Measuring the Probabilistic Powerdomain

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Abstract

In this paper we initiate the study of measurements on the probabilistic powerdomain. We show how measurements on an underlying domain naturally extend to its probabilistic powerdomain, so that the kernel of the extension consists of exactly those normalized measures on the kernel of the measurement on the underlying domain. This result is combined with now-standard results from the theory of measurements to obtain a new proof that the fixed point associated with a weakly hyperbolic IFS with probabilities is the unique invariant measure whose support is the attractor of the underlying IFS.

1 Introduction

A relatively recent discovery [16] in domain theory is that most domains come equipped with a natural *measurement*: a Scott continuous map into the non-negative reals in the dual order which encodes the Scott topology. The existence of measurements was exploited by Martin [15–17,19] to study the space of maximal elements of a domain, and to formulate various fixed point theorems for domains, including fixed point theorems for non-monotonic maps.

The theory of measurements meshes particularly fruitfully with the idea of domains as models of classical spaces. Here we say that a domain D is a *model* of a topological space X if the set of maximal elements of D, equipped with the relative Scott topology, is homeomorphic to X. For instance, a simple model of the unit interval $[0, 1]$ is the interval domain $\mathbf{I}[0, 1]$. This domain consists of the non-empty closed sub-intervals of $[0, 1]$, ordered by reverse inclusion. The idea is that an interval $[a, b]$ represents a partially defined real number — we

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are not sure what the number is, but we know it lies somewhere between a and b. The maximal elements of $\mathbf{I}[0,1]$ are the singleton intervals $[x]$, and these are in bijective correspondence with the elements of $[0,1]$. Furthermore, the map sending an interval to its length is a measurement on $\mathbf{I}[0,1]$. Notice that we can view the measurement as giving a notion of the degree of partiality of an element of $\mathbf{I}[0,1]$. In particular, the collection of maximal elements of $\mathbf{I}[0,1]$ coincides with the *kernel* of the measurement: the set of elements with measurement 0.

Given a domain model D of a space X, under quite mild conditions on D the set of normalized Borel measures[3] on X, equipped with the weak topology, can be embedded into the set of maximal elements of the probabilistic powerdomain $\mathbf{P}D$ (cf. Edalat [6]). This construction was utilized by Edalat [4,5] to provide new results on the existence of attractors for iterated function systems, and to define a generalization of the Riemann integral to functions on metric spaces.

We can begin to appreciate Edalat's idea by taking D to be the interval domain $\mathbf{I}[0,1]$. In this case we have an embedding of the normalized Borel measures on $[0,1]$ in the maximal elements of $\mathbf{PI}[0,1]$. Writing $\delta_I \in \mathbf{PI}[0,1]$ for the point mass concentrated at $I \in \mathbf{I}[0,1]$, and given a positive integer n, the distribution

$$\nu_n = \textstyle\sum_{i=1}^{n} \frac{1}{n}\delta_{I_i},$$

where $I_i = [\frac{i-1}{n}, \frac{i}{n}]$, is an element of $\mathbf{PI}[0,1]$. Intuitively we regard the ν_n as approximations to Lebesgue measure on $[0,1]$, and we might expect that $\bigsqcup_n \nu_n$ is Lebesgue measure on $[0,1]$ (under the above-mentioned embedding of Borel measures on $[0,1]$ into the maximal elements of $\mathbf{PI}[0,1]$).

In fact, a proof of this may be found in Edalat [5]; but it is non-trivial. (In particular, it requires the result that valuations on $\mathbf{PI}[0,1]$ extend to Borel measures.) Indeed, it is not even straightforward that $\bigsqcup_n \nu_n$ is maximal in $\mathbf{PI}[0,1]$. On the other hand, building on the measurement on $\mathbf{I}[0,1]$, there is a very natural candidate for a measurement M on $\mathbf{PI}[0,1]$: simply define

$$M(\textstyle\sum r_i \delta_{[a_i,b_i]}) = \sum r_i(b_i - a_i).$$

Suppose we could prove that M really is a measurement; then, from a basic property of measurements, the simple observation that $M(\nu_n) \to 0$ as n increases, entails that $\bigsqcup_n \nu_n$ defines a Borel measure on $[0,1]$. Thus we replace Edalat's argument from [5] with an argument involving measurements which can also be applied in other settings.

In this paper we show that each measurement m (satisfying a suitable condi-

[3] The notion of a measure is quite separate from the notion of a measurement, despite the similarity of the terminology.

tion, called MP) on a domain D has a natural extension to a measurement M on the probabilistic powerdomain $\mathbf{P}D$. Moreover, we show that the kernel of M, equipped with the relative Scott topology, is homeomorphic to the space of valuations [4] on the kernel of m equipped with the weak topology.

We show that the condition MP, alluded to above, is satisfied by the natural measurements on standard models of metric spaces, such as the interval domain, the formal ball model and the upper space model. We also show that any ω-continuous dcpo D whose Scott and Lawson topologies agree on the subset $\mathsf{max}\, D$ of maximal elements admits such a measurement.

These results can be used to derive facts about domains in general which are *independent* of measurement: for example, if D is an ω-continuous dcpo whose Scott and Lawson topologies agree on $\mathsf{max}\, D$, then the set of normalized Borel measures on $\mathsf{max}\, D$, equipped with the weak topology, can be embedded into the set of maximal elements of $\mathbf{P}D$. They can also be used to derive results which are *independent* of domain theory altogether, such as a new proof that the fixed point associated with a weakly hyperbolic iterated function system with probabilities is the unique measure whose support is the attractor of the underlying iterated function system.

This paper can be seen as a probabilistic analogue of [19]. The latter gives a necessary and sufficient condition for a measurement on a domain D to extend to a measurement on the convex powerdomain $\mathbf{C}D$. Knowing that certain measurements extend to the convex powerdomain enables one to prove that any ω-continuous dcpo D with $\mathsf{max}\, D$ regular satisfies the property that the Vietoris hyperspace of $\mathsf{max}\, D$ embeds into $\mathsf{max}\, \mathbf{C}D$ (as the kernel of a measurement). Further, Edalat's domain theoretic analysis of hyperbolic iterated function systems is then shown to be a consequence of standard results about measurement.

2 Background

In this section we summarize some of the notions from topology, measure theory and domain theory which will be used in this paper.

2.1 Topology and Measure Theory

We assume familiarity with basic topological notions such as closure, basis, neighbourhood, convergence and compactness. Here we just explain some

[4] Roughly speaking, valuations and measures are synonymous.

terms which may be slightly less well known to computer scientists, but which are central to our subject matter. In particular, we outline some of the basic connections between topology and measure theory.

A topological space is *regular* if each neighbourhood of a point x contains a closed neighbourhood of the same point x. A space is *locally compact* if each neighbourhood of a point x contains a compact neighbourhood of the same point x. A space is *completely metrizable* if the topology is generated by a complete metric.

A collection of subsets of a given set X which contains $\emptyset$, and is closed under finite unions and complementation is called a *field*. A topology on a set X generates a field, i.e., the smallest field containing all the open sets. The members of this field can all be written as disjoint unions of *crescents*, where a crescent is the difference between two open sets. A *σ-field* on a set X is a field on X which is also closed under countable unions. The σ-field generated by the open sets of a topological space is called the *Borel σ-field*.

Suppose $\mathcal{F}X$ is the Borel σ-field on a topological space X. A *normalized Borel measure* on X is a function $\mu\colon \mathcal{F}X \to [0,1]$ satisfying $\mu(X) = 1$ and, for any countable pairwise disjoint family $\{T_i\} \subseteq \mathcal{F}X$,

$$\mu(\bigcup T_i) = \sum_i \mu(T_i)\,.$$

The *weak topology* on the space of normalized Borel measures on X is the weakest topology such that, for each bounded continuous function $f\colon X \to \mathbb{R}$, the map $\mu \mapsto \int f\,d\mu$ is continuous. The weak topology can also be characterized independently of a notion of integral. In fact, a net $\langle \mu_i \rangle$ of normalized Borel measures converges to μ in the weak topology iff $\liminf \mu_i(U) \geqslant \mu U$ for each open set $U \subseteq X$.

2.2 Domain Theory

A *poset* $(P, \sqsubseteq)$ is a set P endowed with a partial order $\sqsubseteq$. The least element of P (if it exists) is denoted $\bot$, and the set of maximal elements of P is written $\mathsf{max}\,P$. Given $A \subseteq P$, we write $\uparrow A$ for the set $\{x \in P \mid (\exists a \in A)\, a \sqsubseteq x\}$; similarly, $\downarrow A$ denotes $\{x \in P \mid (\exists a \in A)\, x \sqsubseteq a\}$. A function $f\colon P \to Q$ between posets P and Q is *monotone* if $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ for all $x, y \in P$. A subset $A \subseteq P$ is *directed* if each finite subset $F \subseteq A$ has an upper bound in A. Note that since $F = \emptyset$ is a possibility, a directed subset must be non-empty. A *(directed) complete partial order (dcpo)* is a poset P in which each directed set $A \subseteq P$ has a least upper bound, denoted $\sqcup A$.

If D is a dcpo, and $x, y \in D$, then we say that x is way-below y, denoted $x \ll y$, if for each directed subset $A \subseteq D$, if $y \sqsubseteq \sqcup A$, then $\uparrow x \cap A \neq \emptyset$. Let $\twoheaddownarrow y = \{x \in D \mid x \ll y\}$; we say that D is *continuous* if it has a *basis*, i.e., a subset $B \subseteq D$ such that for each $y \in D$, $\twoheaddownarrow y \cap B$ is directed with supremum y. If D has a countable basis then we say D is *ω-continuous*. The way-below relation on a continuous dcpo has the *interpolation property*: if $x \ll y$ then there exists a basis element z such that $x \ll z \ll y$.

A subset U of a dcpo D is *Scott-open* if it is an upper set (i.e., $U = \uparrow U$) and for each directed set $A \subseteq D$, if $\sqcup A \in U$ then $A \cap U \neq \emptyset$. The collection ΣD of all Scott-open subsets of D is called the *Scott topology* on D. If D is continuous, then the Scott topology on D is locally compact, and the sets $\twoheaduparrow x$ where $x \in D$ form a basis for the topology. If $S \subseteq D$, we write $\mathrm{Cl}_\sigma(S)$ for the closure of S with respect to the Scott topology. Given dcpos D and E, a function $f \colon D \to E$ is continuous with respect the Scott topologies on D and E iff it is monotone and preserves directed suprema: for each directed $A \subseteq D$, $f(\sqcup A) = \sqcup f(A)$. The *Lawson* topology on a dcpo D is a refinement of the Scott topology generated by including the sets $D \setminus \uparrow x$ for $x \in D$ as opens.

Hereafter continuous dcpos will also be referred to as domains.

3 Valuations and the Probabilistic Powerdomain

We recall some basic definitions and results about valuations and the probabilistic powerdomain.

Definition 1 *Let X be a topological space. A (continuous)* valuation *on X is a mapping $\nu \colon (\Omega X, \subseteq) \to ([0,1], \leqslant)$ satisfying:*

(1) Strictness: $\nu(\emptyset) = 0$.
(2) Monotonicity: $U \subseteq V \Rightarrow \nu(U) \leqslant \nu(V)$.
(3) Modularity: for all $U, V \in \Omega X$, $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$.
(4) Continuity: for every directed family $\{U_i\}_{i \in I}$, $\nu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$.

Each element $x \in X$ gives rise to a valuation defined by

$$\delta_x(U) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{otherwise}. \end{cases}$$

A *simple valuation* has the form $\sum_{a \in A} r_a \delta_a$, where A is a finite subset of X, $r_a \geqslant 0$ and $\sum_{a \in A} r_a \leqslant 1$. A valuation ν is *normalized* if $\nu(X) = 1$. For the most part we will consider valuations defined on the Scott topology ΣD of a dcpo D.

Obviously, valuations bear a close resemblance to measures. Lawson [13] showed that any valuation on an ω-continuous dcpo D extends uniquely to a measure on the Borel σ-field generated by the Scott topology (equivalently by the Lawson topology) on D. This result was generalized to continuous dcpos by Alvarez-Manilla, Edalat and Saheb-Djahromi [3]. In this paper we do not use either of these theorems. In Section 8 we use the well-known fact that any valuation on a metric space has a unique extension to a measure (cf. [2, Corollary 3.24]). But this is only used to mediate between the formulation of the main result of that section, and the results of Hutchinson [10], which are stated for measures.

While the problem of extending valuations to measures is non-trivial, it is straightforward to extend a valuation on a topological space X to a finitely additive set function on the field $\mathcal{F}X$ generated by the open sets of X. Recall that each member R of this field can be written as a finite, disjoint union of crescents, i.e., $R = \bigcup_{i=1}^{n} U_i \setminus V_i$ for open $U_i, V_i \subseteq X$. The extension of a valuation ν to $\mathcal{F}X$ assigns to R the value

$$\sum_{i=1}^{n}(\nu(U_i) - \nu(U_i \cap V_i)).$$

Also we recall from Heckmann [9, Section 3.2] that if $E \in \mathcal{F}X$ then we may define a valuation $\nu|_E$ by $\nu|_E\,(O) = \nu(O \cap E)$ for all open $O \subseteq X$.

Next we review from [11, Section 3.9] the definition of the integral of a lower semi-continuous function $f\colon X \to [0,\infty)$ (i.e., a continuous function for the Scott topology on $[0,\infty)$) against a valuation ν on X. This is precisely the construction we need to extend a measurement on a domain D to a measurement on $\mathbf{P}D$.

First, if a lower semi-continuous function $f\colon X \to [0,\infty)$ is simple, i.e., has finite range, then we can write f uniquely as a linear combination of characteristic functions

$$f = \sum_{i=1}^{n} \alpha_i \chi_{f^{-1}(\alpha_i)}\,,$$

with moreover $f^{-1}(\alpha_i) \in \mathcal{F}X$. This leads us to define

$$\int f\,d\nu = \sum_{i=1}^{n} \alpha_i \nu(f^{-1}(\alpha_i)).$$

Now any lower semi-continuous function $f\colon X \to [0,\infty)$ is the uniform limit of the sequence of simple functions $\langle f_n \rangle$, where

$$f_n = \sum_{i=1}^{n2^n} 2^{-n} \chi_{f^{-1}(i2^{-n},\infty)}.$$

The integral $\int f \, d\nu$ is now defined to be $\sup_n \int f_n \, d\nu$.

This is, of course, completely analogous to the way one defines the integral of a non-negative measurable function against a measure. The weak topology on the set of valuations on X is now defined to be the weakest topology such that $f \mapsto \int f \, d\mu$ is lower semi-continuous for each lower semi-continuous map f. (For Hausdorff spaces the same condition characterizes the weak topology on Borel measures.)

Next we recall the probabilistic powerdomain construction from Jones [11].

Definition 2 *Given a dcpo D, the probabilistic powerdomain $\mathbf{P}D$ is the dcpo of all valuations defined on D in its Scott topology, and ordered by $\sigma \sqsubseteq \nu$ if and only if $\sigma(U) \leqslant \nu(U)$ for all $U \in \Sigma D$.*

Theorem 3 (Jones [11]) *If D is a continuous dcpo, then $\mathbf{P}D$ is a continuous dcpo with a basis $\mathcal{B} = \{\Sigma_{i=1}^n r_i \delta_{p_i} \mid p_i \in B\}$, where $B \subseteq D$ is a basis for D.*

The following proposition shows that the Scott topology on $\mathbf{P}D$ is just the weak topology.

Proposition 4 (Edalat [6]) *Suppose D is a continuous dcpo, then a net $\langle \nu_i \rangle_{i \in I}$ in $\mathbf{P}D$ converges to ν in the Scott topology iff*

$$\liminf \nu_i(U) \geqslant \nu(U)$$

for all Scott open subsets $U \subseteq D$.

4 Measurement

Let $m : D \to E$ be a Scott continuous map between domains D and E. We define the *kernel* of m by

$$\mathsf{ker}\, m = \{x \in D : m(x) \in \mathsf{max}\, E\}.$$

Definition 5 *For $\varepsilon \in E$, the ε-*approximations *of $x \in D$ are*

$$m_\varepsilon(x) = \{y \in D : y \sqsubseteq x \; \& \; \varepsilon \ll m(y)\}.$$

We say that m measures *$x \in D$ if, for all open $U \subseteq D$, we have*

$$x \in U \Rightarrow (\exists \varepsilon \in E)\, x \in m_\varepsilon(x) \subseteq U.$$

A helpful intuition is to think of m as an abstraction function, representing elements of D in a (simpler) domain E; the ε-approximations $m_\varepsilon(x)$ are those points in D below x whose measurement is 'ε-close to that of x in E'. From this viewpoint, m measures $x \in D$ just in case this abstraction is faithful to the Scott topology at x. In particular, a sequence $\langle x_n \rangle$ in $\downarrow x$ converges to x in the Scott topology precisely when $m(x_n)$ converges to $m(x)$ in E.

Definition 6 *A* measurement *is a continuous map $m : D \to E$ which measures every element of* $\mathsf{ker}\, m$.

In this paper we will typically take $E = [0, \infty)^*$: the non-negative reals in the opposite order. In this case we can see a measurement as capturing the degree of partiality of elements of D by a single number. Elaborating the measurement condition in this particular instance, we have that $m : D \to [0, \infty)^*$ is a measurement iff for any Scott open U and any ideal element $x \in \mathsf{ker}\, m$,

$$x \in U \Rightarrow (\exists\, \varepsilon > 0)\, \{y \in D : y \sqsubseteq x \ \&\ m(y) < \varepsilon\} \subseteq U.$$

That is, any element below x with sufficiently small measurement lies in U.

It is straightforward to prove that $\mathsf{ker}\, m \subseteq \mathsf{max}\, D$ for a measurement m.

Example 7 *The following examples of measurements are all pertinent to this paper. The first two illustrate the idea that natural models of metric spaces yield canonical measurements into $[0, \infty)^*$.*

(1) If $\langle X, d \rangle$ is a locally compact metric space, then its upper space

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \text{ is compact}, \}$$

ordered by reverse inclusion, is a continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{U}X$ is $\bigcap S$, and the way-below relation is given by $A \ll B$ iff $B \subseteq \operatorname{int} A$. Given $K \in \mathbf{U}X$, defining the diameter of K by

$$|K| = \sup\{d(x, y) : x, y \in K\},$$

it is readily verified that $m(K) = |K|$ is a measurement on $\mathbf{U}X$ whose kernel is $\mathsf{max}\, \mathbf{U}X = \{\{x\} : x \in X\}$.

(2) Given a metric space $\langle X, d \rangle$, the formal ball model *[7] $\mathbf{B}X = X \times [0, \infty)$ is a poset ordered by*

$$(x, r) \sqsubseteq (y, s) \text{ iff } d(x, y) \leqslant r - s.$$

The way-below relation is characterized by

$$(x, r) \ll (y, s) \text{ iff } d(x, y) < r - s.$$

The poset $\mathbf{B}X$ *is a continuous dcpo iff the metric* d *is complete. Moreover* $\mathbf{B}X$ *has a countable basis iff* X *is separable. A natural measurement* m *on* $\mathbf{B}X$ *is given by* $m(x,r)=r$. *Then* $\mathsf{ker}\, m = \mathsf{max}\,\mathbf{B}X = \{(x,0) : x \in X\}$.

(3) Let $X = \{x_n\}_{n\in\mathbb{N}}$ *be a countably infinite set, and* $(\mathbb{P}X, \subseteq)$ *the lattice of subsets of* X *ordered by inclusion. Observe that* $S \ll T$ *in* $\mathbf{P}X$ *iff* S *is a finite subset of* T. *We can define a measurement* $m\colon \mathbb{P}X \to [0,\infty)^*$ *by*

$$m(S) = 1 - \sum_{x_n \in S} 2^{-(n+1)}.$$

One of the motivations behind the introduction of measurement in [16] was to facilitate the formulation of sharper fixed point theorems. The following is a basic example of one such result.

Theorem 8 *Let* $f\colon D \to D$ *be a monotone map on a pointed continuous dcpo* D *equipped with a measurement* $m\colon D \to E$. *If* $\bigsqcup m(f^n(\bot)) \in \mathsf{max}\, E$, *then*

$$x^\star = \bigsqcup_{n\geqslant 0} f^n(\bot) \in \mathsf{ker}\, m$$

is the unique fixed point of f. *Moreover,* $x^\star$ *is an attractor: For all* x, $f^n(x) \to x^\star$ *in the Scott topology on* D. *This convergence restricts to* $\mathsf{ker}\, m$ *if* f *carries* $\mathsf{ker}\, m$ *into* $\mathsf{ker}\, m$.

5 Lebesgue Measurements and MP-Measurements

Martin [19] gives a necessary and sufficient condition for a measurement on a domain D to extend to a measurement on the convex powerdomain $\mathbf{C}D$, thereby uncovering the class of *Lebesgue measurements*. Before defining this class we first extend the definition of m_ε to arbitrary sets $S \subseteq D$ by writing

$$m_\varepsilon(S) = \bigcup_{s\in S} m_\varepsilon(s).$$

Definition 9 *A continuous map* $m : D \to [0,\infty)^*$ *is a* Lebesgue measurement *if for all compact* $K \subseteq \mathsf{ker}\, m$ *and all open* $U \subseteq D$,

$$K \subseteq U \Rightarrow (\exists\, \varepsilon > 0)(m_\varepsilon(K) \subseteq U).$$

Clearly any Lebesgue measurement is indeed a measurement according to Definition 6. We ought also to mention that Lebesgue *measurements* have nothing to do with Lebesgue *measure*. The name arises because a measurement m induces a distance map on $\mathsf{ker}\, m$, and this map has the Lebesgue covering property precisely when m satisfies the conditions laid out in Definition 9, cf. Martin [15].

Knowing that Lebesgue measurements extend to the convex powerdomain enables one to prove that any ω-continuous dcpo D with $\mathsf{max}D$ regular satisfies the property that the Vietoris hyperspace of $\mathsf{max}\, D$ embeds into $\mathsf{max}\, \mathbf{C}D$ (as the kernel of a measurement). Further, Edalat's domain theoretic analysis of hyperbolic iterated function systems is then shown to be a consequence of standard results from measurement. In the same setting, the *necessity* of complete metrizability becomes apparent.

Theorem 10 (Martin [15]) *A space is completely metrizable iff it is the kernel of a Lebesgue measurement $m : D \to [0, \infty)^*$ on a continuous dcpo.*

Here we seek analogous results, with the probabilistic powerdomain in place of the convex powerdomain, and the weak topology on Borel measures in place of the Vietoris topology on compact sets. We now identify a condition which ensures that a measurement on a domain D extends to a measurement on the probabilistic powerdomain $\mathbf{P}D$.

Given a continuous map $m \colon D \to E$, we consider the following condition:

$$a \ll b \in D \Rightarrow (\exists\, \varepsilon \in E) m_\varepsilon(\uparrow b \cap \mathsf{ker}\, m) \subseteq \twoheaduparrow a. \qquad \text{(MP)}$$

In words, the condition states that whenever b is way-above a, then there exists ε such that all the ε-approximations of elements in the kernel above b are still way-above a.

Proposition 11 *Suppose $m \colon D \to [0, \infty)^*$ satisfies condition* (MP)*; then m is a Lebesgue measurement.*

PROOF. Let $U \subseteq D$ be Scott open and K a compact subset of $\mathsf{ker}\, m$ with $K \subseteq U$. For each $x \in K$ let $a_x \ll b_x \ll x$ with $a_x \in U$. Since m satisfies condition (MP), for each $x \in K$ there exists $\varepsilon_x > 0$ such that $m_{\varepsilon_x}(\uparrow b_x \cap \mathsf{ker}\, m) \subseteq \twoheaduparrow a_x$. Furthermore, by compactness of K, we have $x_1, ..., x_n \in K$ such that $K \subseteq \uparrow b_{x_1} \cup \cdots \cup \uparrow b_{x_n}$. But then, taking $\varepsilon = \min \varepsilon_i$, we have that $m_\varepsilon(K) \subseteq U$. $\square$

In particular, $m \colon D \to [0, \infty)^*$ satisfying (MP) is a measurement. We call such a map an *MP-measurement.* (We explain the terminology below.)

Example 12 *It turns out that all the measurements we have considered thus far are MP-measurements.*

(1) Recall from Example 7 the definition of a measurement m on the upper space of a locally compact metric space X. Suppose $E \ll K \in \mathbf{U}X$, i.e., there exists an open set $U \subseteq X$ with $K \subseteq U \subseteq E$. Using the compactness

of K it is straightforward that $(\exists\,\varepsilon > 0)(\forall x \in K) B_\varepsilon(x) \subseteq U$, where $B_\varepsilon(x)$ is the open ball of radius ε centred at x. It follows that $m_\varepsilon(\uparrow K \cap \mathsf{ker}\, m) \subseteq \twoheaduparrow E$.

(2) Suppose $\langle X, d\rangle$ is a metric space, and let m denote the measurement on the formal ball model $\mathbf{B}X$ defined in Example 7. Suppose $(x,r) \ll (y,s) \in \mathbf{B}X$. Let $\varepsilon = \dfrac{(r-s-d(x,y))}{2} > 0$. We show that

$$m_\varepsilon(\uparrow (y,s) \cap \mathsf{ker}\, m) \subseteq \twoheaduparrow(x,r).$$

To this end, suppose $(y,s) \sqsubseteq (w,0)$ and $(z,t) \sqsubseteq (w,0)$ for some $w,z \in X$ and $t < \varepsilon$. Then

$$\begin{aligned} d(x,z) &\leqslant d(x,y) + d(y,w) + d(w,z) \\ &\leqslant d(x,y) + s + t \\ &< (r - s - 2t) + s + t \\ &= r - t\,. \end{aligned}$$

Thus $(x,r) \ll (z,t)$. $\square$

The name (MP) for the condition described above arises from the notion of an *MP-hull* in Lawson [14]. That paper was concerned with determining which spaces arise as the space of maximal points of an ω-continuous dcpo.

Definition 13 *An* MP-hull *is an ω-continuous dcpo such that the relative Scott topology coincides with the relative Lawson topology on $\mathsf{max}\, D$.*

Theorem 14 *Every MP-hull D admits an MP-measurement $m\colon D \to [0,\infty)^*$.*

PROOF. Suppose D is an MP-hull with countable basis B, and let

$$I = \{(a,b) \in B \times B \mid a \ll b\}.$$

It is straightforward that $m\colon D \to (\mathbb{P}I, \subseteq)$ defined by

$$m(x) = \{(a,b) \mid x \in \twoheaduparrow a \;\vee\; x \notin \mathrm{Cl}_\sigma(\uparrow b)\}, \tag{1}$$

is a Scott continuous map. Next we show that $\mathsf{max}\, D = \mathsf{ker}\, m$.

Let $x \in \mathsf{max}\, D$. If $(a,b) \in I$, then $\mathrm{Cl}_\sigma(\uparrow b) \cap \mathsf{max}\, D = \uparrow b \cap \mathsf{max}\, D \subseteq \twoheaduparrow a$, since $\uparrow b$ is Lawson closed and the Scott and Lawson topologies agree on $\mathsf{max}\, D$. It follows that $(a,b) \in m(x)$. But the choice of (a,b) was arbitrary so we have that $x \in \mathsf{ker}\, m$. Conversely, suppose $x \in \mathsf{ker}\, m$ with $x \sqsubseteq y$. If $a \ll b \sqsubseteq y$, then the fact that $(a,b) \in m(x)$ and $x \in \mathrm{Cl}_\sigma(\uparrow b)$implies $a \ll x$. Thus $x = y$.

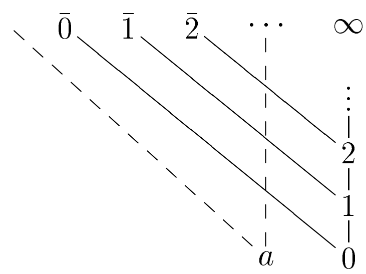

Fig. 1.

Now we show that m satisfies condition (MP). If $a \ll b$, then taking $\varepsilon = \{(a,b)\}$ we have

$$m_\varepsilon(\uparrow b \cap \mathsf{ker}\, m) = m_\varepsilon(\uparrow b \cap \mathsf{max}\, D) = m_\varepsilon(\mathrm{Cl}_\sigma(\uparrow b) \cap \mathsf{max}\, D) \subseteq \twoheaduparrow a.$$

It is now straightforward that composing m with the measurement in Example 7(iii) yields an MP-measurement $m\colon D \to [0,\infty)^*$. $\square$

Our main result, Theorem 30, says that an MP-measurement $m\colon D \to [0,\infty)^*$ extends in a natural way to a measurement $M\colon \mathbf{P}D \to [0,\infty)^*$ on the probabilistic powerdomain of D. Furthermore, in this case, $\mathsf{ker}\, M$ is homeomorphic to the set of normalized valuations on $\mathsf{ker}\, m$ in the weak topology. Edalat's domain theoretic analysis of hyperbolic iterated function systems [4] can then be shown to be a consequence of standard results using measurement [16].

We conclude this section with an example showing that the class of MP-measurements is strictly smaller than the class of Lebesgue measurements.

Example 15 *Let $D = \{n, \bar{n} : n \in \mathbb{N}\} \cup \{a, \infty\}$, with order generated by $n \sqsubseteq \bar{n}$, $a \sqsubseteq \bar{n}$ and $n \sqsubseteq m \sqsubseteq \infty$ for all $n, m \in \mathbb{N}$ with $n \leqslant m$ in the usual order (see Figure 1). We define a measurement $m\colon D \to [0,\infty)^*$ by requiring that $m(\bar{n}) = m(\infty) = 0$, $m(n) = 2^{-n}$ and $m(a) = 1$. Then m fails to satisfy MP, since $a \ll a$ but for no $\varepsilon > 0$ is it the case that $m_\varepsilon(\uparrow a \cap \mathsf{ker}\, m) \subseteq \twoheaduparrow a$. On the other hand, m is a Lebesgue measurement: in particular, the only compact subsets of $\mathsf{ker}\, m$ contained in $\twoheaduparrow a$ are finite.*

6 Comparing Valuations

One of the most elegant results about the probabilistic powerdomain is the Splitting Lemma. This bears a close relationship to a classic problem in probability theory: find a joint distribution with given marginals.

Lemma 16 (Jones [11]) *Let $\sigma = \sum_{a \in A} r_a \delta_a$ and $\nu = \sum_{b \in B} s_b \delta_b$ be simple* *valuations. Then $\sigma \ll \nu$ if and only if there exists a family of non-negative transport (or flow) numbers $\{u_{a,b}\}_{a\in A, b\in B}$ satisfying*

(1) For each $a \in A$, $\sum_{b\in B} u_{a,b} = r_a$.
(2) For each $b \in B$, $\sum_{a\in A} u_{a,b} < s_b$.
(3) $u_{a,b} \neq 0$ implies $a \ll b$.

We can picture the situation above as a network flow diagram with a set A of sources, a set B of sinks, and an edge from each source to each sink. Each source $a \in A$ has value r_a, each sink $b \in B$ has value s_b, and $u_{a,b}$ indicates the value of the mass flowing from a to b.

In the remainder of this section we give a characterization of when a simple valuation lies way-below an arbitrary valuation.

Proposition 17 (Kirch [12]) *If ν is a valuation on D, then $\sum_{a\in A} r_a\delta_a \ll \nu$ if and only if $\forall S \subseteq A$, $\sum_{a\in S} r_a < \nu(\Uparrow S)$.*

Definition 18 *Fix a finite subset $A \subseteq D$, and for each $S \subseteq A$ define*

$$\mathrm{Cr}(A, S) = \bigcap_{a\in S} \Uparrow a \setminus \bigcup_{a'\in A\setminus S} \Uparrow a'.$$

Observe that $\{\mathrm{Cr}(A, S)\}_{S\subseteq A}$ is a family of crescents partitioning D.

Proposition 19 *Let ν be a valuation on D, $\sum_{a\in A} r_a\delta_a$ a simple valuation on D, and $\{E_i\}_{i\in I} \subseteq \mathcal{F}D$ a finite partition of D refining $\{\mathrm{Cr}(A, S)\}_{S\subseteq A}$. Then $\sum_{a\in A} r_a\delta_a \ll \nu$ iff there exists a relation $R \subseteq A \times I$ such that*

(1) $(a, i) \in R$ implies $E_i \subseteq \Uparrow a$;
(2) for all $S \subseteq A$, $\sum_{a\in S} r_a < \sum_{i\in R(S)} \nu(E_i)$.

PROOF. ($\Rightarrow$) Suppose $\sum_{a\in A} r_a\delta_a \ll \nu$. Define R by $R(a, i)$ just in case $E_i \subseteq \Uparrow a$. Then, given $S \subseteq A$, by Proposition 17,

$$\sum_{a\in S} r_a < \nu(\Uparrow S) = \sum_{i\in R(S)} \nu(E_i).$$

($\Leftarrow$) Given a relation R satisfying conditions (1) and (2) above, then for all $S \subseteq A$ we have

$$\sum_{a\in S} r_a < \sum_{i\in R(S)} \nu(E_i) \leqslant \nu(\Uparrow S).$$

Thus $\sum_{a\in A} r_a\delta_a \ll \nu$ by Proposition 17. $\square$

Next we give an alternate characterization of the way-below relation on $\mathbf{P}D$. This is a slight generalization of the Splitting Lemma, and should be seen as dual to Proposition 19.

Proposition 20 *Suppose $\sum_{a\in A} r_a\delta_a$ and ν are valuations on D and $\{E_i\}_{i\in I} \subseteq \mathcal{F}D$ is a partition of D refining $\{\mathrm{Cr}(A,S)\}_{S\subseteq A}$. Then $\sum_{a\in A} r_a\delta_a \ll \nu$ iff there exists a family of 'transport numbers' $\{t_{a,i}\}_{a\in A, i\in I}$ where*

(1) For each $a \in A$, $\sum_{i\in I} t_{a,i} = r_a$.
(2) For each $i \in I$, $\sum_{a\in A} t_{a,i} < \nu(E_i)$.
(3) $t_{a,i} > 0$ implies $E_i \subseteq \Uparrow a$.

PROOF. ($\Leftarrow$) Given the existence of a family of transport numbers $\{t_{a,i}\}$, define $R \subseteq A \times I$ by $R(a,i)$ iff $t_{a,i} > 0$. Then R satisfies (1) and (2) in Proposition 19.

($\Rightarrow$) By Proposition 19 there exists a relation $R \subseteq A \times I$ satisfying conditions (1) and (2) thereof. The proof that such a relation yields transport numbers as required uses the max-flow min-cut theorem from graph theory. The basic idea is due to Jones [11], but we refer the reader to the formulation of Heckmann [9, Lemma 2.7] which is general enough to apply to the present setting. $\square$

6.1 Splittings as Stochastic Relations

Next we define a composition of two splittings with a common index set. This is nothing but (the discrete case of) composition in the category of stochastic relations considered in [1].

Definition 21 *Suppose $u = \{u_{a,b}\}_{a\in A, b\in B}$ and $v = \{v_{b,c}\}_{b\in B, c\in C}$ are families of non-negative real numbers, where A, B and C are finite. Assuming that $\sum_{c\in C} v_{b,c} > 0$ for each $b \in B$, we define $u \mathbin{⨟} v$ to be an $(A \times C)$-indexed family where*

$$(u \mathbin{⨟} v)_{a,c} = \sum_{b\in B} u_{a,b} \left(\frac{v_{b,c}}{\sum_{c'\in C} v_{b,c'}} \right).$$

Furthermore, we define u^{-1} to be the $(B \times A)$-indexed family $(u^{-1})_{b,a} = u_{a,b}$.

The idea that one can compose splittings leads to the following question. Suppose $\sigma = \sum_{a\in A} r_a\delta_a$, $\nu = \sum_{b\in B} s_b\delta_b$ and $\rho = \sum_{c\in C} t_c\delta_c$ are simple valuations with $\sigma \ll \nu \ll \rho$. If $u = \{u_{a,b}\}$ is a splitting between σ and ν, and $v = \{v_{b,c}\}$ is a splitting between ν and ρ, then is $u \mathbin{⨟} v$ a splitting between σ and ρ? (That is, does $u \mathbin{⨟} v$ satsify conditions (1-3) in Lemma 16?) The following proposition answers this question in the affirmative.

Proposition 22 *Let u and v be as above. Then for each $a \in A$,*

$$\sum_{c\in C} (u \fatsemi v)_{a,c} = \sum_{b\in B} u_{a,b}\,. \tag{2}$$

Furthermore, if $\sum_{a\in A} u_{a,b} < \sum_{c\in C} v_{b,c}$ for each $b \in B$, it follows that

$$\sum_{a\in A} (u \fatsemi v)_{a,c} < \sum_{b\in B} v_{b,c} \tag{3}$$

for each $c \in C$.

PROOF. Simple algebra. $\square$

7 Measuring the Probabilistic Powerdomain

Until now, all of the concrete instances of measurement that we have considered have been maps into $[0,\infty)^*$. Henceforth we consider measurements into $[0,1]$. There is no loss of generality here, since $[0,\infty)^*$ can be order-embedded in $[0,1]$. We used $[0,\infty)^*$ in the preceding sections since this choice is both simpler and more conventional (see [15]). However, for the extension of a measurement to the probabilistic powerdomain it is more convenient to use $[0,1]$. Note that the condition MP is generic. The specialization to a measurement $m\colon D \to [0,1]$ says that whenever $a \ll b \in D$, then there exists $\varepsilon > 0$ such that $m_{1-\varepsilon}(\uparrow b \cap \mathsf{ker}\, m) \subseteq \twoheaduparrow a$.

Definition 23 *If $m\colon D \to [0,1]$ is a measurement on a continuous dcpo D, then we define $M\colon \mathbf{P}D \to [0,1]$ by $M(\nu) = \int m\, d\nu$.*

The Scott continuity of M follows directly from the continuity of the integral. In particular, we have that

$$M(\nu) = \sup\{\textstyle\sum_{i=1}^{n} r_i m(a_i) : \sum_{i=1}^{n} r_i \delta_{a_i} \ll \nu\}.$$

The next few propositions describe the kernel of M. It is worth remarking that in proving Proposition 24 we do not assume that valuations on continuous dcpos extend to measures.

Proposition 24 *Let $\nu \in \mathsf{ker}\, M$, i.e., $\int m\, d\nu = 1$. Then for a crescent $E = U \setminus V$, where $U, V \in \Sigma D$, we have that $\nu(E) > 0$ implies $E \cap \mathsf{ker}\, m \neq \emptyset$.*

PROOF. We construct an increasing sequence $\langle x_n | n \in \mathbb{N}\rangle$ in E with $m(x_n) \geqslant n/(n+1)$. It follows that $\bigsqcup x_n \in E \cap \mathsf{ker}\, m$.

Firstly, since $\nu|_E$ is a non-zero valuation on D, we may choose $x_1 \in E$ such that $\nu|_E(\Uparrow x_1) > 0$. Thus, defining $E_1 = E \cap \Uparrow x_1$, we have $\nu(E_1) > 0$.

Next, assume x_n has been defined such that $E_n = E \cap \Uparrow x_n$ has $\nu(E_n) > 0$. Let $\rho = \frac{1}{\nu(E_n)}\nu|_{E_n}$. Since

$$\nu = \nu|_{E_n} + \nu|_{E_n^c},$$

the inequality $M(\nu|_{E_n^c}) \leqslant \nu(E_n^c)$ forces $M(\nu|_{E_n}) = \nu(E_n)$, whence $M(\rho) = 1$.

We may choose a simple valuation $\sigma \ll \rho$ such that $M(\sigma) > n/(n+1)$. Thus there exists $y \in D$ (namely one of the mass points of σ) such that $m(y) \geqslant n/(n+1)$ and $\nu(E_n \cap \Uparrow y) > 0$. Now pick $x_{n+1} \in E_n \cap \Uparrow y$ such that $\nu(E_n \cap \Uparrow x_{n+1}) > 0$. $\square$

Proposition 25 *Let $\nu \in \mathsf{ker}\, M$. If $U_1, U_2 \in \Sigma D$ with $U_1 \cap \mathsf{ker}\, m = U_2 \cap \mathsf{ker}\, m$, then $\nu(U_1) = \nu(U_2)$.*

PROOF. Since neither of the crescents $U_1 \setminus U_2$ and $U_2 \setminus U_1$ meets $\mathsf{ker}\, m$ it follows that

$$\begin{aligned}
\nu(U_1) &= \nu(U_1 \cap U_2) + \nu(U_1 \setminus U_2) \\
&= \nu(U_1 \cap U_2) \quad \text{(by Proposition 24)} \\
&= \nu(U_1 \cap U_2) + \nu(U_2 \setminus U_1) \quad \text{(by Proposition 24)} \\
&= \nu(U_2).
\end{aligned}$$

$\square$

Theorem 26 *The space of normalized valuations on $\mathsf{ker}\, m$ in the weak topology is homeomorphic to $\mathsf{ker}\, M$ equipped with the relative Scott topology.*

PROOF. Suppose ν is a normalized valuation on $\mathsf{ker}\, m$. Then we easily see that $\nu^* \colon \Sigma D \to [0,1]$ defined by $\nu^*(O) = \nu(O \cap \mathsf{ker}\, m)$ is a valuation on ΣD. For all positive integers n, since

$$\nu^*(\{x : m(x) > n/(n+1)\}) = \nu(\mathsf{ker}\, m) = 1,$$

$M(\nu^*) \geqslant n/(n+1)$. Thus $\nu^* \in \mathsf{ker}\, M$.

Conversely, suppose $\nu \in \mathsf{ker}\, M$. We define a valuation ν_* on the open sets of $\mathsf{ker}\, m$ as follows. For an open set $O \subseteq \mathsf{ker}\, m$ we define $\nu_*(O) = \nu(O^\dagger)$ where $O^\dagger$ is the greatest Scott open subset of D such that $O^\dagger \cap \mathsf{ker}\, m = O$. Now for all open subsets O_1, O_2 of $\mathsf{ker}\, m$,

$$\begin{aligned}
\nu_*(O_1 \cup O_2) + \nu_*(O_1 \cap O_2) &= \nu((O_1 \cup O_2)^\dagger) + \nu((O_1 \cap O_2)^\dagger) \\
&= \nu(O_1^\dagger \cup O_2^\dagger) + \nu(O_1^\dagger \cap O_2^\dagger) \quad \text{(by Proposition 25)} \\
&= \nu(O_1^\dagger) + \nu(O_2^\dagger) \quad \text{(by modularity of } \nu\text{)} \\
&= \nu_*(O_1) + \nu_*(O_2).
\end{aligned}$$

Thus ν_* is modular. By similar reasoning it also follows that ν_* is Scott continuous. One easily sees that the maps $\nu \mapsto \nu^*$ and $\nu \mapsto \nu_*$ are inverse.

Recall that a net $\langle \nu_i \rangle$ of normalized valuations on $\mathsf{ker}\, m$ converges to ν in the weak topology iff $\liminf \nu_i(O) \geqslant \nu(O)$ for all open $O \subseteq \mathsf{ker}\, m$. Using Proposition 4 it is routine to show that the bijection above is a homeomorphism. □

Corollary 27 *If m satisfies MP and D is an ω-continuous dcpo, then the space of normalized Borel measures on $\mathsf{ker}\, m$ in the weak topology is homeomorphic to $\mathsf{ker}\, M$ in the relative Scott topology.*

PROOF. Since an MP-measurement is a Lebesgue measurement, $\mathsf{ker}\, m$ is a separable metric space by Theorem 10. In this case, as we remarked earlier, valuations and Borel measures are in one-to-one correspondence. □

We now begin the build-up to our main result, Theorem 30, showing that M is a measurement. In particular, it is this result which allows us to conclude that $\mathsf{ker}\, M \subseteq \mathsf{max}\, \mathbf{P}D$. The following proposition and lemma contain most of the work involved in proving Theorem 30.

Proposition 28 *Let $\rho = \sum_{c \in C} t_c \delta_c$ and $\varepsilon > 0$ be such that $M(\rho) > 1 - \varepsilon$. If $C' = \{c \in C : m(c) > 1 - \sqrt{\varepsilon}\}$, then $\rho' = \sum_{c \in C'} t_c \delta_c$ satisfies $M(\rho') > 1 - 2\sqrt{\varepsilon}$.*

PROOF. From $\sum_{t \in C} t_c(1 - m(c)) \leqslant 1 - M(\rho) < \varepsilon$, it follows that

$$M(\rho) - M(\rho') \leqslant \sum_{c \in C \setminus C'} t_c \leqslant \sum_{c \in C \setminus C'} \frac{t_c(1 - m(c))}{\sqrt{\varepsilon}} < \sqrt{\varepsilon}\,.$$

□

Lemma 29 *Suppose $m \colon D \to [0,1]$ is an MP-measurement and $M : \mathbf{P}D \to [0,1]$ is the map given in Definition 23. Let $\nu \in \mathsf{ker} M$ and $\sigma = \sum_{a \in A} r_a \delta_a \ll \nu$. Then there exists $\varepsilon > 0$ such that whenever $\rho = \sum_{c \in C} t_c \delta_c \sqsubseteq \nu$ and $M(\rho) > 1 - \varepsilon$, then $\sigma \ll \rho$.*

PROOF. Let $\lambda = \sum_{b \in B} s_b \delta_b$ be such that $\sigma \ll \lambda \ll \nu$ and let $u = \{u_{a,b}\}$ be a splitting between σ and λ as in Lemma 16. Applying Proposition 20 with

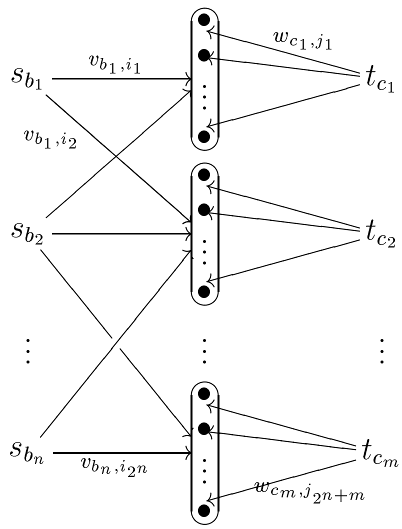

Fig. 2.

the partition $\{E_i\}_{i\in I}$, where $I = \mathbb{P}B$ and $E_i = \mathrm{Cr}(B, i)$, we obtain a splitting $v = \{v_{b,i}\}$ between λ and ν.

We now define $\varepsilon > 0$ in terms of u and v. First we choose $\varepsilon_1 > 0$ to be a lower bound on the unfulfilled demand at each of the sink nodes for the $v_{b,i}$:

$$\varepsilon_1 = \min_{i\in I}(\nu(E_i) - \textstyle\sum_{b\in B} v_{b,i}). \tag{4}$$

Since m satisfies (MP) we may choose $\varepsilon_2 > 0$ such that for all $a \in A$ and $b \in B$ with $a \ll b$ it holds that $m_{1-\varepsilon_2}(\uparrow b \cap \mathsf{ker}\, m) \subseteq \twoheaduparrow a$. Now we define $\varepsilon > 0$ by $\varepsilon = \min(\varepsilon_1, \varepsilon_2)^2/4$.

Suppose we are given $\rho = \sum_{c\in C} t_c\delta_c \sqsubseteq \nu$ with $M(\rho) > 1 - \varepsilon$. Let $C' \subseteq C$ and $\rho' \sqsubseteq \rho$ be as in Proposition 28. In particular, we have $m(c) > 1 - \varepsilon_2$ for each $c \in C'$, and $M(\rho') > 1 - \varepsilon_1$.

Applying Proposition 20 once again, with the partition $\{F_j\}_{j\in J}$, where $J = \mathbb{P}(B \cup C')$ and $F_j = \mathrm{Cr}(B \cup C', j)$, we obtain a splitting $w = \{w_{c,j}\}$ between ρ' and ν. Notice that the partition $\{F_j\}$ refines $\{E_i\}$. We write $j \equiv i$ whenever $j \cap B = i$, so $E_i = \bigcup_{j\equiv i} F_j$. We illustrate the splittings v and w in the network flow diagram in Figure 2. Roughly speaking, we would like obtain a splitting between σ and ρ' as $u \mathbin{⨾} v \mathbin{⨾} w^{-1}$. Notice, however, that w^{-1} and v are not even composable as it stands: they do not have a common index set. We first have to amalgamate the flow numbers $w_{c,j}$ which go into the same group of circled nodes. Formally, we define the $(C' \times I)$-indexed set $\bar{w}$ by $\bar{w}_{c,i} = \sum_{j\equiv i} w_{c,j}$.

We claim that $u \mathbin{⨾} v \mathbin{⨾} \bar{w}^{-1}$ defines a splitting between σ and ρ' in the sense of Lemma 16, so that $\sigma \ll \rho' \sqsubseteq \rho$. We verify condition 3 of the lemma as follows.

$$
\begin{aligned}
(u \fatsemi v \fatsemi \bar{w}^{-1})_{a,c} > 0 &\Rightarrow (\exists i)(\exists b)(u_{a,b} > 0 \;\&\; v_{b,i} > 0 \;\&\; \bar{w}_{c,i} > 0) && (5)\\
&\Rightarrow (\exists i)(\exists j \equiv i)(\exists b)(u_{a,b} > 0 \;\&\; v_{b,i} > 0 \;\&\; w_{c,j} > 0) && (6)\\
&\Rightarrow (\exists i)(\exists j \equiv i)(\exists b)(a \ll b \;\&\; E_i \subseteq \Uparrow b \;\&\; F_j \subseteq \Uparrow c). && (7)
\end{aligned}
$$

Now $w_{c,j} > 0$ in (6) implies $\nu(F_j) > 0$ (cf. Proposition 20(2)). Thus, by Proposition 24, there exists $z \in F_j \cap \mathsf{ker}\, m$. Also, from (7), we have that $a \ll b \sqsubseteq z$ and $c \sqsubseteq z$. Since $z \in \mathsf{ker} m$ and $m(c) > 1 - \varepsilon_2$, from $m_{1-\varepsilon_2}(\uparrow b \cap \mathsf{ker} m) \subseteq \Uparrow a$ it follows that $a \ll c$.

To verify the condition in Lemma 16(1), observe that two applications of Proposition 22 yield

$$\sum_{c \in C'} (u \fatsemi v \fatsemi \bar{w}^{-1})_{a,c} = \sum_{i \in I} (u \fatsemi v)_{a,i} = \sum_{b \in B} u_{a,b} = r_a\,.$$

It remains to verify the condition in Lemma 16(2). Now

$$\sum_{j \in J} \left(\nu(F_j) - \sum_{c \in C'} w_{c,j} \right) = \nu(D) - \rho'(D) \leqslant 1 - M(\rho') < \varepsilon_1\,. \tag{8}$$

The parenthesized term above is always positive. Thus, for each $i \in I$, taking the partial sum in (8) over those $j \in J$ with $j \equiv i$, we get

$$\nu(E_i) - \sum_{c \in C'} \bar{w}_{c,i} < \varepsilon_1\,.$$

From the definition of ε_1 in (4) it follows that for each $i \in I$,

$$\sum_{b \in B} v_{b,i} < \sum_{c \in C'} \bar{w}_{c,i}.$$

Now two applications of Proposition 22 yields $\sum_{a \in A} (u \fatsemi v \fatsemi \bar{w}^{-1})_{a,c} < \sum_{i \in I} \bar{w}^{-1}_{i,c} = t_c$ for each $c \in C'$. $\Box$

Having proved Lemma 29, the result that M is a measurement follows from general domain theory.

Theorem 30 *Suppose m is an MP-measurement on a domain D, and let M be the extension of m to a Scott continuous map $\mathbf{P}D \to [0,1]$ as given in Definition 23. Then M is a measurement.*

PROOF. Let $\nu \in \mathsf{ker}\, M$ and $\sigma \ll \nu$. We have to show that there exists $\varepsilon > 0$ such that whenever $\rho \sqsubseteq \nu$ and $M(\rho) > 1 - \varepsilon$, then $\sigma \ll \rho$.

By the interpolation property of $\ll$ there exists a simple valuation σ' with $\sigma \ll \sigma' \ll \nu$. By Lemma 29 there exists $\varepsilon > 0$ such that whenever $\rho' \sqsubseteq \nu$ is simple and $M(\rho') > 1-\varepsilon$, then $\sigma' \ll \rho'$. But if $\rho \sqsubseteq \nu$ is an arbitrary valuation with $M(\rho) > 1-\varepsilon$, then there is a simple valuation $\rho' \ll \rho$ with $M(\rho') > 1-\varepsilon$. Thus $\sigma \ll \sigma' \ll \rho' \ll \rho$. $\square$

Corollary 31 *If D is an ω-continuous dcpo whose Scott and Lawson topologies agree at the top, then the space of normalized Borel measures on $\mathsf{max}\, D$ in the weak topology embeds as a subspace of $\mathsf{max}\, \mathbf{P}D$.*

PROOF. The Lawson topology is metrizable in any ω-continuous dcpo, so $\mathsf{max}\, D$ is metrizable. Since valuations and Borel measures are in one-to-one correspondence in any separable metric space, it suffices to prove the result above with valuations in place of measures.

By Theorem 14, D admits an MP-measurement m with $\mathsf{ker}\, m = \mathsf{max}\, D$. Since the extension $M : \mathbf{P}D \to [0,1]$ is a measurement, it follows that $\mathsf{ker} M \subseteq \mathsf{max} D$. Now the result directly follows from Corollary 27. $\square$

This result is also implied by [6, Corollary 4.1]. The proof of that corollary uses the fact that valuations on $\mathbf{P}D$ extend uniquely to measures. In [18] we prove a more general version of Corollary 31, assuming only that D is an ω-continuous dcpo with $\mathsf{max}\, D$ regular. This depends on a more general version of Theorem 30. Here we avoid the extra generality in order to give a less technical presentation.

8 Iterated Function Systems

Definition 32 *An* iterated function system *(IFS) on a complete metric space X is a collection of continuous maps $f_i : X \to X$ indexed over a finite set I. Such an IFS is denoted $\langle X, \{f_i\}_{i\in I}\rangle$. If each map f_i is contracting, then the IFS is said to be* hyperbolic.

A hyperbolic IFS induces a contraction F on the complete metric space of non-empty compact subsets of X equipped with the Hausdorff metric. F is defined by

$$F(K) = \bigcup_{i \in I} f_i(K).$$

By Banach's contraction mapping theorem, F has a unique fixed point: the *attractor* of the IFS. An alternate domain-theoretic proof this result, due to Hayashi [8], involves considering F as a continuous selfmap of $\mathbf{CU}X$ and deducing that the least fixed point of F is maximal in $\mathbf{CU}X$, and therefore is a unique fixed point. Many different fractal sets arise as, or can be approximated by, attractors of IFSs.

Definition 33 *A* weighted IFS $\langle X, \{(f_i, p_i)\}_{i\in I}\rangle$ *consists of an IFS* $\langle X, \{f_i\}_{i\in I}\rangle$ *and a family of weights* $0 < p_i < 1$*, where* $\sum_{i\in I} p_i = 1$*. These data induce a so-called Markov operator* $G\colon \mathcal{M}X \to \mathcal{M}X$ *on the set* $\mathcal{M}X$ *of normalized Borel measures on* X*, given by*

$$G(\nu)(B) = \sum_{i\in I} p_i \nu(f_i^{-1}(B)) \tag{9}$$

for each Borel subset $B \subseteq X$*.*

The space $\mathcal{M}X$ equipped with the weak topology can be metrized by the *Hutchinson metric* [10]. Furthermore, if a weighted IFS is hyperbolic then the map G is contracting with respect to the Hutchinson metric. In this case the unique fixed point of G, obtained by the contraction mapping theorem, defines a normalized measure called an *invariant measure* for the IFS. The support of the invariant measure is the attractor of the underlying IFS. This construction is an important method of defining fractal measures. Next we outline a domain-theoretic construction, due to Edalat [4], of invariant measures for so-called weakly hyperbolic IFSs on compact metric spaces.

Edalat's approach involves embedding the set of measures on a compact metric space X in the domain $\mathbf{PU}X$ of valuations on the upper space of X. Recall from Section 4 that $\mathbf{U}X$ admits an MP-measurement $m\colon \mathbf{U}X \to [0,1]$, where $m(K) = 2^{-|K|}$; in turn this yields a measurement M on $\mathbf{PU}X$. Next, a weighted IFS $\langle X, \{(f_i, p_i)\}_{i\in I}\rangle$ induces a continuous map $T\colon \mathbf{PU}X \to \mathbf{PU}X$ – the domain theoretic analogue of the Markov operator – defined by

$$T(\nu)(O) = \sum_{i\in I} p_i \nu((\mathbf{U}f_i)^{-1}(O)) \tag{10}$$

where $\mathbf{U}f_i\colon \mathbf{U}X \to \mathbf{U}X$ is the map $K \mapsto f_i(K)$.

Applying T to δ_X, the point valuation concentrated at $X \in \mathbf{U}X$, one obtains $T(\delta_X) = \sum_{i\in I} p_i \delta_{f_i(X)}$. Iterating, it follows that

$$T^n(\delta_X) = \sum_{i_1,\dots,i_n\in I} p_{i_1}\dots p_{i_n}\delta_{f_{i_1}\dots f_{i_n}(X)}. \tag{11}$$

Thus $MT^n(\delta_X)$, the measurement of the n-th iterate, equals

$$\sum_{i_1,\dots,i_n\in I} p_{i_1}\dots p_{i_n} m(f_{i_1}\dots f_{i_n}(X)).$$

To ensure that $MT^n(\delta_X) \to 1$ as $n \to \infty$ it is sufficient to require that for all $\varepsilon > 0$, there exists $n \geqslant 0$ such that $|f_{i_1} \dots f_{i_n}(X)| < \varepsilon$ for all sequences $i_1 i_2 ... i_n \in I^n$. In fact, by König's lemma, it is sufficient that for each infinite sequence $i_1 i_2 ... \in I^\omega$, $|f_{i_1} \dots f_{i_n}(X)| \to 0$ an $n \to \infty$. Edalat calls an IFS satisfying the latter condition *weakly hyperbolic*. It is clearly the case that every hyperbolic IFS is weakly hyperbolic.

Theorem 34 (Edalat [4]) *A weakly hyperbolic weighted IFS $\langle X, \{(f_i, p_i)\}_{i \in I}\rangle$ on a compact metric space X has a unique invariant measure which is moreover an attractor for the Markov operator (9).*

PROOF. Every valuation on a compact metric space extends to a Borel measure, and conversely every Borel measure restricts to a valuation. Thus, to prove the existence of a unique invariant measure, it suffices to prove that there is a unique valuation ν on X such that $\nu(O) = \sum_{i \in I} p_i \nu(f_i^{-1}(O))$ for all open $O \subseteq X$.

Let D be the sub-dcpo of $\mathbf{PU}X$ consisting of valuations with mass 1. Then D is pointed and continuous, and T restricts to a monotone map $D \to D$. Thus we may apply Theorem 8 to deduce that T has a unique fixed point on D, and this point lies in $\mathsf{ker}\, M$.

By an obvious identification of X with $\mathsf{ker}\, m$ we may regard the Markov operator G, defined in (9), as a selfmap of the set of valuations on $\mathsf{ker}\, m$. Next we show that G, so regarded, agrees with T. Formally, if $O \subseteq \mathbf{U}X$ is Scott open, then, using the notation of Theorem 26, we have

$$\begin{aligned}
G(\nu_*)^*(O) &= G(\nu_*)(O \cap \mathsf{ker}\, m) \\
&= \sum_{i \in I} p_i \nu_*(f_i^{-1}(O \cap \mathsf{ker}\, m)) \\
&= \sum_{i \in I} p_i \nu((\mathbf{U} f_i)^{-1}(O)) \quad (f_i^{-1}(O \cap \mathsf{ker}\, m) = (\mathbf{U} f_i)^{-1}(O) \cap \mathsf{ker}\, m) \\
&= T(\nu)(O).
\end{aligned}$$

Since $T = G$ on $\mathsf{ker}\, m$, we know that the unique fixed point of T is a unique invariant measure. Furthermore, it also follows that T takes $\mathsf{ker}\, M$ into $\mathsf{ker}\, M$, and so, by Theorem 8, the fixed point of T is an attractor for T in the relative Scott topology on $\mathsf{ker}\, M$. But $\mathsf{ker}\, M \simeq \mathcal{M}X$, so the invariant measure for G is also an attractor in the weak topology. $\square$

The construction of the unique invariant measure here is essentially the same as in Edalat [4]. However it is justified in a different way. Edalat deduces that the least fixed point of T is a unique fixed point by proving that it is maximal. This observation depends on a characterization of the maximal elements of $\mathbf{PU}X$ in terms of their supports. This last requires some more measure-theoretic machinery than we have used here: in particular he uses the result of Lawson [13] on extending valuations on ω-continuous dcpos to Borel measures over the Lawson topology.

9 Summary and Future Work

We introduced the class of MP-measurements: a strict subclass of the Lebesgue measurements from [19]. We showed that the natural measurements on the upper space and formal ball models are MP-measurements, and that any domain which is an MP-hull in the sense of Lawson [14] admits an MP-measurement.

Our main result, Theorem 30, showed that an MP-measurement $m\colon D \to [0,1]$ extends in a natural way to a measurement $M\colon \mathbf{P}D \to [0,1]$ on the probabilistic powerdomain. As an application of this result we showed how Edalat's domain theoretic construction of unique invariant measures for IFS's can be justified by standard results about measurements.

Martin [19] proves that the requirement that $m\colon D \to [0,1]$ be a Lebesgue measurement is both necessary and sufficient for the natural extension of M to the convex powerdomain $\mathbf{C}D$ to define a measurement. The corresponding result does not hold for MP-measurements and the probabilistic powerdomain: there are measurements which do not satisfy MP and yet extend to the probabilistic powerdomain. For example, it turns out that the measurement in Example 15, which is Lebesgue but not MP, extends to the probabilistic powerdomain. The question of obtaining a necessary condition remains open.

An interesting problem is to characterize the maximal elements of the probabilistic powerdomain. In particular, for an MP-measurement m with $\mathsf{ker}\, m = \mathsf{max}\, D$, do we have $\mathsf{ker}\, M = \mathsf{max}\, \mathbf{P}D$?

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