Cantor Groups, Haar Measure and Lebesgue Measure on $[0, 1]$

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Lebesgue Measure and Unit Interval

- $[0, 1] \subseteq \mathbb{R}$ inherits *Lebesgue measure*: $\lambda([a, b]) = b - a$.
- *Translation invariance:* $\lambda(A + x) = \lambda(A)$ for all (Borel) measurable $A \subseteq \mathbb{R}$ and all $x \in \mathbb{R}$.

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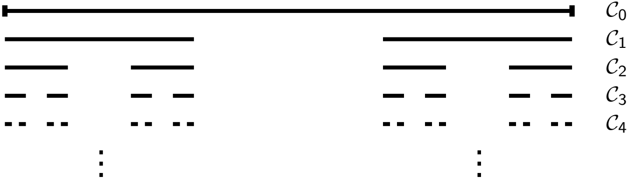
- *Translation invariance:* $\lambda(A+x) = \lambda(A)$ for all (Borel) measurable $A \subseteq \mathbb{R}$ and all $x \in \mathbb{R}$.
- **Theorem** (Haar, 1933) Every locally compact group G has a unique (up to scalar constant) left-translation invariant regular Borel measure μ_G called *Haar measure*.

  If G is compact, then $\mu_G(G) = 1$.

  Example: $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ with quotient measure from λ.

  If G is finite, then μ_G is normalized counting measure.

The Cantor Set



$$\mathcal{C} = \bigcap_n \mathcal{C}_n \subseteq [0,1] \text{ compact 0-dimensional, } \lambda(\mathcal{C}) = 0.$$

Theorem: $\mathcal{C}$ is the unique compact Hausdorff 0-dimensional second countable perfect space.

Cantor Groups

- *Canonical Cantor group:*
  $\mathcal{C} \simeq {\mathbb{Z}_2}^{\mathbb{N}}$ is a compact group in the product topology.

  $\mu_{\mathcal{C}}$ is the product measure $(\mu_{\mathbb{Z}_2}(\mathbb{Z}_2) = 1)$

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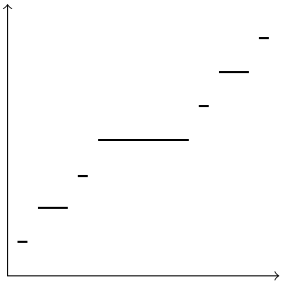

Theorem: (Schmidt) The Cantor map $\mathcal{C} \to [0,1]$ sends Haar measure on $\mathcal{C} = \mathbb{Z}_2{}^{\mathbb{N}}$ to Lebesgue measure.

Goal: Generalize this to *all* group structures on $\mathcal{C}$.

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  Definition: A *Cantor group* is a compact 0-dimensional second countable perfect space endowed with a topological group structure.

- $G = \prod_{n>1} \mathbb{Z}_n$ is also a Cantor group.

  μ_G is the product measure $(\mu_{\mathbb{Z}_n}(\mathbb{Z}_n) = 1)$

- $\mathbb{Z}_{p^\infty} = \varprojlim_n \mathbb{Z}_{p^n}$ – p-adic integers.
- $H = \prod_n S(n)$, where each $S(n)$ is the symmetric group on n letters.

Two Theorems and a Corollary

- **Theorem:** If G is a compact 0-dimensional group, then G has a neighborhood basis at the identity of clopen normal subgroups.
- *Proof:*
  1. G is a Stone space, so there is a basis $\mathcal{O}$ of clopen neighborhoods of e.
     If $O \in \mathcal{O}$, then $e \cdot O = O \;\Rightarrow\; (\exists U \in \mathcal{O})\, U \cdot O \subseteq O$
     $U \subseteq O \;\Rightarrow\; U^2 \subseteq U \cdot O \subseteq O$. So $U^n \subseteq O$.
     Assuming $U = U^{-1}$, the subgroup $H = \bigcup_n U^n \subseteq O$.
  2. Given $H < G$ clopen, $\mathcal{H} = \{xHx^{-1} \mid x \in G\}$ is compact.
     $G \times \mathcal{H} \to \mathcal{H}$ by $(x, K) \mapsto xKx^{-1}$ is continuous.
     $K = \{x \mid xHx^{-1} = H\}$ is clopen since H is, so G/K is finite.
     Then $|G/K| = |\mathcal{H}|$ is finite, so $L = \bigcap_{x \in G} xHx^{-1} \subseteq H$ is clopen and normal.

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- **Theorem:** (Fedorchuk, 1991) If $X \simeq \varprojlim_{i \in I} X_i$ is a strict projective limit of compact spaces, then $Prob(X) \simeq \varprojlim_{i \in I} Prob(X_i)$.

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- **Lemma:** If $\varphi \colon G \to H$ is a surmorphism of compact groups, then $Prob(\varphi)(\mu_G) = \mu_H$.

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  In particular, if $X = G, X_i = G_i$ are compact groups, then $\mu_G = \lim_{i \in I} \mu_{G_i}$.

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- **Corollary:** If G is a Cantor group, then $G \simeq \varprojlim_n G_n$ with G_n finite for each n.
  Moreover, $\mu_G = \lim_n \mu_n$, where μ_n is normalized counting measure on G_n.

It's all about Abelian Groups

- **Theorem:** If $G = \varprojlim_n G_n$ is a Cantor group, there is a sequence $(\mathbb{Z}_{k_i})_{i>0}$ of cyclic groups so that $H = \varprojlim_n (\oplus_{i \leq n} \mathbb{Z}_{k_i})$ has the same Haar measure as G.

  Proof: Let $G \simeq \varprojlim_n G_n$, $|G_n| < \infty$.

  Assume $|H_n| = |G_n|$ with H_n abelian.

  If $\phi_n \colon G_{n+1} \to G_n$ is surjective, then $G_{n+1}/\ker \phi_n \simeq G_n$. So $|G_{n+1}| = |G_n| \times |\ker \phi_n|$.

  Define $H_{n+1} = H_n \times \mathbb{Z}_{|\ker \phi_n|}$. Then $|H_{n+1}| = |G_{n+1}|$,

  so $\mu_{H_n} = \mu_n = \mu_{G_n}$ for each n, and $H = \varprojlim_n H_n$ is abelian.

  Hence $\mu_H = \lim_n \mu_n = \mu_G$.

Combining Domain Theory and Group Theory

$\mathcal{C} = \varprojlim_n H_n$, $H_n = \oplus_{i \leq n} \mathbb{Z}_{k_i}$

Endow H_n with *lexicographic order* for each n; then

$\pi_n \colon H_{n+1} \to H_n$ by $\pi_n(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ &

$\iota_n \colon H_n \hookrightarrow H_{n+1}$ by $\iota_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$ form

embedding-projection pair.

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$\mathcal{C} \simeq \text{bilim}\,(H_n, \pi_n, \iota_n)$ is bialgebraic chain:

- $\mathcal{C}$ totally ordered, has all sups and infs
- $K(\mathcal{C}) = \bigcup_n \{(x_1, \ldots, x_n, 0, \ldots) \mid (x_1, \ldots, x_n) \in H_n\}$
- $K(\mathcal{C}^{op}) = \{\sup\,(\downarrow k \setminus \{k\}) \mid k \in K(\mathcal{C})\}$

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$\mathcal{C} \simeq \text{bilim}\,(H_n, \pi_n, \iota_n)$ is bialgebraic chain:

$\varphi \colon K(\mathcal{C}) \to [0,1]$ by $\varphi(x_1, \ldots, x_n, 0, 0, \ldots) = \sum_{i \leq n} \frac{x_i}{k_1 \cdots k_i}$ strictly monotone

induces $\widehat{\varphi} \colon \mathcal{C} \to [0,1]$ monotone, Lawson continuous.

Direct calculation shows:

$\mu_{\mathcal{C}}(\widehat{\varphi}^{-1}(a,b)) = \lambda((a,b))$ for $a \leq b \in [0,1]$; i.e., $Prob(\widehat{\varphi})(\mu_{\mathcal{C}}) = \lambda$.

Alternative Proof

1. *Cantor Tree:* $\mathcal{CT} \simeq \Sigma^\infty = \Sigma^* \cup \Sigma^\omega, \quad \Sigma = \{0, 1\}$

   $s \leq t \iff (\exists u)\, su = t$. Then $\mathsf{Max}\,\mathcal{CT} \simeq \mathcal{C}$.

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2. *Interval domain:* $\mathrm{Int}([0,1]) = (\{[a,b] \mid 0 \leq a \leq b \leq 1\}, \supseteq)$

   $\widehat{\phi}\colon \mathcal{C} \to [0,1]$ extends to $\Phi\colon \mathcal{CT} \to \mathrm{Int}([0,1])$ Scott continuous.

   Then $\mathit{Prob}(\Phi)\colon \mathit{Prob}(\mathcal{CT}) \to \mathit{Prob}(\mathrm{Int}([0,1]))$, so

   $\lambda = \mathit{Prob}(\Phi)(\mu_{\mathcal{C}}) = \lim \mathit{Prob}(\Phi)(\mu_n)$, where

   $\mathit{Prob}(\Phi)(\mu_n) = \sum_{1 \leq i \leq 2^n} \frac{1}{2^n} \cdot \delta_{[\frac{i-1}{2^n}, \frac{i}{2^n}]}$

Skorohod's Theorem

Theorem: (Skorohod) If ν is a Borel probability measure on a Polish space X, then there is a measurable map $\xi_\nu \colon [0,1] \to X$ satisfying $\mathit{Prob}(\xi_\nu)(\lambda) = \nu$; i.e., $\nu(A) = \lambda(\xi_\nu^{-1}(A))$.

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Proof using Domain Theory:

1° : $X \hookrightarrow \overline{X} \subseteq [0,1]^{\mathbb{N}}$ & X is dense G_δ in $\overline{X}$; $\mathit{Prob}(X) \hookrightarrow \mathit{Prob}(\overline{X})$.

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Now $\phi \colon \mathcal{C} \to [0,1]$ has an upper adjoint $\phi^* \colon [0,1] \to \mathcal{C}$.

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Then $\xi_\nu = \phi \circ \chi_\nu\colon \mathcal{C} \to \mathcal{C}$ is lower adjoint to $\phi^* \circ F_\nu\colon \mathcal{C} \to \mathcal{C}$, so it preserves all sups and gives the desired result. □

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Remark: Notice that, in the case of $\nu \in Prob(\mathcal{C})$, we have proved more than claimed, since ξ_ν preserves all suprema, and so it is Scott continuous, rather than being just measurable.