Cantor Groups, Haar Measure and Lebesgue Measure on $[0, 1]$

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Lebesgue Measure and Unit Interval

- $[0, 1] \subseteq \mathbb{R}$ inherits *Lebesgue measure*: $\lambda([a, b]) = b - a$.
- *Translation invariance*: $\lambda(A + x) = \lambda(A)$ for all (Borel) measurable $A \subseteq \mathbb{R}$ and all $x \in \mathbb{R}$. 
Lebesgue Measure and Unit Interval

- [0, 1] ⊆ R inherits Lebesgue measure: \( \lambda([a, b]) = b - a \).
- Translation invariance: \( \lambda(A + x) = \lambda(A) \) for all (Borel) measurable \( A \subseteq \mathbb{R} \) and all \( x \in \mathbb{R} \).
- Theorem (Haar, 1933) Every locally compact group \( G \) has a unique (up to scalar constant) left-translation invariant regular Borel measure \( \mu_G \) called Haar measure.
  
  If \( G \) is compact, then \( \mu_G(G) = 1 \).

  Example: \( \mathbb{T} \simeq \mathbb{R}/\mathbb{Z} \) with quotient measure from \( \lambda \).

  If \( G \) is finite, then \( \mu_G \) is normalized counting measure.
The Cantor Set

\[ C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n = \bigcap_n C_n \subseteq [0, 1] \text{ compact } 0\text{-dimensional, } \lambda(C) = 0. \]

**Theorem:** \( C \) is the unique compact Hausdorff 0-dimensional second countable perfect space.
Cantor Groups

- **Canonical Cantor group:**
  \( \mathcal{C} \cong \mathbb{Z}_2^\mathbb{N} \) is a compact group in the product topology.

\( \mu_\mathcal{C} \) is the product measure \( (\mu_{\mathbb{Z}_2}(\mathbb{Z}_2) = 1) \)
**Cantor Groups**

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**Theorem:** (Schmidt) The Cantor map \( \mathcal{C} \rightarrow [0, 1] \) sends Haar measure on \( \mathcal{C} = \mathbb{Z}_2^\mathbb{N} \) to Lebesgue measure.

**Goal:** Generalize this to all group structures on \( \mathcal{C} \).
Cantor Groups

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*Definition:* A Cantor group is a compact 0-dimensional second countable perfect space endowed with a topological group structure.

- \( G = \prod_{n>1} \mathbb{Z}_n \) is also a Cantor group.
  \( \mu_G \) is the product measure \((\mu_{\mathbb{Z}_n}(\mathbb{Z}_n) = 1)\)

- \( \mathbb{Z}_{p^\infty} = \lim_{\leftarrow} \mathbb{Z}_{p^n} – p\)-adic integers.

- \( H = \prod_n S(n) \), where each \( S(n) \) is the symmetric group on \( n \) letters.
Two Theorems and a Corollary

- **Theorem:** If $G$ is a compact 0-dimensional group, then $G$ has a neighborhood basis at the identity of clopen normal subgroups.

- **Proof:**
  1. $G$ is a Stone space, so there is a basis $\mathcal{O}$ of clopen neighborhoods of $e$.
     If $O \in \mathcal{O}$, then $e \cdot O = O \Rightarrow (\exists U \in \mathcal{O}) U \cdot O \subseteq O$
     $U \subseteq O \Rightarrow U^2 \subseteq U \cdot O \subseteq O$. So $U^n \subseteq O$.
     Assuming $U = U^{-1}$, the subgroup $H = \bigcup_n U^n \subseteq O$.
  2. Given $H < G$ clopen, $\mathcal{H} = \{xHx^{-1} | x \in G\}$ is compact.
     $G \times \mathcal{H} \to \mathcal{H}$ by $(x, K) \mapsto xKx^{-1}$ is continuous.
     $K = \{x | xHx^{-1} = H\}$ is clopen since $H$ is, so $G/K$ is finite.
     Then $|G/K| = |\mathcal{H}|$ is finite, so $L = \bigcap_{x \in G} xHx^{-1} \subseteq H$ is clopen and normal.
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- **Theorem:** If $G$ is a compact 0-dimensional group, then $G$ has a neighborhood basis at the identity of clopen normal subgroups.

- **Corollary:** If $G$ is a Cantor group, then $G \simeq \lim_{\leftarrow n} G_n$ with $G_n$ finite for each $n$. 
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▶ **Theorem:** If $G$ is a compact 0-dimensional group, then $G$ has a neighborhood basis at the identity of clopen normal subgroups.

▶ **Corollary:** If $G$ is a Cantor group, then $G \cong \lim_{\leftarrow n} G_n$ with $G_n$ finite for each $n$.

▶ **Theorem:** (Fedorchuk, 1991) If $X \cong \lim_{\leftarrow i \in I} X_i$ is a strict projective limit of compact spaces, then $\text{Prob}(X) \cong \lim_{\leftarrow i \in I} \text{Prob}(X_i)$. 
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- **Lemma:** If $\varphi : G \to H$ is a surmorphism of compact groups, then $\text{Prob}(\varphi)(\mu_G) = \mu_H$. 


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▶ **Theorem:** (Fedorchuk, 1991) If $X \simeq \lim_{\leftarrow i \in I} X_i$ is a strict projective limit of compact spaces, then $\text{Prob}(X) \simeq \lim_{\leftarrow i \in I} \text{Prob}(X_i)$.

In particular, if $X = G, X_i = G_i$ are compact groups, then $\mu_G = \lim_{i \in I} \mu_{G_i}$.
Two Theorems and a Corollary

- **Theorem**: If $G$ is a compact 0-dimensional group, then $G$ has a neighborhood basis at the identity of clopen normal subgroups.

- **Corollary**: If $G$ is a Cantor group, then $G \simeq \lim_{\leftarrow} G_n$ with $G_n$ finite for each $n$. Moreover, $\mu_G = \lim_n \mu_n$, where $\mu_n$ is normalized counting measure on $G_n$. 
It’s all about Abelian Groups

**Theorem:** If $G = \lim_{\leftarrow n} G_n$ is a Cantor group, there is a sequence $(\mathbb{Z}_{k_i})_{i>0}$ of cyclic groups so that $H = \lim_{\leftarrow n} (\bigoplus_{i \leq n} \mathbb{Z}_{k_i})$ has the same Haar measure as $G$.

**Proof:** Let $G \simeq \lim_{\leftarrow n} G_n, \ |G_n| < \infty$.

Assume $|H_n| = |G_n|$ with $H_n$ abelian.

If $\phi_n : G_{n+1} \to G_n$ is surjective, then $G_{n+1}/\ker \phi_n \simeq G_n$. So $|G_{n+1}| = |G_n| \times |\ker \phi_n|$.

Define $H_{n+1} = H_n \times \mathbb{Z}_{|\ker \phi_n|}$. Then $|H_{n+1}| = |G_{n+1}|$, so $\mu_{H_n} = \mu_n = \mu_{G_n}$ for each $n$, and $H = \lim_{\leftarrow n} H_n$ is abelian.

Hence $\mu_H = \lim_n \mu_n = \mu_G$. 
Combining Domain Theory and Group Theory

$C = \varprojlim_n H_n$, $H_n = \bigoplus_{i \leq n} \mathbb{Z}_{k_i}$

Endow $H_n$ with lexicographic order for each $n$; then

$\pi_n : H_{n+1} \rightarrow H_n$ by $\pi_n(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ &

$\iota_n : H_n \hookrightarrow H_{n+1}$ by $\iota_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$ form embedding-projection pair.
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form embedding-projection pair.

\( C \cong \text{bilim} (H_n, \pi_n, \iota_n) \) is bialgebraic chain:

- \( C \) totally ordered, has all sups and infs
- \( K(C) = \bigcup_n \{ (x_1, \ldots, x_n, 0, \ldots) \mid (x_1, \ldots, x_n) \in H_n \} \)
- \( K(C^{op}) = \{ \sup (\downarrow k \setminus \{ k \}) \mid k \in K(C) \} \)
Combining Domain Theory and Group Theory

\[ C = \lim \downarrow_n H_n, \ H_n = \bigoplus_{i \leq n} \mathbb{Z}_{k_i} \]

Endow \( H_n \) with \textit{lexicographic order} for each \( n \); then

\[ \pi_n: H_{n+1} \to H_n \text{ by } \pi_n(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n) \]

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\( C \cong \text{bilim} (H_n, \pi_n, \iota_n) \) is bialgebraic chain:

\[ \varphi: K(C) \to [0, 1] \text{ by } \varphi(x_1, \ldots, x_n, 0, 0, \ldots) = \sum_{i \leq n} \frac{x_i}{k_1 \cdots k_i} \text{ strictly monotone} \]

induces \( \hat{\varphi}: C \to [0, 1] \) monotone, Lawson continuous.

Direct calculation shows:

\[ \mu_C(\hat{\varphi}^{-1}(a, b)) = \lambda((a, b)) \text{ for } a \leq b \in [0, 1]; \text{ i.e., } \text{Prob}(\hat{\varphi})(\mu_C) = \lambda. \]
Alternative Proof

1. Cantor Tree: \( \mathcal{CT} \simeq \Sigma^\infty = \Sigma^* \cup \Sigma^\omega, \quad \Sigma = \{0, 1\} \)

   \( s \leq t \iff (\exists u) su = t. \) Then \( \text{Max} \mathcal{CT} \simeq \mathcal{C}. \)
Alternative Proof

1. **Cantor Tree**: $\mathcal{C}T \simeq \Sigma^\infty = \Sigma^* \cup \Sigma^\omega$, $\Sigma = \{0, 1\}$

\[ s \leq t \iff (\exists u) su = t. \] Then $\text{Max } \mathcal{C}T \simeq \mathcal{C}$.

2. **Interval domain**: $\text{Int}([0, 1]) = (\{[a, b] \mid 0 \leq a \leq b \leq 1\}, \subseteq)$

\[ \hat{\phi} : \mathcal{C} \rightarrow [0, 1] \] extends to $\Phi : \mathcal{C}T \rightarrow \text{Int}([0, 1])$ Scott continuous.

Then $\text{Prob}(\Phi) : \text{Prob}(\mathcal{C}T) \rightarrow \text{Prob}(\text{Int}([0, 1]))$, so

\[ \lambda = \text{Prob}(\Phi)(\mu_{\mathcal{C}}) = \lim \text{Prob}(\Phi)(\mu_n), \] where

\[ \text{Prob}(\Phi)(\mu_n) = \sum_{1 \leq i \leq 2^n} \frac{1}{2^n} \cdot \delta_{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]} \]
Skorohod’s Theorem

**Theorem:** (Skorohod) If \( \nu \) is a Borel probability measure on a Polish space \( X \), then there is a measurable map \( \xi_\nu : [0, 1] \rightarrow X \) satisfying
\[
\text{Prob}(\xi_\nu)(\lambda) = \nu; \text{ i.e., } \nu(A) = \lambda(\xi_\nu^{-1}(A)).
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**Corollary:** If \( \nu \) is a Borel probability measure on a Polish space \( X \), then there is a measurable map \( \xi_\nu : C \rightarrow X \) satisfying \( \text{Prob}(\xi_\nu)(\mu_C) = \nu \).
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**Proof using Domain Theory:**

1° : \( X \hookrightarrow \overline{X} \subseteq [0, 1]^\mathbb{N} \) & \( X \) is dense \( G_\delta \) in \( \overline{X} \); \( \text{Prob}(X) \hookrightarrow \text{Prob}(\overline{X}) \).
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**Corollary:** If \( \nu \) is a Borel probability measure on a Polish space \( X \), then there is a measurable map \( \xi_\nu : \mathcal{C} \to X \) satisfying \( \text{Prob}(\xi_\nu)(\mu_\mathcal{C}) = \nu \).

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2°: \( \overline{X} \) is a continuous image of \( \mathcal{C} \): \( \exists \psi : \mathcal{C} \to \overline{X} \).
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3°: $\text{Prob}(\psi) : \text{Prob}(C) \rightarrow \text{Prob}(\overline{X})$. 
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4°: So it suffices to show every $\nu \in \text{Prob}(C)$ satisfies the Corollary.
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$\nu \in \text{Prob}(C)$ qua valuation on $\Sigma(C)$ preserves all suprema, so $F_\nu: C \to [0, 1]$ by $F_\nu(x) = \nu(\downarrow x)$ preserves all infima -- $F_\nu$ is the ‘cdf’ of $\nu$. 
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\( F_\nu \) has a lower adjoint \( \chi_\nu : [0, 1] \to C \) preserving all suprema;
\( \chi_\nu(r) \leq x \text{ iff } r \leq F_\nu(x) \).
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\( \lambda(\{ r \mid \chi_\nu(r) \leq x \}) = \lambda(\{ r \mid r \leq F_\nu(x) \}) = \lambda([0, F_\nu(x)]) = F_\nu(x) \).

Now \( \phi : C \to [0, 1] \) has an upper adjoint \( \phi^* : [0, 1] \to C \).
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**Theorem:** (Skorohod) If $\nu$ is a Borel probability measure on a Polish space $X$, then there is a measurable map $\xi_\nu : [0, 1] \to X$ satisfying $\text{Prob}(\xi_\nu)(\lambda) = \nu$; i.e., $\nu(A) = \lambda(\xi_\nu^{-1}(A))$.

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$F_\nu$ has a lower adjoint $\chi_\nu : [0, 1] \to C$ preserving all suprema; $\chi_\nu(r) \leq x$ iff $r \leq F_\nu(x)$. So

$$\lambda(\{r \mid \chi_\nu(r) \leq x\}) = \lambda(\{r \mid r \leq F_\nu(x)\}) = \lambda([0, F_\nu(x)]) = F_\nu(x).$$

Now $\phi : C \to [0, 1]$ has an upper adjoint $\phi^* : [0, 1] \to C$.

Then $\xi_\nu = \phi \circ \chi_\nu : C \to C$ is lower adjoint to $\phi^* \circ F_\nu : C \to C$, so it preserves all sups and gives the desired result. \qed
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**Theorem:** (Skorohod) If $\nu$ is a Borel probability measure on a Polish space $X$, then there is a measurable map $\xi_\nu : [0, 1] \to X$ satisfying $\text{Prob}(\xi_\nu)(\lambda) = \nu$; i.e., $\nu(A) = \lambda(\xi_\nu^{-1}(A))$.

**Corollary:** If $\nu$ is a Borel probability measure on a Polish space $X$, then there is a measurable map $\xi_\nu : C \to X$ satisfying $\text{Prob}(\xi_\nu)(\mu_C) = \nu$.

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Then $\xi_\nu = \phi \circ \chi_\nu : C \to C$ is lower adjoint to $\phi^* \circ F_\nu : C \to C$, so it preserves all sups and gives the desired result. \(\square\)

**Remark:** Notice that, in the case of $\nu \in \text{Prob}(C)$, we have proved more than claimed, since $\xi_\nu$ preserves all suprema, and so it is Scott continuous, rather than being just measurable.