposition:** $(\Theta \mathit{Prob}(\Sigma^\infty), \leq)$ is a bounded complete domain: all nonempty subsets have infima. Moreover, given $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \Theta \mathit{Prob}(\Sigma^\infty)$ directed and $\mu \in \Theta \mathit{Prob}(\Sigma^\infty)$, TAE:

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A Random Variables Model

$\Theta Prob(\Sigma^\infty)$ with $\mu \leq \nu \quad$ iff $\quad \pi_{\mathsf{supp}_\Lambda \mu}(\nu) = \mu$.

Define the space of *continuous random variables over P* to be:

$$
\begin{aligned}
\Theta RV(\Sigma^\infty, P) \quad = \quad & \{(\mu, f) \mid \mu \in \Theta Prob(\Sigma^\infty) \ \& \\
& \quad f\colon \mathsf{supp}_\Lambda \mu \to P \ \Lambda\text{-continuous}\}
\end{aligned}
$$

$$
(\mu, f) \leq (\nu, g) \quad \text{iff} \quad \pi_{\mathsf{supp}_\Lambda \mu}(\nu) = \mu \ \& \ f \circ \pi_{\mathsf{supp}_\Lambda \mu} \leq g.
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$$(\mu, f) \le (\nu, g) \quad \text{iff} \quad \pi_{\mathsf{supp}_\Lambda \mu}(\nu) = \mu\ \&\ f \circ \pi_{\mathsf{supp}_\Lambda \mu} \le g.$$

Idea for this due to Goubault-Larrecq and Varacca (LICS 2011).

Now we need to show this is in BCD if P is.

Function Space Component

- $X \sqsubseteq_C Y \in AC(A^\infty), P \in \mathrm{BCD} \implies$
  $f \mapsto f \circ \pi_X \colon [X \longrightarrow P] \hookrightarrow [Y \longrightarrow P]$ &
  $g \mapsto \widehat{g} \colon [Y \longrightarrow P] \twoheadrightarrow [X \longrightarrow P]$ by $\widehat{g}(x) = \inf g(\pi_X^{-1}(x))$.

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  $g \mapsto \widehat{g} \colon [Y \longrightarrow P] \longrightarrow\!\!\!\!\!\rightarrow [X \longrightarrow P]$ by $\widehat{g}(x) = \inf g(\pi_X^{-1}(x))$.
- $X \in AC(A^\infty), P \in \mathrm{BCD} \implies [X \longrightarrow P] \in \mathrm{BCD}$:

$$[X \longrightarrow P] \simeq \lim_n [\pi_n(X) \longrightarrow P] \simeq \lim_n P^{\pi_n(X)}.$$

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- $(\bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P], \leq_R) \in \mathrm{BCD}$:

$$f \leq_R g \quad \text{iff} \quad \mathrm{dom}\, f \sqsubseteq_C \mathrm{dom}\, g \ \&\ f \circ \pi_{\mathrm{dom}\, f} \leq g.$$

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Defining the Model

- $\Theta Prob(A^\infty) \times \bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P] \in \mathrm{BCD}$ if $P \in \mathrm{BCD}$.

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Defining the Model

- $\Theta Prob(A^\infty) \times \bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P] \in \mathrm{BCD}$ if $P \in \mathrm{BCD}$.
- For $P \in \mathrm{BCD}$

  $\Theta RV(A^\infty, P) = \{(\mu, f) \mid \mu \in \Theta Prob(A^\infty), f \colon \mathrm{supp}_\Lambda \mu \longrightarrow P\}$

  is a retract of $\Theta Prob(A^\infty) \times \bigoplus_{X \in AC(A^\infty)} [X \longrightarrow P]$:

  $(\mu, f) \mapsto (\pi_Y(\mu), f \circ \pi_Y)$ is the projection

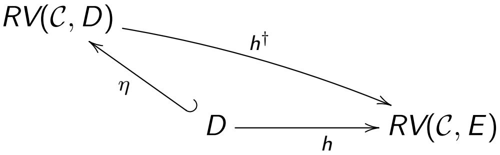
  $$Y = \mathrm{supp}_\Lambda \mu \wedge \mathrm{dom}\, f.$$

But, this is NOT a Monad

Given $P \in \mathrm{BCD}$, define

$$RV(\mathcal{C}, D) = \Theta Prob(A^{\infty}) \times \bigoplus_{X \in AC(A^{\infty})} [X \longrightarrow P]$$

Then there are maps satisfying the monad laws



$\eta(x) = (\delta_{\epsilon}, \mathrm{const}_x)$

But $h^{\dagger}$ is not monotone. :-(

Alternative Model

- *Basic idea:* Flatten model so concatenation doesn't need to be monotone in first component.
- Leads to model which looks like

$$1 \text{ flip} \oplus 2 \text{ flips} \oplus 3 \text{ flips} \oplus \cdots \oplus n \text{ flips} \oplus \cdots$$

- Begin with $\mathsf{SProb}(n) = \left\{ \sum_{i<n} r_i \delta_i \mid 0 \leq r_i \ \& \ \sum_i r_i \leq 1 \right\}$
  - $\sum_i r_i \delta_i \leq \sum_i s_i \delta_i$ iff $r_i \leq s_i$ $(\forall i)$.
  - $\sum_i r_i \delta_i \wedge \sum_i s_i \delta_i = \sum_i (r_i \wedge s_i) \delta_i$
  - $\bot = 0$
  - A directed $\Rightarrow (\sup A)(i) = \sup_{\mu \in A} \mu(i)$.

Alternative Model

- Flat random variable domain:
  $$RV^\flat(D) = \bigoplus_n \left(\mathrm{SProb}(\{0,1\}^n) \times D^{\{0,1\}^n}\right)$$
  - $(\mu_n, X_n) \le (\nu_m, X_m)$ iff $m = n, \mu_n \le \nu_n$, and $X_n(i) \le X_m(i)$ $(\forall i)$.
- $f\colon D \to E \;\Rightarrow\; RV^\flat(f)\colon RV^\flat(D) \to RV^\flat(E)$

  by $RV^\flat(f)(\mu_n, X_n) = (\mu_n, f \circ X_n)$.
- $RV^\flat(D)$ forms a monad on BCD.
- *Problem:* $RV^\flat(D)$ makes too many distinctions:
  $(\frac{1}{3}\delta_0 + \frac{2}{3}\delta_1, (a,b)) \ne (\frac{2}{3}\delta_0 + \frac{1}{3}\delta_1, (b,a))$, etc.
- Solution requires some background work.

Free ordered semigroup

- $P^* = \bigoplus_{n>0} P^n$ is free ordered semigroup over poset P:
  - $w \leq w'$ iff $|w| = |w'|$ & $w_i \leq_P w'_i$ $(\forall i \leq |w|)$.
  - $ww' \in P^{m+n}$ if $w \in P^m$ & $w' \in P^n$.
  - *Note (J.-E. Pin):* Free ordered *monoid* is flat.
- Also works for P in BCD, FS, or RB.
- To obtain the free *commutative* semigroup, we take a quotient:
  - $S(n)$ acts on P^n by permuting the components.
  - $P^n/S(n)$ is the set of n-bags over P.
  - $\pi_n \colon P^n \to P^n/S(n)$ is monotone.
- $COS(P) = \bigoplus_{n>0} P^n/S(n)$ – free commutative ordered semigroup over P.

Free ordered domain

- Rudin's Lemma implies this also works in domains.
- $CDS(P) = \bigoplus_{n>0} P^n/S(n)$ – free commutative domain semigroup over domain P.

Apply this to $RV^\flat(D)$ to obtain flat "commutative" random variable domain:

- $CRV^\flat(D) = \bigoplus_n \left(\mathrm{SProb}(2^n) \times D^{2^n}\right)/S(2^n)$
- Now $(\frac{1}{3}\delta_0 + \frac{2}{3}\delta_1, (a, b)) \equiv (\frac{2}{3}\delta_0 + \frac{1}{3}\delta_1, (b, a))$, etc.
- *But:* $(\frac{1}{3}\delta_0 + \frac{2}{3}\delta_1, (a, b)) \not\equiv (\frac{1}{3}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{3}\delta_1, (a, b, b))$
- Still a monad over RB and FS (but not over BCD).

Some Additional Comments

- Work was inspired by Varacca's *indexed valuations* (2004) and Goubault-Larrecq & Varacca's work on the first model.
- Jean Goubault-Larrecq is working on a patch to the first model.
  - Refines the order
- Tyler Barker also working on a patch
  - Redefines the Kleisli lift – somewhat akin to conditional probability
- Remaining question: Can the second model be extended to include recursion on the number of flips?