Random Bits of Noise

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Channels

- Mechanism for communication
  - *Overt:* used as intended to exchange information
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  Note: We assume C is lossless: $\sum_i P(y_j|x_i) = 1$ $(\forall i)$.
- For $p = (p_0\ p_1\ \ldots\ p_{m-1})$ probability distribution on X we get

$$p \cdot M_C = \left(\sum_i p_i P(y_0|x_i)\ \ \sum_i p_i P(y_1|x_i)\ \ldots\ \sum_i p_i P(y_{n-1}|x_i)\right)$$

  corresponding distribution on Y.

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More generally,

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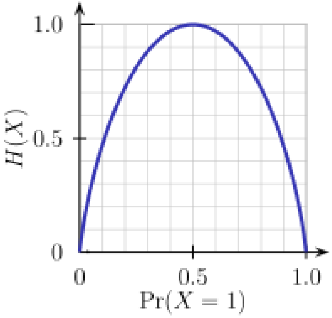
Entropy function

$H(X) = -\sum_i p(s_i) \log_2 p(s_i)$

Average information in X

Binary case:
$H(X) = -\sum_{i=0}^{1} p(s_i) \log_2 p(s_i)$



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  $p(X,Y)$ – joint probability distribution on $X \times Y$

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Conditional Entropy

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  $= -\sum_j p(s_j)H(X|Y=s_j)$

Mutual Information

- $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$
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Channel Capacity

- $Cap = \sup_Y I(X;Y)$
  $= \sup_X H(Y) - H(Y|X)$

From Noise Matrices to Random Variables

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  - $X = p = [x \;\; 1-x]$ - input distribution
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Capacity of Channel with Noise Matrix M_C

- $$Cap(M) = \sup_p \; H(p \cdot M) - H(p \cdot M \mid p)$$
  $$= \sup_x \; H(xa + (1-x)b) - (xH(a) + (1-x)H(b))$$
  $$(p = [x \quad 1-x]; \;\; H(z) \equiv H([z \;\; 1-z])$$

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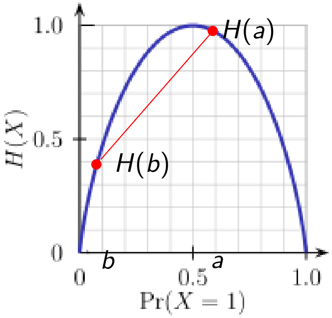

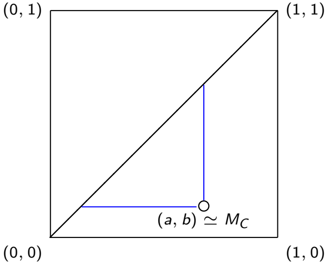

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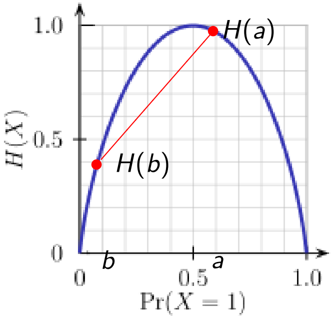

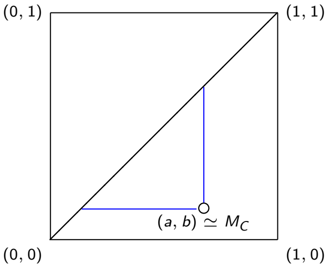

Mean Value Theorem $\Longrightarrow$

$$Cap(M_C) = H(x_0 a + (1-x_0)b) - (x_0 H(a) + (1-x_0)H(b))$$

where $H'(x_0) = \frac{H(a)-H(b)}{a-b}$.

$m \times n$-matrices

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$Cap(M_C) = H(p_0 \cdot M_C) - p_0 \cdot \langle H(M_C) \rangle$, where $\nabla H(p_0 \cdot M_C) = \overrightarrow{n}_P$

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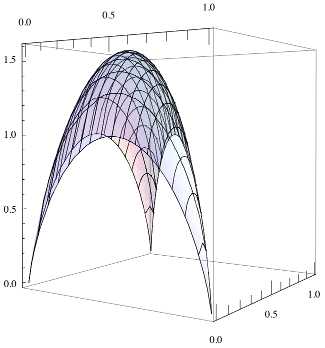

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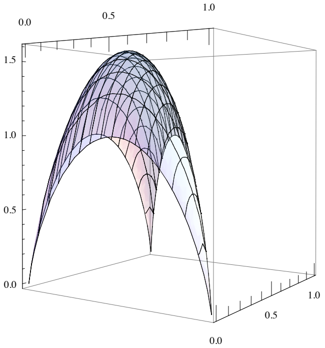

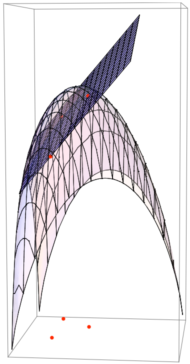

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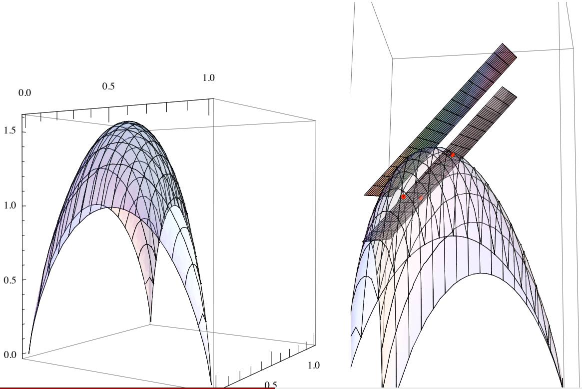

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- *Question:* What is the *optimal* map $F \colon (\mathcal{C}([0,1]^{m(n-1)}), \supseteq) \to \mathbb{R}^{\mathrm{op}}_{\geq 0}$?

  $F(A) \leq F(A') \Leftrightarrow Cap'(A) \leq Cap'(A')$

m-dimensional stochastic matrices

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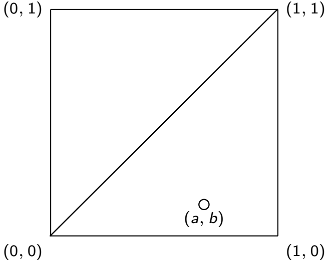
- $Cap(M) = Cap(N)$ if $M(\mathbb{ST}_m)M = M(\mathbb{ST}_m)N \;\Leftrightarrow\; \phi(M) = \phi(N)$.

Binary Noise Matrices - Martin, Moskowitz & Allwein



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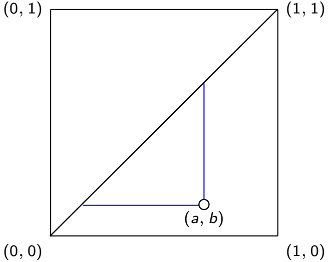

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$- - \equiv (\begin{smallmatrix} a \\ b \end{smallmatrix}) \leq (\begin{smallmatrix} c \\ d \end{smallmatrix}) \;\Leftrightarrow\; (\begin{smallmatrix} c \\ d \end{smallmatrix}) \in N(\begin{smallmatrix} a \\ b \end{smallmatrix})$

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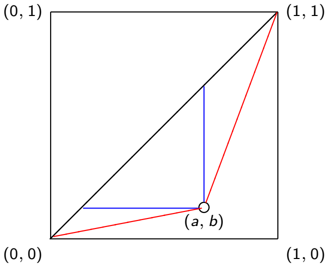

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– – $\equiv \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \Leftrightarrow \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \in N \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$

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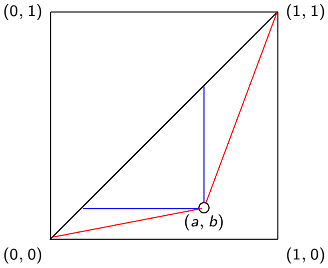
Binary Noise Matrices - Martin, Moskowitz & Allwein



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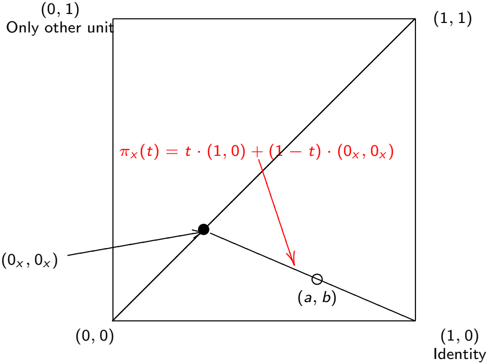
$$\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} c \\ c \end{smallmatrix}\right) = \begin{pmatrix} a(c-c)+c \\ b(c-c)+c \end{pmatrix} = \left(\begin{smallmatrix} c \\ c \end{smallmatrix}\right)$$

$$(\forall M)(\exists M' \in \mathcal{N}) \; Cap(M) = Cap(M').$$

Binary Noise Matrices - Martin, Moskowitz & Allwein

$$x = \binom{a}{b} \quad 0_x = \frac{b}{1-\det x} \quad \binom{0_x}{0_x} = \lim_n x^n$$



One-parameter Semigroup

A *one-parameter semigroup* is a monoid homomorphism $\phi\colon ([0,1),+) \to S$ into a monoid S.

Generalizing to higher dimensions

One parameter semigroups

S compact, affine, $e = e^2, f = f^2 \in S$, $ef = fe = e$ implies $\{(1-\lambda)f + \lambda e \mid 0 \leq \lambda \leq 1\}$ is a one-parameter semigroup from f to e.

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Example. But:

$$\left\{\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\lambda)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mid 0 \leq \lambda \leq 1\right\}$$

goes through

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

Generating $\mathbb{ST}_m$

The Group of Units

- A monoid S has an identity – associated to it is the *group of units* $G(S)$ of the monoid.
- $G(S)$ - compact group if S compact.
- $G(\mathbb{ST}_m) = S_m$ – symmetric group on m letters.

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First approximate:

$\langle G(\mathbb{ST}_m), M(\mathbb{ST}_m)\rangle$ monoid generated by group of units and minimal ideal.

- Compact affine monoid.
- Every element in $\langle Id_m, M(\mathbb{ST}_m)\rangle$ lies on a unique one-parameter semigroup – all straight lines.
- $\cup_{g\in S_m}\langle g, M(\mathbb{ST}_m)\rangle / \equiv_{Cap}$ identifies each element with some element in $M(\mathbb{ST}_m)$; i.e, with some compact, convex subset of $[0,1]^{m(m-1)}$.

A smaller submonoid:

Doubly Stochastic Matrices

$M \in \mathbb{ST}_m$ is *doubly stochastic* if each column also sums to 1.

$\mathbb{DST}_m$ – doubly stochastic $m \times m$ matrices.

- Again a monoid with same group of units.
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- *NOT* a group!!

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Theorem: Any compact affine group is a point.

Proof: [Hofmis] Suppose G is such a group. Let $x \in G$ have order n.

Then $\sum_{i \leq n} \frac{x^i}{n} = (\sum_{i \leq n} \frac{x^i}{n})^2$ by Fubini.

So, $\sum_{i \leq n} \frac{x^1}{n} = e$; then $x^i = e$ as e is extreme. □

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Structure of $\langle \mathbb{DST}_m \rangle$: Compact monoid; $G(\langle \mathbb{DST}_m \rangle) = S_m$;
$M(\langle \mathbb{DST}_m \rangle) = \left\{ \frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \right\}$. Each element of S_3 lies on a line to $\frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$.

Universal affine semigroups:

Generalizing to Stochastic Relations

$M_C \colon X \to Y$ generalizes to $f \colon X \to \mathrm{Prob}(Y)$.

For Y compact, T_2, so is $\mathrm{Prob}(Y)$ in weak*-topology.

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Here's an explanation:

$$X \mapsto C(X,\mathbb{R}) \colon \mathsf{Comp} \to \mathsf{Ban}$$

is contravariant: $f \colon X \to Y \;\Rightarrow\; C(f) \colon C(Y,\mathbb{R}) \to C(X,\mathbb{R})$

$$C(X,\mathbb{R}) \mapsto C(C(X,\mathbb{R}),\mathbb{R}) \colon \mathsf{Ban} \to \mathsf{Ban}.$$

But $C^2(X,\mathbb{R}) \simeq \mathsf{Meas}(X,\mathbb{R})$. Extract $\mathrm{Prob}(X)$ by restriction.

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If $(S, \cdot)$ is a compact semigroup, then

$$\cdot : S \times S \to S \;\Rightarrow\; * : \mathrm{Prob}(S) \times \mathrm{Prob}(S) \to \mathrm{Prob}(S)$$

by $(\mu * \nu)(A) = (\mu \times \nu)(\{(x, y) \in S \times S \mid x \cdot y \in A\})$.

Then $(\mathrm{Prob}(S), *)$ is a compact semigroup.

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Moreover, $x \mapsto \delta_x\colon S \to \mathrm{Prob}(S)$ sends S into the set of extreme points of $\mathrm{Prob}(S)$.

If S is a compact monoid, then $\langle\{\delta_g \mid g \in G(S)\}\rangle$ corresponds to the doubly stochastic matrices.

For G a compact group, $M(\langle\{\delta_g \mid g \in G\}\rangle) = \{\mu_G\}$ – Haar measure on G.