Local DCPOs, Local CPOs and Local Completions

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EXTENDED ABSTRACT

Abstract

We use a subfamily of the Scott-closed sets of a poset to form a *local completion* of the poset. This is simultaneously a topological analogue of the ideal completion of a poset and a generalization of the sobrification of a topological space. After we show that our construction is the object level of a left adjoint to the forgetful functor from the category of local cpos to the category of posets and Scott-continuous maps, we use this completion to show how local domains can play a role in the study of domain-theoretic models of topological spaces. Our main result shows that any topological space that is homeomorphic to the maximal elements of a continuous poset that is weak at the top also is homeomorphic to the maximal elements of a bounded complete local domain. The advantage is that continuous maps between such spaces extend to Scott-continuous maps between the modeling local domains.

1 Introduction

The Scott topology is of fundamental importance in domain theory. It lies at the heart of the structure of domains, and of how to define the appropriate morphisms between domains: they are exactly those maps that are continuous with respect to the Scott topology. Even more, the family of Scott closed sets is one representation of the lower or Hoare power domain of a domain. Another application using Scott-closed sets was first revealed in [9], where it was shown that the sobrification of an algebraic poset in its Scott topology is an algebraic dcpo with the same set of compact elements. This result was extended to continuous posets in [12]. The sobrification of any topological

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space is just the set of irreducible closed subsets of the space, and any T_0-space is homeomorphic to its image in its sobrification, and that image is dense in the sobrification. One way to view these results is that the sobrification of a continuous poset completes the poset into a continuous dcpo whose way-below relation – and hence Scott topology – is completely determined by the original poset; this provides a topological generalization of the ideal completion of a poset into an algebraic dcpo.

In this paper we find another use for the family of Scott-closed sets. Here we show how a generalization of the sobrification of a continuous poset forms a *local cpo* which again contains the original poset densely. In fact, this is the object-level of the left adjoint to the forgetful functor from the category of local cpos and Scott continuous maps to the category of posets and Scott continuous maps. This local cpo – the *local completion* of the underlying poset – is simply the closure of the image of the poset within the family of bounded, irreducible Scott-closed sets, where a subset is *bounded* if it has an upper bound in the underlying poset. The sub-cpo of the family of closed sets generated by the family of bounded Scott-closed sets of a continuous cpo was first investigated in [11], where it was shown to form the object level of a left adjoint to the forgetful functor from a subcategory of Scott domains to the category of SFP objects and Scott-continuous maps.

After we establish the adjunction between the categories of local dcpos and posets, we turn our attention to using our local completion to model topological spaces. This line of inquiry has its roots in the work of Edalat [2–4] who found numerous applications of domain theory to areas beyond programming semantics. A recurring theme of Edalat's work has been the question of which topological spaces can be represented as the family of maximal elements of a domain. Previous results have shown that an important property for such models is that the inherited Scott topology on the set of maximal elements agrees with the inherited weak-dual topology on the maximal elements; we call continuous posets that satisfy this property *weak at the top*. In his seminal paper [7], Lawson characterized the spaces that arise as the maximal elements of ω-continuous dcpos which are weak at the top as being exactly the Polish spaces. Lawson also showed that domains that are weak at the top are exactly those for which the mapping into the family of Scott closed sets preserves maximal elements. We extend this result to the case of continuous posets [2], and then use it to show that any topological space that is homeomorphic to the maximal elements of a continuous poset that is weak at the top also can be represented as the space of maximal elements of a local domain that is bounded complete. This applies in particular to both Lawson's model of Polish spaces and to the formal ball model of Edalat and Heckmann [5]. We also show how continuous mappings between spaces which can be so

[2] Unfortunately, our terminology, taken from [9], differs from that of [7]. For us, a continuous poset need not be directed complete, while the posets referred to in [7] are what now are commonly called *continuous dcpos*.

modeled naturally extend to their local domain models.

2 Local dcpos and local cpos

We begin by recalling some basics about partial orders. If P is a partially ordered set, then a subset $D \subseteq P$ is *directed* if each finite subset of D has an upper bound in D. P is a *directed complete partial order (dcpo)* if each directed subset of P has a least upper bound in P. P is a *local dcpo* if each directed subset of P with an upper bound has a least upper bound; it is easy to see that this implies that $\downarrow x = \{y \in P \mid y \sqsubseteq x\}$ is a dcpo for each $x \in P$, but the converse may fail (cf. [9]). A *local cpo* is a local dcpo which also has a least element.

If P is a partial order and $x, y \in P$, then we write $x \ll y$ if for each directed subset $D \subseteq P$ for which $\sqcup D$ exists, if $y \sqsubseteq \sqcup D$, then there is some $d \in D$ with $x \sqsubseteq d$. We let $\Downarrow y = \{x \in P \mid x \ll y\}$, and we say P is a *continuous poset* if $\Downarrow y$ is directed and $y = \sqcup \Downarrow y$ for each $y \in P$. A *domain* is a continuous cpo, while a *local domain* is a continuous local cpo. An element $x \in P$ is *compact* if $x \ll x$; we let $K(P)$ denote the set of compact elements of P, and we say P is an *algebraic poset* if $K(y) = \downarrow y \cap K(P)$ is directed and $y = \sqcup K(y)$ for each $y \in P$.

A mapping between partial orders is *Scott continuous* if it is monotone and preserves suprema of directed sets. More precisely, if $f \colon P \to Q$ is a monotone map between partial orders P and Q, then f is *Scott continuous* if for each directed subset $D \subseteq P$ for which $\sqcup D$ exists in P, then $f(D)$ has a supremum in Q and $f(\sqcup D) = \sqcup f(D)$. Equivalently, we can define a *Scott-closed* subset of a partial order to be a lower set which is closed under (existing) suprema of directed subsets. It is easy to show that the intersection of any family of such sets is another such, as is the finite union, so the family $\Gamma(P)$ of such sets forms the closed sets of a topology. It also is easy to show that a monotone mapping $f \colon P \to Q$ between partial orders is continuous with respect to this topology if and only i f preserves suprema of those directed subsets of P that have suprema.

The following results are from [9,12]:

Theorem 2.1 *If P is a continuous poset, then the family* $\mathrm{Spec}\Gamma(P)$ *of sup-primes of the Brouwerian lattice* $\Gamma(P)$ *is a continuous dcpo. Moreover for $X, Y \in \mathrm{Spec}\Gamma(P)$, $X \ll Y$ in $\mathrm{Spec}\Gamma(P)$ if and only if there are $x \ll y \in P$ with $X \subseteq \downarrow x \subseteq \downarrow y \subseteq Y$.* □

This result can be raised to the level of an adjunction:

Theorem 2.2 *The forgetful functor from the category of continuous dcpos and Scott-continuous maps to the category of continuous posets and Scott-continuous mappings has a left adjoint. This left adjoint associates to a continuous poset P the family* $\mathrm{Spec}\Gamma(P)$*. This adjunction also restricts to an* *adjunction between the full subcategories of algebraic posets and algebraic dcpos.*

Proof. The previous theorem shows that the set of sup-primes of the Brouwerian lattice $\Gamma(P)$ is a dcpo which is continuous (algebraic) when P is. Since posets are always T_0 spaces in their Scott topology, we know from basic results that the mapping $\eta_P : P \to \mathrm{Spec}\Gamma(P)$ by $\eta_P(x) = \downarrow x = \overline{\{x\}}^{\sigma}$ is a homeomorphism onto its image. Now, each continuous map $f : P \to Q$ between continuous posets satisfies $f^{-1} : \Gamma(Q) \to \Gamma(P)$ preserves all infima and all finite suprema. Hence, f^{-1} has a lower adjoint $\Gamma(f) : \Gamma(P) \to \Gamma(Q)$ which preserves all suprema and all sup-primes. It is routine to show the restriction of this map to the set of sup-primes is then the desired extension of f, and it is uniquely determined since every closed set is the supremum of elements from the image of P in $\mathrm{Spec}\Gamma(P)$. □

2.1 From Posets to Local DCPOs

Our goal in this section is to extend the results at the end of the previous subsection to categories of posets and continuous maps.

Notation: We let

$$\mathrm{B}\Gamma(P) = \{X \in \Gamma(P) \mid (\exists x \in P)\ X \subseteq \downarrow x\}$$

denote the family of *bounded Scott closed sets* in the poset P. We also let

$$\mathrm{BSpec}(P) = \mathrm{Spec}\Gamma(P) \cap \mathrm{B}\Gamma(P)$$

to be the (sup-)prime closed subsets of P which also are bounded.

Lemma 2.3 *If P is a local dcpo, and $Q = \downarrow Q \subseteq P$ is a lower set in P, then the Scott topology that Q inherits from P coincides with the Scott topology defined by the order on Q.*

Proof. If $D \subseteq Q \subseteq P$ is directed, and if $\sqcup_Q D$ and $\sqcup_P D$ both exist, then $\sqcup_P D \sqsubseteq \sqcup_Q D$. But if Q is a lower set, then $\sqcup_P D \in Q$, and so the two must coincide. So, for a lower set $X = \downarrow X \subseteq Q$, X is Scott-closed in Q iff $\sqcup_Q D \in X$ for any $D \subseteq Q$ directed for which $\sqcup_Q D$ exists. But $\sqcup_Q D$ exists implies $\sqcup_P D$ exists since P is a local dcpo, and so $\sqcup_Q D \in X$ iff $\sqcup_P D \in X$ for any such $D \subseteq Q$, iff $X = \overline{X}^{\sigma_P} \cap Q$. Hence the Scott-closed subsets of Q are the intersections of Scott-closed subsets of P with Q, which means the intrinsically defined Scott topology on Q is the one it inherits from P. □

Proposition 2.4 *Let P be a poset. Then*

(i) *The Scott topology on* $\mathrm{B}\Gamma(P)$ *is the inherited Scott topology from* $\Gamma(P)$.

(ii) *The Scott topology on* $\mathrm{BSpec}(P)$ *is the one it inherits from* $\mathrm{Spec}\Gamma(P)$.

Moreover, if P is a continuous poset, then

(iii) *The Scott topology on* $\mathrm{Spec}\Gamma(P)$ *is the one it inherits from* $\Gamma(P)$.

(iv) *The Scott topology on* $\mathrm{BSpec}(P)$ *is the inherited Scott topology from* $\mathrm{B}\Gamma(P)$*, which also coincides with the inherited Scott topology from* $\Gamma(P)$*.*

Proof. Part (i) follows since $\mathrm{B}\Gamma(P) = \downarrow \eta_P(P)$ is a lower set in $\Gamma(P)$, so Lemma 2.3 implies that its inherited Scott topology from $\Gamma(P)$ is the one its order defines.

For part (ii), $\mathrm{BSpec}(P)$ is the intersection of a lower set in $\Gamma(P)$ with $\mathrm{Spec}\Gamma(P)$, and so this is a lower set in $\mathrm{Spec}\Gamma(P)$. But it is well-known that the supremum of a directed set of sup-primes in a complete lattice is another such (cf. [6]), so $\mathrm{Spec}\Gamma(P)$ is a dcpo. Hence Lemma 2.3 once again implies the Scott topology on this subset is the same as the one it inherits from $\mathrm{Spec}\Gamma(P)$.

Now assume that P is a continuous poset. Then it is easy to show that the family

$$\{\downarrow F \mid F \text{ is finite and } \Uparrow x \neq \emptyset \ (\forall x \in F)\}$$

is a basis for $\Gamma(P)$. Theorem 2.1 then implies (iii) holds.

Part (iv) now follows from te previous results: (ii) and (iii) imply the Scott topology on $\mathrm{BSpec}(P)$ is the one it inherits from $\Gamma(P)$, and (i) implies the Scott topology on $\mathrm{B}\Gamma(P)$ is the one it inherits from $\Gamma(P)$, so the Scott topology on $\mathrm{BSpec}(P)$ is the one it inherits from $\mathrm{B}\Gamma(P)$. □

We recall that a poset is *bounded complete* if every subset that has an upper bound has a least upper bound. This definition traditionally applies to dcpos, but the conditional for the existence of suprema seems tailor-made for our setting of local dcpos. In any case, if the poset is also an ω-algebraic cpo, then it is a *Scott domain.*

Theorem 2.5 *Let* P *be a poset endowed with its Scott topology. Then*

(i) $\mathrm{B}\Gamma(P)$ *is a bounded complete local cpo.*

(ii) $\mathrm{BSpec}(P)$ *is a local dcpo, and if* P *has a least element, then it is a local cpo.*

Moreover, if P *is continuous, then so are* $\mathrm{B}\Gamma(P)$ *and* $\mathrm{BSpec}(P)$ *and in this case,*

$$X \ll Y \in \mathrm{BSpec}(P) \quad \text{iff} \quad (\exists x \ll y \in P)\ X \subseteq \downarrow x \subseteq \downarrow y \subseteq Y.$$

Proof. Since the Scott topology on a poset is always T_0, the mapping $\eta_P\colon P \to \Gamma(P)$ by $\eta_P(x) = \downarrow x$ is a homeomorphism and order isomorphism onto its image. Since $\mathrm{B}\Gamma(P)$ is the lower set of this image in $\Gamma(P)$, it follows that $\mathrm{B}\Gamma(P)$ is a bounded complete local cpo: If $\mathcal{X} \subseteq \mathrm{B}\Gamma(P)$ has an upper bound, then $\exists Y \in \mathrm{B}\Gamma(P)$ with $X \subseteq Y$ $(\forall X \in \mathcal{X})$. Then $(\exists x \in P)\, Y \subseteq \downarrow x$, so $X \subseteq \downarrow x$ for all $X \in \mathcal{X}$. It follows that $\sqcup_{\Gamma(P)}\mathcal{X} = \overline{\cup \mathcal{X}}^{\sigma} \subseteq \downarrow x$, so $\sqcup_{\Gamma(P)}\mathcal{X} \in \mathrm{B}\Gamma(P)$. This proves (i).

For any poset P, the family $\mathrm{Spec}\Gamma(P)$ is closed under directed suprema in $\Gamma(P)$, and therefore the supremum in $\mathrm{B}\Gamma(P)$ of a directed subset of $\mathrm{BSpec}(P)$ also lies in $\mathrm{BSpec}(P)$ if it exists in $\mathrm{B}\Gamma(P)$. Since the latter is a local dcpo, it follows that $\mathrm{BSpec}(P)$ is one as well.

It is shown in [12] that $\Gamma(P)$ is a completely distributive (and hence continuous) lattice if P is a continuous poset. Since $\mathrm{B}\Gamma(P)$ is a lower set in $\Gamma(P)$, it follows readily that $\mathrm{B}\Gamma(P)$ is a continuous poset.

Furthermore, we know from Theorem 2.1 that, for a continuous poset P, the dcpo $\mathrm{Spec}\Gamma(P)$ is continuous and satisfies $X \ll Y$ iff $(\exists x \ll y \in P)\, X \subseteq {\downarrow}x \subseteq {\downarrow}y \subseteq Y$. Proposition 2.4 implies the Scott topology on $\mathrm{BSpec}(P)$ is the one it inherits from $\mathrm{Spec}\Gamma(P)$, and since $\eta_P(P) \subseteq \mathrm{BSpec}(P)$, it follows that this same relation holds in $\mathrm{BSpec}(P)$; from this it follows that $\mathrm{BSpec}(P)$ is a continuous poset. $\square$

Each of the structures $\mathrm{BSpec}(P)$ and $\mathrm{B}\Gamma(P)$ forms the object level of an adjunction, as we now demonstrate. We define

- Pos - the category of posets and *Scott continuous* maps.
- $\mathsf{Pos}_\perp$ - the full subcategory of Pos of posets with least element.
- LD - the category of local dcpos and Scott-continuous maps.
- LC - the full subcategory of LD of local cpos.
- BCLC - the category of bounded complete local cpos and maps preserving all *existing* suprema.

Theorem 2.6

(i) *The object map* $P \mapsto \mathrm{BSpec}(P)$ *extends to a functor* $\mathrm{BSpec}\colon \mathsf{Pos} \to \mathsf{LD}$ *which is a left adjoint to the forgetful functor* $\mathsf{LD} \to \mathsf{Pos}$.

(ii) *The functor from* (i) *restricts to a left adjoint* $\mathrm{BSpec}_\perp\colon \mathsf{Pos}_\perp \to \mathsf{LC}$ *to the forgetful functor* $\mathsf{LC} \to \mathsf{Pos}_\perp$.

(iii) *The object map* $P \mapsto \mathrm{B}\Gamma(P)$ *extends to a functor* $\mathrm{B}\Gamma\colon \mathsf{Pos}_\perp \to \mathsf{BCLC}$ *which is a left adjoint to the forgetful functor* $\mathsf{BCLC} \to \mathsf{Pos}_\perp$.

Proof. Any continuous mapping $f\colon X \to Y$ between topological spaces gives rise to a mapping $f^{-1}\colon \Gamma(Y) \to \Gamma(X)$ preserving all infima and all finite suprema. Then f^{-1} has a lower adjoint $\Gamma(f)^*\colon \Gamma(X) \to \Gamma(Y)$ by $\Gamma(f)^*(X) = \overline{f(X)}$, which preserves all suprema and all sup-primes. If $X = P$ is a poset equipped with its Scott topology, then the fact that $\Gamma(f)^*$ preserves all suprema means $\Gamma(f)^*$ is continuous with respect to the Scott topology on $\Gamma(P)$, and since $\Gamma(f)^*$ preserves all sup-primes, we conclude that $\Gamma(f)^*$ is the unique such mapping, since any closed set is the supremum of the sets ${\downarrow}x = \overline{x}^\sigma$ that it contains.

Now, since $\mathrm{BSpec}(P)$ is a local cpo which is dense in $\mathrm{Spec}\Gamma(P)$, we can simply restrict the functor Γ^* to a functor $\mathrm{BSpec}\colon \mathsf{Pos} \to \mathsf{LD}$. Clearly we can further restrict these functors to the subcategory $\mathsf{Pos}_\perp$ on the poset side, and to LC on the local cpo side.

Finally, the family $\mathrm{B}\Gamma(P)$ of bounded Scott-closed sets is a bounded complete local cpo, and we can just as well restrict the upper adjoint $\Gamma(f)^*\colon \Gamma(P) \to \Gamma(Q)$ to this domain. Since $\Gamma(f)^* \circ \eta_P = \eta_Q \circ f$ for any continuous map $f\colon P \to$

Q, and since $\Gamma(f)^*$ is order-preserving, it follows easily that $\Gamma(f)^*(\mathrm{B}\Gamma(P)) \subseteq \mathrm{B}\Gamma(Q)$. Moreover, $\mathrm{B}\Gamma(P)$ is closed in $\Gamma(P)$ under all existing suprema and all infima, so Γ^* restricts to a functor $\mathrm{B}\Gamma\colon \mathsf{Pos}_\perp \to \mathsf{BCLC}$. This is left adjoint to the forgetful functor because all closed sets are the suprema of the point-closures they contain. □

2.2 From Local DCPOs to CPOs

Our next goal is to relate local dcpos to the better-known directed complete partial orders.

Theorem 2.7

(i) *There is a left adjoint* $\mathrm{BSpec}^*\colon \mathsf{LD} \to \mathsf{DCPO}$ *to the forgetful functor from the category* DCPO *of dcpos and continuous maps to the category* LD *of local dcpos and Scott continuous maps.*

(ii) *The functor from* (i) *restricts to a left adjoint* $\mathrm{BSpec}^*_\perp\colon \mathsf{LC} \to \mathsf{CPO}$.

(iii) *There is a left adjoint* $\mathrm{B}\Gamma^*\colon \mathsf{LC} \to \mathsf{BCPO}$ *to the forgetful functor from the category of bounded complete cpos and maps preserving all existing suprema to the category of local cpos and Scott-continuous maps.*

Proof. If P is a local dcpo, then the family $\mathrm{Spec}\Gamma(P)$ of sup-primes in the Brouwerian lattice of Scott-closed subsets of P is a dcpo, as is true of the sup-primes of any complete distributive lattice. Moreover, the same arguments given in the proofs of Theorem 2.6 show that any Scott-continuous mapping $f\colon P \to Q$ between local dcpos extends uniquely to a Scott-continuous mapping $\mathrm{BSpec}^*(f)\colon \mathrm{Spec}\Gamma(P) \to \mathrm{Spec}\Gamma(Q)$. This is all we need to guarantee that BSpec^* is left adjoint to the forgetful functor. This proves (i), and (ii) is a simple corollary.

For (iii), we note that the family $\mathrm{Spec}\Gamma(P)$ of sup-primes of a local cpo is a subcpo of $\Gamma(P)$, and so its lower set,

$$\mathrm{B}\Gamma^*(P) = \downarrow \mathrm{Spec}\Gamma(P) = \{X \in \Gamma(P) \mid (\exists Y \in \mathrm{Spec}\Gamma(P))\ X \subseteq Y\}$$

also is a cpo, and it is this set which forms the object level of our left adjoint. Moreover, this lower set is closed under all non-empty infima, and so it is a bounded complete cpo. Once again, if $f\colon P \to Q$ is a Scott-continuous mapping between local cpos, then the mapping $\Gamma(f)^*\colon \Gamma(P) \to \Gamma(Q)$ preserves all suprema and all sup-primes. This means $\Gamma(f)^*(\mathrm{Spec}\Gamma(P)) \subseteq \mathrm{Spec}\Gamma(Q)$, and so $\Gamma(f)^*(X) \in \downarrow_{\Gamma(Q)} \mathrm{Spec}\Gamma(Q)$ for any $X \in \downarrow_{\Gamma(P)} \mathrm{Spec}\Gamma(P)$. That is, $\Gamma(f)^*(\mathrm{B}\Gamma^*(P)) \subseteq \mathrm{B}\Gamma^*(Q)$, and since the mapping $\mathrm{B}\Gamma^*(f) = \Gamma(f)^*|_{\mathrm{B}\Gamma^*(P)}$ preserves existing suprema, it is the mapping we seek. Finally, this is the unique mapping extending $f\colon P \to Q$ since since any closed set is the supremum of the point closures it contains. □

Of course, whenever one talks about categories of cpos, then fixed points of continuous selfmaps immediately spring to mind, as do questions of domain constructors and cartesian closed categories. We now consider these issues.

Theorem 2.8 *The categories* LD *of local dcpos and* LC *of local cpos are closed under the following:*

- *Products.*
- *Sums.*
- *Lift.*
- *Continuous function spaces.*

In particular, since the one-point local cpo is a terminal object in both these categories, both of them are cartesian closed.

Proof. Only the closure under function space merits comment. If P and Q are local dcpos, then the space $[P \to Q]$ of Scott-continuous maps is partially ordered under the pointwise ordering. Suppose $F \subseteq [P \to Q]$ is a directed family of continuous functions and $f \sqsubseteq g$ for all $f \in F$, for some $g \in [P \to Q]$. For any $x \in P$, $\{f(x) \mid f \in F\} \subseteq \downarrow g(x)$, and since Q is a local dcpo, $\downarrow g(x)$ is a dcpo, so there is a supremum $f'(x) := \sqcup_Q \{f(x) \mid f \in F\}$. Hence $f' \colon P \to Q$ with $f' = \sqcup F$ is well-defined.

The proof that f' is continuous is the same as for continuous maps on dcpos: Given any directed subset $D \subseteq P$ for which $\sqcup_P D$ exists, then $g(\sqcup_P D) = \sqcup_Q g(D)$. Then $g|_{\downarrow \sqcup_P D} \colon \downarrow \sqcup_P D \to \downarrow \sqcup_Q g(D)$ is continuous, and since $f \sqsubseteq g$ for all $f \in F$, it follows that each mapping $f|_{\downarrow \sqcup_P D} \colon \downarrow \sqcup_P D \to \downarrow \sqcup_Q g(D)$ is continuous. Since P and Q are local dcpos, both $\downarrow \sqcup_P D$ and $\downarrow \sqcup_Q g(D)$ are dcpos, and so the directed family of $\{f|_{\downarrow \sqcup_P D} \mid f \in F\}$ has least upper bound $f'|_{\downarrow \sqcup_P D}$ which is continuous. But $f'(x) = \sqcup_Q \{f(x) \mid f \in F\}$ for each $x \in \downarrow \sqcup_P D$, and so $f'(\sqcup_P D) = \sqcup_Q f'(D)$, so $f' = \sqcup F \colon P \to Q$ is continuous.

The remarks about cartesian closure now readily follow – in fact, they also can be gleaned from the fact that we are working within subcategories of the category of posets and monotone maps, which itself is cartesian closed. □

Proposition 2.9 *A continuous selfmap $f \colon P \to P$ of a local cpo has a least fixed point if and only if there is a subcpo $Q \subseteq P$ and a continuous selfmap $g \colon Q \to Q$ such that $f(x) \sqsubseteq g(x)$ for all $x \in Q$.*

Proof. If $f \colon P \to P$ has a fixed point x, then $f(y) \sqsubseteq f(x) = x$ for any $y \in \downarrow x$. Hence $\downarrow x$ is a subcpo of P and $f|_{\downarrow x} \colon \downarrow x \to \downarrow x$ dominates f on $\downarrow x$.

Conversely, suppose $g \colon Q \to Q$ is a continuous selfmap of a subcpo of P such that $f|_Q \sqsubseteq g$. Since g is continuous, there is a (least) fixed point x for g. Then $y \sqsubseteq x$ implies $f(y) \sqsubseteq f(x) \sqsubseteq g(x) = x$, and so $f(\downarrow x) \subseteq \downarrow x$. That is, $f|_{\downarrow x} \colon \downarrow x \to \downarrow x$ is continuous, and so this mapping has a least fixed point. Now, for any fixed point y of f, $f(\downarrow y) \subseteq \downarrow y$, so $f(\downarrow x \cap \downarrow y) \subseteq \downarrow x \cap \downarrow y$. This implies that f has a least fixed point on the subcpo[3] $\downarrow x \cap \downarrow y$, and this least fixed point must be in $\downarrow x$, so it coincides with the least fixed point of $f|_{\downarrow x}$. That is, this is the least fixed point of f. □

[3] The definition of a local cpo requires that each directed set with an upper bound has a least upper bound. It is this assumption that allows us to conclude that $\downarrow x \cap \downarrow y$ is a cpo.

At this point, one might begin to investigate the possibility of solving domain equations within the categories LD and LC. But, we do not see a simple way to guarantee that fixed points exist for enough continuous maps between local cpos, other than to restrict to subcategories of cpos, which means we would not provide any new solutions to the equations one might first examine. However, this clearly is a topic for future research.

Another worthwhile area for further research would be to extend our results from posets to topological spaces in general. That is, we believe the results described above should admit a generalization that would provide the notion of a *local sobrification* of a topological space. This notion may prove useful for studying the representation theory of non-commutative rings, in particular for studying non-commutative C^*-algebras.

3 Modeling Topological Spaces

One of the most important initiatives in domain theory in recent years has been the work of Abbas Edalat and the members of the Comprox group [2–4]. This work has provided many innovative applications of domain theory to areas where it formerly had no apparent application, as well as providing new methods to attack long-standing problems in those areas [2]. One outgrowth of this work has been interest in which topological spaces are homeomorphic to the maximal elements of the posets that arise as models in the varying areas under study. The definitive results along this line are the *formal ball model* of Edalat and Heckmann [5], which shows that any metric space is homeomorphic to the maximal elements of an ω-continuous poset, and Lawson's result [7] that characterizes the maximal elements of certain ω-continuous domains as exactly the Polish spaces. Other results that have emerged recently are due to Martin [8] who has shown any ω-continuous poset admits a natural mapping, called a *measurement*, which gives rise to the Scott topology, and the set of maximal elements of such a poset is always metrizable.

Several questions remain open in this area. For example, only limited progress has been made on relations between *categories* of topological spaces and those of domains that serve as models: in the case of the formal ball model, it has only been shown that the Lipschitz continuous functions can be extended to Scott continuous functions between the associated models. In addition, exactly which spaces can arise as the maximal elements of a domain also is unsettled.

This work has focused on using domains or continuous posets as models for topological spaces. We propose to show how *bounded complete local domains* also provide interesting models for topological spaces. Local domains are local cpos that are continuous posets, so they satisfy all the properties of a domain, except they are not necessarily directed complete. As we mentioned in the introduction, these objects have been used to give an explanation of how certain models of CSP [1] arise from a domain-theoretic viewpoint [9]. But our interest here is on using these structures for modeling topological spaces. Our main result shows that any topological space that can be modeled in a continuous poset that is weak at the top also can be modeled as the set of maximal elements of a bounded complete local domain – the local domain analogue of a Scott domain. This includes the formal ball model of Edalat and Heckmann [5] and Lawson's model for Polish spaces [7]. In addition, we show that continuous mappings between spaces that admit such models extend to Scott-continuous mappings between the modeling local domains. This resolves a problem that has arisen in the formal ball model, since only Lipschitz continuous functions are known to extend to Scott-continuous mappings between those models.

Recall that the *weak^d topology* on a poset P has the family $\{\uparrow x \mid x \in P\}$ as a subbasis of closed sets.

Definition 3.1 A continuous dcpo P is *weak at the top* if the weak^d topology and the Scott topology agree on the maximal elements $\mathrm{Max}(P)$.

It is shown in [5] how to construct a model for any metric space X in an ω-continuous poset; i.e. an ω-continuous poset P which is weak at the top and for which $\mathrm{Max}(P)$ is homeomorphic to X. In [7], it is shown how to model any Polish space (i.e. complete, separable metric space) as the maximal elements of an ω-continuous cpo that is weak at the top, and it is proved that the maximal elements of any such domain form a Polish space. But the models that are constructed in both cases are arcane and not well-understood. A much better-understood family of domains is the family of bounded complete domains – cpos for which any non-empty subset has an infimum.

Definition 3.2 We say that topological space X *has a domain-theoretic model* if X is homeomorphic to the maximal elements of some continuous poset P that is weak at the top and that has a least element.

Note that the assumption about P having a least element is only a technical requirement; if a space X is homeomorphic to the maximal elements of a continuous poset which is weak at the top, but which lacks a least element, clearly adding a least element as compact element of the poset meets our needs without significantly altering the structure of the modeling poset.

Recall that the *Lawson topology* on poset P, denoted $\lambda(P)$, is the common refinement of the Scott and weak^d topologies. On a continuous poset P, $\lambda(P)$ is Hausdorff and has the family of sets $\{\Uparrow x \setminus \uparrow F \mid F \cup \{x\} \subseteq P \text{ finite}\}$ as a basis.

Proposition 3.3 *If P is a continuous poset then the map $\eta_P\colon P \to \mathrm{BSpec}(P)$ by $\eta_P(x) = \downarrow x$ is a homeomorphism with respect to the Scott topologies. Moreover, the following are equivalent:*

(i) *P is weak at the top.*

(ii) *There is an order-preserving continuous injection of $(P, \lambda(P))$ into a com-*

pact pospace, i.e., into a compact Hausdorff space endowed with a (topologically) closed partial order.

Proof. It follows from Theorem 2.1 that η_P is a homeomorphism with respect to the Scott topologies.

The final equivalence is shown for domains in [7], but the result does not require that the poset in question be directed complete, but only continuous. $\square$

Example 3.4 For a continuous poset P with least element, it is not necessarily true that $\mathrm{B}\Gamma(P)$ is weak at the top. Indeed, the following example essentially due to Martin [8] shows this may fail. Let

$$P = \{a_m \mid m \geq 0\} \cup \{b_m \mid m \geq 0\} \cup \{b, x\},$$

where $\{a_m \mid m \geq 0\} \cup \{b\}$ is an antichain, $\{b_m \mid m \geq 0\}$ is a chain (increasing with m) with supremum b and satisfying $b_m \sqsubseteq a_n$ whenever $m \leq n$, and $x \sqsubseteq a_m$ for each $m \geq 0$. Then each element except b is compact, and $\Downarrow b$ is the chain $\{b_m \mid m \geq 0\} \cup \{b\}$. Also, $\downarrow x = \{x\}$ is Scott closed, and $b \not\in \uparrow_{\mathrm{B}\Gamma(P)}(\downarrow x)$, but each Scott-open set containing $\Downarrow b$ in $\mathrm{B}\Gamma(P)$ must contain residually many of the sets $\downarrow a_m$, each of which contains x.

Proposition 3.5 *For a continuous poset P, the following are equivalent:*

(i) *P is weak at the top.*

(ii) $\mathrm{B}\Gamma(P)$ *is weak at the top.*

Proof. The definition of $\mathrm{B}\Gamma(P)$ implies that $\eta_P \colon P \to \mathrm{B}\Gamma(P)$ is a bijection of $\mathrm{Max}(P)$ onto $\mathrm{Max}(\mathrm{B}\Gamma(P))$. If P is continuous, then $\eta_P \colon P \to \eta_P(P)$ is a homeomorphism with respect to the Scott topologies. It also is a homeomorphism with respect to the weak^d topologies: If $\downarrow p$ is a maximal element of $\mathrm{B}\Gamma(P)$ and $X \in \mathrm{B}\Gamma(P)$, then $\downarrow p \in \uparrow X$ iff $X \subseteq \downarrow p$ iff $p \in \bigcap_{x \in X} \uparrow x$. Hence $\mathrm{Max}(P)$ and $\mathrm{Max}(\mathrm{B}\Gamma(P))$ are homeomorphic in both the inherited Scott- and the inherited weak^d topologies. From this, the equivalence follows. $\square$

Theorem 3.6 *For a topological space X, the following are equivalent:*

(i) *X has a domain-theoretic model.*

(ii) *X is homeomorphic to the maximal elements of a bounded complete local domain which is weak at the top.*

Proof. Note that the mapping $\eta_P \colon P \to \mathrm{B}\Gamma(P)$ preserves maximal elements, and for P continuous, this mapping is a homeomorphism onto its image. In particular, $\mathrm{Max}(P) \simeq \mathrm{Max}(\mathrm{B}\Gamma(P))$. Moreover, $\mathrm{B}\Gamma(P)$ is bounded complete, and if P is weak at the top, then so is $\mathrm{B}\Gamma(P)$. So, if X has a domain-theoretic model P, then $\mathrm{B}\Gamma(P)$ is a domain-theoretic model for X that also is bounded complete.

The converse is obvious. $\square$

The reason to transfer spaces which have a domain-theoretic model in *some* local domain into a model that is bounded complete is the ease with which continuous mappings can be extended in the latter situation.

Lemma 3.7 *Let $f: P \to Q$ be a monotone mapping between local domains. Then the mapping $F: P \to Q$ by $F(x) = \sqcup_Q f(\Downarrow x)$ is the largest continuous mapping satisfying $F(x) \sqsubseteq f(x)$ for all $x \in P$.* □

Proposition 3.8 *Let P and Q be bounded complete local domains and let $f: \mathrm{Max}(P) \to \mathrm{Max}(Q)$ be a continuous map with respect to the relative Scott topologies. Then there is a Scott-continuous map $F: P \to Q$ with $F|_{\mathrm{Max}(P)} = f$.*

Proof. We define $F_1(x) = \wedge f(\uparrow x \cap \mathrm{Max}(P))$. This mapping is well-defined since $\uparrow x \cap \mathrm{Max}(P) \neq \emptyset$ for all $x \in P$. It also is clear that the mapping F_1 is monotone and extends f. Then Lemma 3.7 implies $F: P \to Q$ by $F(x) = \sqcup_Q F_1(\Downarrow x)$ is Scott-continuous.

We next show $F|_{\mathrm{Max}(P)} = f$. If $x \in \mathrm{Max}(P)$, let $y \ll w \ll f(x)$ in Q. Then $x \in f^{-1}(\Uparrow w \cap \mathrm{Max}(Q))$ is open in $\mathrm{Max}(P)$. So there is some $z \in P$ with $z \ll x$ and $\Uparrow z \cap \mathrm{Max}(P) \subseteq f^{-1}(\Uparrow w \cap \mathrm{Max}(Q))$. This implies $y \sqsubseteq F_1(z)$, and so $y \sqsubseteq F_1(z) \sqsubseteq \sqcup_Q F_1(\Downarrow p) = F(p)$ for any $p \in P$ with $z \ll p$. In particular, $y \ll F(x)$, and since $y \ll f(x)$ is arbitrary and $f(x)$ is maximal, $F(x) = f(x)$. □

Corollary 3.9 *Suppose that $f: X \to Y$ is a continuous mapping between spaces X and Y, and suppose $\phi_X: X \simeq \mathrm{Max}(P_X)$ and $\phi_Y: Y \simeq \mathrm{Max}(P_Y)$ are homeomorphisms onto the maximal elements of bounded complete local domains P_X and P_Y, respectively. Then there is a Scott-continuous mapping $F: P \to Q$ which extends the mapping $\phi_Y \circ f \circ \phi_X^{-1}$.* □

Corollary 3.10 *Let $f: X \to X$ be a continuous selfmap of the space X and suppose that $x \in X$ is a fixed point for f. If X has a domain-theoretic model P which is a bounded complete local domain, then the extension F of f has a least fixed point.*

Proof. That $F|_{\mathrm{Max}(P)} = f$ implies $F(x) = x$, and so $F(\downarrow x) \subseteq \downarrow x$ since F is monotone. But $\downarrow x$ is a cpo since P is a local domain with least element, and so the continuous selfmap $F|_{\mathrm{Max}(P)}$ has a least fixed point on $\downarrow x$. It is routine to show that this is the least fixed point of F on P. □

We can apply our results to either the Edalat-Heckmann formal ball model to conclude that any metric space admits a domain-theoretic model into a bounded complete, ω-continuous local domain, or to Lawson's model for Polish spaces, so that they, too, can be modeled with bounded complete, ω-continuous local domains. In each case, we also can conclude that continuous mappings between such spaces extend to Scott-continuous mappings between the local domain models. It would be nice to have categorical results here – i.e. to extend this association to the level of a functor. But it is not clear that Proposition 3.8 yields an extension process that is compositional on maps.

Our results show that continuous maps between spaces that admit domain-theoretic models that are bounded complete local domains extend to continuous maps between the domains. Moreover, the extensions have least fixed points if the maps being extended have fixed points. We believe this is a faithful representation of the situation among topological spaces – even metric spaces – and continuous maps between them. Namely, the goal of modeling spaces with cpos so that all continuous maps extend would result in every extending map having a (least) fixed point regardless of whether the map being extended had one or not. While we cannot show that this might not happen here, we also have no method to *prove* that the extension we define has a least fixed point unless the original map being extended has a fixed point, precisely because we use local domains as the models. These models require that the mapping have a sub-cpo that it leaves invariant before we can prove any fixed points exist.

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