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Monoids Over Domains

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Dedicated to Klaus Keimel on the occasion of his 65th birthday

In this paper, we describe three distinct monoids over domains, each with a commutative analog, the latter of which define bag domain monoids. Our results were inspired by work by Varacca (Varacca 2003), and they lead to a constructive approach to his *Hoare indexed valuations* over a continuous poset P. We use our constructive approach to describe an analog of the probabilistic power domain and the laws which characterize it that forms a Scott-closed subset of Varacca's construct. These we call the *Hoare random variables over* P.

1. Introduction

Adding probabilistic choice as an operator has long been a goal of those working in process algebras. Despite numerous attempts—(Morgan *et al* 1994; Lowe 1993) to cite only two—it has been difficult to identify an approach to modeling probabilistic choice that melds well with the established operators in process algebra – nondeterministic choice has proved to be particularly difficult, as has the hiding operator of *CSP* (Roscoe 1997), the theory of *communicating sequential processes* first proposed by C. A. R. Hoare. The approach to adding probabilistic choice to *CSP* taken in (Morgan *et al* 1994) has been one of the most successful to date, but it requires sacrificing the idempotence of nondeterministic choice. Some progress towards a general denotational model supporting both forms of choice and retaining the idempotence of nondeterminism (an approach hinted at in (Morgan *et al* 1994)) was presented in (Mislove 2000), with a more elaborate mathematical presentation in (Tix 1999); a limited operational justification of the model was presented in (Mislove, Ouaknine and Worrell). In this approach, one first forms the probabilistic power domain over a domain, then applies one of the standard nondeterministic power domains, finally extracting the order-convex and geometrically convex subsets to achieve the model. Even though both forms of choice exist in the model and all the

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expected laws for each are retained, the model itself has proved less than persuasive in practice. In particular, the resulting model imposes a relation between probabilistic choice and nondeterministic choice; e.g., in the case of the upper power domain, the nondeterministic choice of two processes is below any probabilistic choice of these processes. So even though the constructions are monadic, they introduce new inequations not originally specified in the construction.

Another approach to the interaction of nondeterminism and probabilistic choice was taken up by Varacca (Varacca 2003), who sought a model by weakening the laws for probabilistic choice. The motivation was a result of Plotkin and Varacca that showed there is no distributive law between the probabilistic power domain monad and any of the standard nondeterministic choice power domain monads; this result implies that the composition of the associated monads would not yield a monad. Varacca called his construction, "indexed valuations" because they distinguish simple valuations with the same support according to the number of times a given point mass is used and the individual assignments of mass to each point. Varacca was able to show that the indexed valuation monad analogous to each power domain monad enjoys a distributive law over that power domain monad. This means the composition forms a monad, resulting in a model that supports both his version of probabilistic choice and the analogous version of nondeterministic choice. However, his construction proceeds by writing down an abstract basis for each of his constructs in terms of a basis for the underlying domain, and then imposing identifications between the resulting basis elements. This makes it difficult to unravel the constructions and to penetrate the internal structure of the model. On the other hand, Varacca does establish equational characterizations for his constructs.

In this paper we present a construction of one of Varacca's models from first principles, showing how it can be built up incrementally. This allows a better understanding of the structure of the model. One of the results we obtain using this approach is an analog of the probabilistic power domain that arises as a Scott-closed subset of his construction. We call this construct the *Hoare random variables* over the continuous poset P, and we provide a characterization of the construct in terms of the inequations they satisfy.

Our approach to Varacca's indexed valuations is via monoids and especially commutative monoids over posets. It is well known that the commutative monoid over a set can be obtained as a quotient of the free semigroup of words over the set by the family of symmetric groups, $S(n)$, where $S(n)$ acts on the set of n-letter words by permuting the letters. What has not been realized before is that this same construction can be applied to ordered sets, and to domains. Indeed, we show that the free commutative poset (continuous poset, dcpo, domain) monoid over a poset (continuous poset, dcpo, domain) is obtained in the same way. We also show that, in the continuous case, the way-below relation on the commutative monoid is the quotient of the way-below on the monoid. We also discover that, in analogy to the three free ordered semilattices over a poset, there are three ordered commutative monoids over a poset, and each of these constructs extends to continuous posets, dcpos, and domains. We also investigate the closure of various cartesian closed categories of domains under the formation of n-bags, the family of bags having n members. Ultimately, though, our results about commutative monoids are aimed at providing an alternative approach to constructing Varacca's indexed valuations, but as we show in the case we focus on – that of the Hoare indexed valuations – there is a novel, added twist from domain theory that enters the picture.

The rest of the paper is organized as follows. In the next section we recall some basic facts about domains. In the following section, we present new results about ordered monoids, including commutative ordered monoids over various categories of dcpos and domains. Next we outline Varacca's construction of indexed valuations, and recall his main results. Next come the main results of the paper. We use our results about free commutative monoids over domains, extended to include the action of the nonnegative reals, to derive an alternate presentation of Varacca's Hoare indexed valuations construction. Having recaptured his construction, we then single out the Hoare random variables over a domain P, which are an analog of the probabilistic power domain over P, and we characterize them by the inequations they satisfy. We close the paper with some ideas about further work.

2. Preliminaries

In this section we recall some basic definitions we will find useful. A standard reference for this material is (Keimel, *et al* 2003) or (Abramsky and Jung 1994). To begin, a *partially ordered set* or *poset* for short, is a set equipped with a reflexive, antisymmetric and transitive relation. We let Pos denote that category of posets and monotone maps.

A subset $A \subseteq P$ of a partially ordered set is *directed* if each finite subset of A has an upper bound *in* A. A poset in which every directed subset has a least upper bound is called *directed complete* or a *dcpo* for short. A *cpo* is a dcpo that also has a least element.

If P is a poset and $x, y \in P$, then we write $x \ll y$, and say x *is way-below* y if for every directed subset $A \subseteq P$, if $\sqcup A$ exists and $y \leq \sqcup A$, then $x \leq a$ for some $a \in A$. P is a *continuous poset* if $\Downarrow y = \{x \in P \mid x \ll y\}$ is directed and $y = \sqcup \Downarrow y$ holds for all $y \in P$. A continuous dcpo is called a *domain*.

An *abstract basis* is a pair $(P, \ll)$ where $\ll$ is a transitive relation on P satisfying the *interpolation property:*

$$F \ll x \ \&\ F \subseteq P \text{ finite } \Rightarrow (\exists y \in P)\ F \ll y \ll x.$$

By $F \ll x$ we mean $z \ll x\ \forall z \in F$. If $(P, \ll)$ is an abstract basis, then $I \subseteq P$ is a *round ideal* if I is a directed $\ll$-lower set, and $x \in I \Rightarrow (\exists y \in I)\ x \ll y$. The *round-ideal completion* of an abstract basis $(P, \ll)$ is the family of round ideals, under inclusion. This forms a domain.

If $f \colon P \to Q$ is a monotone map between posets, then f is *Scott continuous* if for all $A \subseteq P$ directed for which $\sqcup A$ exists, $f(\sqcup A) = \sqcup f(A)$. (Note that the monotonicity of f implies that $f(A)$ is directed if A is.) We let DCPO denote the category of dcpos and Scott-continuous maps, and CPO the subcategory of cpos and *strict maps:* ones that preserve least elements. We also let ConPos denote the category of continuous posets and Scott-continuous maps, and DOM the full subcategory of domains.

One of the fundamental results about dcpos is that the family of Scott continuous maps between two dcpos is another dcpo in the pointwise order. Since it's easy to show that the finite product of a family of dcpos or continuous posets is another such, and the one-point poset is a terminal object in each of the relevant categories, a central question is which categories of dcpos or domains are cartesian closed. This is true for DCPO, and there are several categories of domains and Scott continuous maps between them that are ccc's. These include:

— RB *domains*, which are retracts of *bifinite domains*, themselves limits of families of finite posets under embedding-projection pairs of maps. Bifinite domains can also be described as those domains P for which the identity 1_P is the directed supremum of a family $\{f_k\}_{k\in K} \subseteq \mathsf{DCPO}[P,P]$ satisfying $f_k(P)$ is finite and $f_k \circ f_k = f_k$, while the identity map of an RB domain is the directed supremum of a family $\{f_k\}_{k\in K} \subseteq \mathsf{DCPO}[P,P]$ satisfying $f_k(P)$ is finite.

— FS *domains*, those domains D satisfying the property that the identity map is the directed supremum of selfmaps $f\colon D \to D$ each *finitely separated* from the identity: i.e., for each selfmap f there is a finite subset $M_f \subseteq D$ with the property that, for each $x \in D$, there is some $m \in M_f$ with $f(x) \leq m \leq x$.

The category FS clearly contains RB and it is known to be a maximal ccc of domains. Containing both of these is the category Coh of *coherent domains,* whose objects are compact in the so-called *Lawson topology.* This category is not cartesian closed, but it plays a central role in the theory, especially for the probabilistic power domain.

We also recall some facts about categories; more details can be found in (Mac Lane 1969). A *monad* or *triple* on a category A is a a 3-tuple $\langle T,\mu,\eta\rangle$ where $T\colon \mathsf{A}\to\mathsf{A}$ is an endofunctor, and $\mu\colon T^2 \stackrel{\cdot}{\longrightarrow} T$ and $\eta\colon 1_{\mathsf{A}} \stackrel{\cdot}{\longrightarrow} T$ are natural transformations satisfying the laws:

$$\mu\circ T\mu = \mu\circ\mu_T \quad\text{and}\quad \mu\circ\eta_T = T = \mu\circ T\eta.$$

Equivalently, if $F\colon \mathsf{A}\to\mathsf{B}$ is left adjoint to $G\colon \mathsf{B}\to\mathsf{A}$ with unit $\eta\colon 1_{\mathsf{A}} \stackrel{\cdot}{\longrightarrow} GF$ and counit $\epsilon\colon FG \stackrel{\cdot}{\longrightarrow} 1_{\mathsf{B}}$, then $\langle GF, G\epsilon F, \eta\rangle$ forms a monad on A, and every monad arises in this fashion.

If $\langle T,\mu,\eta\rangle$ is a monad, then a T*-algebra* is a pair (a,h), where $a\in\mathsf{A}$ and $h\colon Ta\to a$ is an A-morphism satisfying $h\circ\eta_a = 1_a$ and $h\circ Th = h\circ\mu_a$.

For example, each of a power domains, $\mathcal{P}_L, \mathcal{P}_U$ and $\mathcal{P}_C$ define monads on DCPO (cf. (Hennessy and Plotkin 1979)), whose algebras are ordered semilattices; another example is the *probabilistic power domain* $\mathbb{V}$ whose algebras satisfy equations that characterize the probability measures over P (cf. (Jones 1989)).

One of the principle impetuses for Varacca's work was to find a model supporting both nondeterministic choice and probabilistic choice, so that the laws characterizing each of these constructs hold. To accomplish this, one needs to combine the appropriate nondeterminism monad with the probabilistic power domain monad, so that the laws of each constructor are preserved in the resulting model. This can be done using a *distributive law*, which is a natural transformation $d\colon ST \stackrel{\cdot}{\longrightarrow} TS$ between monads S and T on A satisfying several identities—cf. (Beck 1969). The significance of distributive laws is the following theorem of Beck:

Theorem 2.1 (Beck 1969). Let (T,η^T,μ^T) and (S,η^S,μ^S) be monads on the category *A*. Then there is a one-to-one correspondence between

(i) Distributive laws $d : ST \stackrel{\cdot}{\longrightarrow} TS$;
(ii) Multiplications $\mu : TSTS \stackrel{\cdot}{\longrightarrow} TS$, satisfying
— $(TS, \eta^T\eta^S, \mu)$ is a monad;
— the natural transformations $\eta^T S : S \stackrel{\cdot}{\longrightarrow} TS$ and $T\eta^S : T \stackrel{\cdot}{\longrightarrow} TS$ are monad morphisms;
— the following *middle unit law* holds:

$$\begin{tikzcd} TS \arrow[r, "T\eta^S\eta^T S"] \arrow[dr, "Id_{TS}"'] & TSTS \arrow[d, "\mu"] \\ & TS \end{tikzcd}$$

(iii) Liftings $\tilde{T}$ of the monad T to A^S, the category of S-algebras in A.

So, one way to know that the combination of the probabilistic power domain and one of the power domains for nondeterminism provides a model satisfying all the needed laws would be to show there is a distributive law of one of these monads over the probabilistic power domain monad. Unfortunately, it was shown by Plotkin and Varacca (Varacca 2003) that there is no distributive law of $\mathbb{V}$ over $\mathcal{P}_X$, or of $\mathcal{P}_X$ over $\mathbb{V}$ for any of the nondeterminism monads $\mathcal{P}_X$. However, Varacca discovered that weakening one of the laws for probabilistic choice would allow him to find such a distributive law for the resulting constructs and the nondeterminism monads. We return to this point near the end of Section 4.

3. Ordered monoids

In this section we present some results about monoids over posets, dcpos and domains. We begin with the following:

Definition 3.1. Let $P_{\mathbb{N}} \mathrel{\widehat{=}} \dot{\bigcup}_{n\geq 0} P^n$ denote the union of the finite powers of the poset P. For $p \in P_{\mathbb{N}}$, we let $|p| = n$ iff $p \in P^n$, and for $i \leq |p|$, we let p_i denote the i^{th} component of p. We also use ϵ to denote the empty tuple, the sole member of P^0.

We define an order on $P_{\mathbb{N}}$ by $p \sqsubseteq_C q$ iff $|p| = |q|$ and $p_i \sqsubseteq q_i$ for all $i \leq |p|$. This defines an ordered monoid over P where

$$p * q = r \quad \text{with} \quad |r| = |p| + |q| \text{ and } r_k = \begin{cases} p_k & \text{if } k \leq |p|, \\ q_k & \text{if } k > |p|. \end{cases}$$

Proposition 3.1. The mapping $P \mapsto P_{\mathbb{N}}$ is the object level of an endofunctor on each of the categories POS, ConPos, DCPO and DOM, and in each case it is left adjoint to the forgetful functor from the category of ordered monoid posets (dcpos, continuous posets, domains) to the underlying category.

Proof. It is well known that each of the indicated categories is closed under finite products, and from this it is easy to show that $P_{\mathbb{N}}$ is in POS (respectively, ConPos, DCPO, Dom) if P is. It also is routine to show that $*$ is Scott continuous, and the initiality of $P_{\mathbb{N}}$ is straightforward to show. $\square$

We call $\sqsubseteq_C$ the *Convex Order* on $P_{\mathbb{N}}$. Actually, $\sqsubseteq_C$ is but one of three ordered monoids one can define over posets and dcpos. We note that in the following, we often identify a natural number $n \in \mathbb{N}$ with the set $\{0, \ldots, n-1\}$; here are the other two orders:

Definition 3.2. Let P be a poset, and let $P_{\mathbb{N}} \mathrel{\widehat{=}} \bigcup_{n\geq 0} P^n$. We define the partial orders

Lower Order:

$$p \sqsubseteq_L q \text{ iff } \begin{cases} p = \epsilon, \text{ the empty word, or} \\ (\exists \text{ monotone } f\colon k \subseteq |q| \twoheadrightarrow |p|)\ p_{f(j)} \sqsubseteq q_j \quad (\forall j \in k). \end{cases}$$

where $f\colon k \subseteq |q| \twoheadrightarrow |p|$ denotes a surjective map from a subset $k \subseteq |q|$ onto $|p|$.

Upper Order:

$$p \sqsubseteq_U q \text{ iff } \begin{cases} q = \epsilon, \text{ the empty word, or} \\ (\exists \text{ monotone } g\colon k \subseteq |p| \twoheadrightarrow |q|)\ p_i \sqsubseteq q_{g(i)} \quad (\forall i \in k). \end{cases}$$

Proposition 3.2.

(i) The Lower Order is a partial order on $P_{\mathbb{N}}$. Concatenation is a Scott-continuous monoid operation wrt $\sqsubseteq_L$ satisfying $p, q \sqsubseteq_L p * q$. Moreover, if P is continuous, then so is $(P_{\mathbb{N}}, \sqsubseteq_L)$.

(ii) The Upper Order is a partial order on $P_{\mathbb{N}}$. Concatenation is a Scott-continuous monoid operation were $\sqsubseteq_U$ satisfying $p * q \sqsubseteq_U p, q$. If P is a dcpo or domain, then so is $P_{\mathbb{N}}$.

Proof. It is routine to show that both the Lower Order and the Upper Order are partial orders, that $p * q \sqsubseteq_U p, q \sqsubseteq_L p * q$ and that $*\colon P_{\mathbb{N}} \times P_{\mathbb{N}} \to P_{\mathbb{N}}$ is monotone with respect to both orders. We also note that, since the only monotone map $f\colon k \subseteq m \twoheadrightarrow m$ is the identity, it follows that $\sqsubseteq_L |_{P^n}$ is the usual product order on P^n, and the same is true of $\sqsubseteq_U |_{P^n}$.

To show the Scott continuity of $*$ on $(P_{\mathbb{N}}, \sqsubseteq_L)$, we first investigate how to compute suprema of directed sets, when they exist. Let $A \subseteq P_{\mathbb{N}}$ be $\sqsubseteq_L$-directed and suppose A has an upper bound $x \in P_{\mathbb{N}}$. Clearly $a \in A$ implies $|a| \leq |x|$. So, $m = \max\{n \in \mathbb{N} \mid A \cap P^n \neq \emptyset\}$ exists, and we let $A_0 = A \cap P^m$ and let $b \in A_0$. If $a \in A$, then since A is directed, there is $c \in A$ with $a, b \sqsubseteq_L c$, and so $|a|, |b| \leq |c|$, which implies $|c| = m$. It follows that A_0 is directed and cofinal in A. This implies $\sqcup A = \sqcup A_0$ if either of these suprema exists. But since $\sqsubseteq_L |_{P^m}$ is the usual product order, we know $\sqcup A_0$ exists in P^m, and so the same is true of $\sqcup A$.

Now, we know $*\colon P_{\mathbb{N}} \times P_{\mathbb{N}} \to P_{\mathbb{N}}$ is monotone with respect to $\sqsubseteq_L$, and we also know $*\colon P^m \times P^n \to P^{m+n}$ is Scott continuous by Proposition 3.1. Since we know $\sqcup A = \sqcup A_0$ and the latter is computed in P^{m_0}, these results imply $*\colon P_{\mathbb{N}} \times P_{\mathbb{N}} \to P_{\mathbb{N}}$ is $\sqsubseteq_L$-continuous.

Next we investigate $\ll$ on $(P_{\mathbb{N}}, \sqsubseteq_L)$. Suppose that $p \sqsubseteq_L \sqcup A \in P^m$. Then we know that $\sqcup A$ is (eventually) computed coordinatewise in P^m, and so $(\sqcup A_0)_j = \sqcup\{a_j \mid a \in A_0\}$ for

$j \le m$. Since $p \sqsubseteq_L \sqcup A_0$, there is some monotone $f\colon k \subseteq m \twoheadrightarrow |p|$ satisfying $p_{f(i)} \sqsubseteq (\sqcup A_o)_i$ for each $i \le k$. If we choose $q_i \ll p_i$ for each $i \le n$, then $q_{f(i)} \ll (\sqcup A_o)_i$ for each $i \le k$. Since A_0 is directed, we can find $a(i) \in A_0$ with $q_{f(i)} \sqsubseteq a(i)_i$ for each $i \le k$. Since k is finite and A_0 is directed, it follows that there is some $a \in A_0$ with $q_{f(i)} \sqsubseteq a_i$ for all $i \le k$, which implies $q \sqsubseteq_L a$. This implies $q \ll_L p$.

From the preceding paragraph we conclude that, if P is continuous, then $\Downarrow_{P^n} p$ is cofinal in $\Downarrow_{\sqsubseteq_L} p$, so $(P_{\mathbb{N}}, \sqsubseteq_L)$ is continuous.

The arguments for the Upper Order are similar to those for the Lower Order, and, in particular, we have $(P_{\mathbb{N}}, \sqsubseteq_U)$ is continuous if P is, and $\Downarrow_{P^n} p$ is cofinal in $\Downarrow_{\sqsubseteq_U} p$ in this case. The last point – that $(P_{\mathbb{N}}, \sqsubseteq_U)$ is a dcpo if P is one follows from the definition of the order: indeed, the definition of the order implies that $|\ |\colon P_{\mathbb{N}} \to \mathbb{N}$ is antitone, so if $A \subseteq P_{\mathbb{N}}$ is $\sqsubseteq_U$-directed, then there is some n with $A \cap P^n$ is cofinal in A. And, as with the Lower Order, $\sqsubseteq_U |_{P_n}$ is the usual product order on P^n, from which the result follows. $\square$

Theorem 3.1. Each of the assignments $P \mapsto (P_{\mathbb{N}}, \sqsubseteq_C)$, $P \mapsto (P_{\mathbb{N}}, \sqsubseteq_L)$, and $P \mapsto (P_{\mathbb{N}}, \sqsubseteq_U)$ defines the object level of an endofunctor on Pos whose image categories are ordered monoids and ordered monoids satisfying $p, q \sqsubseteq_l p * q$, and ordered monoids satisfying $p * q \sqsubseteq_U p, q$, respectively. In fact, each gives rise to a left adjoint to the inclusion functor.

Moreover, ConPos is invariant under the lower order endofunctor, while each of ConPos, DCPO and DOM are invariant under the upper order endofunctor. In each of these cases, we have a left adjoint to the forgetful functor from the appropriate category of ordered monoids and continuous ordered monoid morphisms.

Proof. Consider the case of the lower order, $\sqsubseteq_L$. Let $(S, \sqsubseteq_S, *)$ be an ordered monoid satisfying $s, t \sqsubseteq_S s * t$ for all $s, t \in S$, and suppose that $g\colon P \to S$ is monotone. Define $G\colon P_{\mathbb{N}} \to S$ by $G(\epsilon) = 1_S$, and $G(p) = g(p_1) * \cdots * g(p_{|p|})$ if $n > 0$. If $p \sqsubseteq q \in P_{\mathbb{N}}$, then there is some $f\colon k \subseteq |q| \twoheadrightarrow |p|$ with $p_{f(j)} \sqsubseteq q_j$ for every $j \in k$. Then $g(p_{f(j)}) \sqsubseteq_S g(q_j)$ for each $j \in k$, so $G(p) = g(p_1) * \cdots * g(p_{|p|}) \sqsubseteq_S g(q_{j_1}) * \cdots * g(q_{j_k}) \sqsubseteq_S g(q_1) * \cdots * g(q_{|q|}) = G(q)$, so G is monotone. It's obvious that G preserves the multiplication on $P_{\mathbb{N}}$, and that G is unique.

Now suppose that $g\colon P \to S$ is Scott continuous and that S is an ordered monoid for which $*\colon S \times S \to S$ is continuous. If $A \subseteq P_{\mathbb{N}}$ is directed and $\sqcup A$ exists, then there is some $n \in \mathbb{N}$ for which $A_0 \cap P^n$ is cofinal in A and $\sqcup A = \sqcup A_0$. But then we can restrict G to P^n, and the order $\sqsubseteq_L$ restricted to P^n is simply the product order. Now $g^n\colon P^n \to S^n$ is continuous from the continuity of g, and then $(*^{n-1} \circ g)\colon P^n \to S$ is a composition of continuous maps, and hence also is continuous. So, $G(\sqcup A) = G(\sqcup A_0) = (*^{n-1} \circ g^n)(\sqcup A_0) = \sqcup((*^{n-1} \circ g^n)(A_0)) = \sqcup G(A_0)$. Thus G is continuous.

Similar arguments apply in the other cases. $\square$

Remark 3.1. The names for each of these orders are inspired by the results from (Hennessy and Plotkin 1979) where the three power domain monads were first presented. As we shall see in the next section, each of these ordered monoids has a commutative version, which is closer still to the semilattices defined in (Hennessy and Plotkin 1979).

In the case of the convex order, we also have a result about pointed domains.

Corollary 3.1. Let $P_{\mathbb{N}\perp}$ denote the dcpo $P_{\mathbb{N}}$ *lifted* (i.e., with a least element added), and define $\preceq$ to be the extension of $\sqsubseteq_C$ on $P_{\mathbb{N}\perp}$ so that $\perp \preceq p$ for all p. Moreover, extend $*$ from $P_{\mathbb{N}}$ to $P_{\mathbb{N}\perp}$ by $\perp * p = p * \perp = \perp$ Then $(P_{\mathbb{N}\perp}, \preceq, *, \epsilon)$ is the object level of the left adjoint to the forgetful functor from the category of ordered monoids (resp., monoid cpos, monoid cpo domains) with least element a zero and strict maps.

Proof. The result is straightforward to show, given Proposition 3.1. □

Remark 3.2. It is an unpublished result of Gordon Plotkin's that the free cpo monoid over a poset has a much more complicated structure; in particular, none of the cartesian closed categories of continuous domains is closed under its formation.

3.1. *Bag domains*

A *bag* or *multiset* is a collection of objects in which the same object can appear more than once. The term "bag" stems from the analogy with shopping, where one can place several copies of the same item in the bag; once objects are placed in the bag, the order in which they were placed there is irrelevant. Bags are determined by the objects that are in them, with only the number of copies of an object being important: two bags are equal if they have the same number of copies of each object either contains. A *bag domain* is a domain that also is a bag or multiset of objects from an underlying domain. The question is how to order such an object so that it is again a domain. The key to this is to realize bag domains as free commutative monoids,

The investigation of bag domains originated in the work of Vickers (Vickers 1992), and also have been considered by Johnstone (Johnstone 1992; Johnstone 1994). Those works were inspired by problems arising in database theory, and the goals of their work was to capture the abstract categorical nature of the construction. Here we present results along the same line, but we provide a more direct construction, since it allows us to analyze the internal structure of the objects more closely. It also allows us to capture the constructions of Varacca (Varacca 2003) more concretely and to understand better their internal structure.[†]

Definition 3.3. Let P be a poset, let $n \in \mathbb{N}$ and let $S(n)$ denote the permutation group of n.

For $\phi \in S(n)$, define a mapping $\phi\colon P^n \to P^n$ by $\phi(d)_i = d_{\phi^{-1}(i)}$. Then ϕ permutes the components of d according to ϕ's permutation of the indices $1 = 1, \ldots, n$.

Next, define a preorder $\preceq_n$ on P^n by

$$d \preceq_n e \quad \text{iff} \quad (\exists \phi \in S(n))\ \phi(d) \sqsubseteq e \quad \text{iff} \quad d_{\phi^{-1}(i)} \sqsubseteq e_i\ (\forall i = 1 \ldots, n). \tag{1}$$

Finally, we define the equivalence relation $\equiv$ on P^n by

$$\equiv\ =\ \preceq_n \cap (\preceq_n)^{-1}. \tag{2}$$

[†] Only as this paper was going to press did the author learn of (Heckmann 1995) which considers issues very close to those investigated in this section.

We also define $\sqsubseteq_n = \preceq_n /\equiv$ and we note that $(P^n/\equiv, \sqsubseteq_n)$ is a partial order. We denote by $[d]$ the image of $d \in P^n$ in $P^n/\equiv$.

Lemma 3.1. Let P be a poset, let $n \in \mathbb{N}$, and let $d, e \in P^n$. Then the following are equivalent:

(i) $[d] \sqsubseteq_n [e]$ in $P^n/\equiv$,
(ii) $(\exists \phi \in S(n))(\forall i = 1, \ldots, n)\ d_i \sqsubseteq e_{\phi(i)}$, for $i = 1, \ldots, n$.
(iii) $\uparrow\{\phi(d) \mid \phi \in S(n)\} \supseteq \uparrow\{\phi(e) \mid \phi \in S(n)\}$.

Proof. For (i) implies (ii), we note that, if $\phi \in S(n)$ satisfies $d_{\phi^{-1}(i)} \sqsubseteq e_i$, then $d_i \sqsubseteq e_{\phi(i)}$, for each $i = 1, \ldots, n$, so (ii) holds. Next, (ii) implies $\phi^{-1}(e) \in \uparrow d$, and then $\psi(e) \in \uparrow\{(\phi(d) \mid \phi \in S(n)\}$ for each $\psi \in S(n)$ by composing permutations, from which (iii) follows. Finally, (iii) implies (i) is clear. □

We also need a classic result due to M.-E. Rudin (Keimel, *et al* 2003, Lemma III-3.3)

Lemma 3.2 (Rudin). Let P be a poset and let $\{\uparrow F_i \mid i \in I\}$ be a filter basis of non-empty, finitely generated upper sets. Then there is a directed subset $A \subseteq \cup_i F_i$ with $A \cap F_i \neq \emptyset$ for all $i \in I$. □

Next, let P be a dcpo and let $n > 0$. We can apply Rudin's Lemma to derive the following:

Proposition 3.3. Let P be a poset, and let $n > 0$.

— If $A \subseteq P^n/\equiv$ is directed, then there is a directed subset $B \subseteq \bigcup_{[a]\in A}\{\phi(a) \mid \phi \in S(n)\} \subseteq P^n$ satisfying

$$\bigcap_{b\in B} \uparrow\{\phi(b) \mid \phi \in S(n)\} = \bigcap_{[a]\in A} \uparrow\{\phi(a) \mid \phi \in S(n)\} \tag{3}$$

and if $\sqcup B$ exists, then so does $\sqcup A$, in which case $[\sqcup B] = \sqcup A$.

— In particular, the mapping $x \mapsto [x]\colon P^n \to P^n/\equiv$ is Scott continuous, and $(P^n/\equiv, \sqsubseteq_n)$ is a dcpo if P is one.

Proof. We first show the claim about directed subsets of $A \subseteq P^n/\equiv$ and $B \subseteq P^n$. Indeed, if $A \subseteq P^n/\equiv$ is directed, then Lemma 3.1 implies that $\{\cup_{\phi\in S(n)} \uparrow \phi(a) \mid [a] \in A\}$ is a filter basis of finitely generated upper sets, and so by Lemma 3.2 there is a directed set $B \subseteq \bigcup_{[a]\in A}\{\phi(a) \mid \phi \in S(n)\}$ with $B \cap \{\phi(a) \mid \phi \in S(n)\} \neq \emptyset$ for each $[a] \in A$.

Now, let $x \in \bigcap_{b\in B} \uparrow\{\phi(b) \mid \phi \in S(n)\}$. If $[a] \in A$, then $B \cap \{\phi(a) \mid \phi \in S(n)\} \neq \emptyset$ means there is some $\phi \in S(n)$ with $\phi(a) \in B$, so $\phi(a) \sqsubseteq x$. Hence $x \in \bigcap_{[a]\in A} \uparrow\{\phi(a) \mid [a] \in A\}$.

Conversely, if $x \in \bigcap_{[a]\in A} \uparrow\{\phi(a) \mid \phi \in S(n)\}$, then for $b \in B$, $[b] \in A$, so $x \in \uparrow\{\phi(b) \mid \phi \in S(n)\}$. This shows Equation 3 holds.

We now show the claims about $\sqcup B$ and $\sqcup A$. Suppose $x = \sqcup B$ exists. If $[a] \in A$, then $B \cap \{\phi(a) \mid \phi \in S(n)\} \neq \emptyset$ means there is some $\phi \in S(n)$ with $\phi(a) \in B$, so $\phi(a) \sqsubseteq x$ by Lemma 3.1. Hence $[a] \sqsubseteq_n [x]$ for each $[a] \in A$, so $[x]$ is an upper bound for A.

We also note that, since $\sqcup B = x$,

$$\bigcap_{b\in B} \uparrow\{\phi(b) \mid \phi \in S(n)\} = \uparrow\{\phi(x) \mid \phi \in S(n)\}.$$

Indeed, the right hand side is clearly contained in the left hand side since $b \sqsubseteq x$ for all $b \in B$. On the other hand, if y is in the left hand side, then $b \sqsubseteq y$ for each $b \in B$. Now, since $S(n)$ is finite, there is some $\phi \in S(n)$ and some cofinal subset $B' \subseteq B$ with $\phi(b) \sqsubseteq y$ for each $b \in B'$. But then $\sqcup B' = \sqcup B$, and so $\sqcup\{\phi(b) \mid b \in B'\} = \phi(x)$, from which we conclude that $\phi(x) \sqsubseteq y$. Thus y is in the right hand side, so the sets are equal.

Now, if $y \in P^n$ satisfies $[a] \sqsubseteq_n [y]$ for each $[a] \in A$, then since $B \subseteq \bigcup_{[a]\in A}\{\phi(a) \mid \phi \in S(n)\}$, it follows that $[b] \sqsubseteq_n [y]$ for each $b \in B$. Then $y \in \bigcap_{b\in B} \uparrow\{\phi(b) \mid \phi \in S(n)\} = \uparrow\{\phi(x) \mid \phi \in S(n)\}$, and so $[x] \sqsubseteq_n [y]$. Thus $[x] = \sqcup A$ in $P^n/\equiv$, which concludes the proof of the claims about $\sqcup B$ and $\sqcup A$.

Finally, for the second itemized claim, what we have just proved shows that directed sets $B \subseteq P^n$ satisfy $[\sqcup B] = \sqcup_{b\in B}[b]$, which means the quotient map is Scott continuous. Moreover, the argument also shows that $P^n/\equiv$ is a dcpo if P is one. □

Proposition 3.4. Let P be a domain and let $n \in \mathbb{N}$. Then

(i) $(P^n/\equiv, \sqsubseteq_n)$ is a domain.
(ii) If P is RB, respectively, FS, then so is $P^n/\equiv$.
(iii) If P is coherent, then so is $P^n/\equiv$.

Proof. *$P^n/\equiv$ is a domain:* Proposition 3.3 shows that $(P^n/\equiv, \sqsubseteq_n)$ is directed complete. To characterize the way-below relation on $P^n/\equiv$, let $x, y \in P^n$ with $x \ll y$. Then $x_i \ll y_i$ for each $i = 1, \ldots, n$, and it follows that $\phi(x) \ll \phi(y)$ for each $\phi \in S(n)$. If $A \subseteq P^n/\equiv$ is directed and $[y] \sqsubseteq_n \sqcup A$, then there is some $\phi \in S(n)$ with $\phi(y) \sqsubseteq z$, where $[z] = \sqcup A$. Then Proposition 3.3 shows there is a directed set $B \subseteq \cup_{[a]\in A} \uparrow\{\phi(a) \mid \phi \in S(n)\}$ with $\sqcup B \equiv z$. Hence, there is some $\psi \in S(n)$ with $\psi(y) \sqsubseteq \sqcup B$. Since $\psi(x) \ll \psi(y)$, it follows that there is some $b \in B$ with $\psi(x) \sqsubseteq b$, so $[x] \sqsubseteq_n [b]$. Hence $[x] \ll [y]$ in $P^n/\equiv$.

We have just shown that $x \ll y$ in P^n implies that $[x] \ll [y]$ in $P^n/\equiv$. Since P^n is a domain, $\twoheaddownarrow y$ is directed with $y = \sqcup \twoheaddownarrow y$, and so the same is true for $\twoheaddownarrow [y] \in P^n/\equiv$. Thus $P^n/\equiv$ is a domain.

$P^n/\equiv$ is RB if P is: Now suppose the P is in RB. Then, by Theorem 4.1 of (Jung 1989) there is a directed family $f_k \colon P \to P$ of Scott continuous maps with $1_P = \sqcup_k f_k$ and $f_k(P)$ finite for each $k \in K$. Then the mappings $(f_k)^n \colon P^n \to P^n$ also form such a family, showing P^n is in RB.

Next, given $k \in K, x \in P^n$ and $\phi \in S(n)$, we have $\phi(f_k^n(x)) = f_k^n(\phi(x))$ since f_k^n is f_k acting on each component of x. It follows that there is an induced map $[f_k^n] \colon P^n/\equiv \to P^n/\equiv$ satisfying $[f_k^n]([x]) = [f_k^n(x)]$, and this map is continuous since $[\ ]$ is a quotient map. Finally, $[f_k^n](P^n/\equiv)$ is finite since $f_k^n(P^n)$ is finite, and that $\sqcup_k [f_k^n] = 1_{P^n/\equiv}$ follows from $\sqcup_k f_k^n = 1_{P^n}$. Thus, $P^n/\equiv$ is RB is P is.

$P^n/\equiv$ is FS if P is: The domain P is FS if there is a directed family of selfmaps $f_k \colon P \to P$ satisfying $\sqcup_k f_k = 1_P$ and for each $k \in K$, there is some finite $M_k \subseteq P$ with $f_k(x) \sqsubseteq m_x \sqsubseteq x$ for some $m_x \in M_k$, for each $x \in P$. As in the case of RB, the mappings $[f_k^n]$ are a directed family of continuous selfmaps of $P^n/\equiv$ whose supremum is the identity, and the subset $[M_k^n]$ is finite and separates $[f_k^n]$ from the identity for each $k \in K$. It follows that $P^n/\equiv$ also are FS domains if P is one.

$P^n/\equiv$ is coherent if P is: Last, we consider coherent domains. Recall a domain is

coherent if the Lawson topology is compact, where the Lawson topology has for a basis the family $\{U \setminus \uparrow F \mid F \subseteq P \text{ finite } \& \ U \text{ Scott open}\}$. Now, if $x \in P^n$, then $\{\phi(x) \mid \phi \in S(n)\}$ is finite, and so if $F \subseteq P^n/\!\equiv$ is finite, then $[\uparrow F]^{-1} = \cup_{[x]\in F} \uparrow\{\phi(x) \mid \phi \in S(n)\}$ is finitely generated. It follows that $[\ ]\colon P^n \to P^n/\!\equiv$ is Lawson continuous, so if P is coherent, then so are P^n and $P^n/\!\equiv$. $\square$

3.2. *Bag domain monoids*

We now investigate commutative monoids over domains, which we call bag domain monoids. This also requires considering how to relate bags of different cardinalities. As we found for the case of ordered monoids, there are three possible ways to do this.

Definition 3.4. Let P be a poset and let $P_{\mathbb{N}}$ denote the disjoint sum of the P^n. We regard $P_{\mathbb{N}}$ as a poset in the convex order defined earlier. We also recall the rank function $|\ |\colon P_{\mathbb{N}} \to \mathbb{N}$ by $|d| = n$ if and only if $d \in P^n$. We now define three "commutative" pre-orders on $P_{\mathbb{N}}$. Let $d, e \in P_{\mathbb{N}}$.

Commutative lower order: Define

$$d \preceq_{CL} e \quad \text{iff} \quad (\exists f\colon k \subseteq |e| \twoheadrightarrow |d|)\ d_{f(i)} \sqsubseteq e_i,\ i \in k.$$

We let $\equiv_L \, = \, \sqsubseteq_{CL} \cap \sqsubseteq_{CL}^{-1}$ and $\sqsubseteq_{CL} = \preceq_{CL}/\!\equiv$.

Commutative upper order: Define

$$d \preceq_{CU} e \quad \text{iff} \quad (\exists f\colon k \subseteq |d| \twoheadrightarrow |e|)\ d_i \sqsubseteq e_{f(i)},\ i \in k.$$

We let $\equiv_U \, = \, \sqsubseteq_{CU} \cap \sqsubseteq_{CU}^{-1}$ and $\sqsubseteq_{CU} = \preceq_{CU}/\!\equiv$.

Commutative convex order: Define

$$d \preceq_{CC} e \quad \text{iff} \quad |d| = |e| \ \& \ (\exists \phi \in S(n))\ d_{\phi(i)} \sqsubseteq e_i,\ i = 1, \dots, |d|.$$

We let $\equiv_C \, = \, \sqsubseteq_{CC} \cap \sqsubseteq_{CC}^{-1}$ and $\sqsubseteq_{CC} = \preceq_{CC}/\!\equiv$.

Remark 3.3.

— Note that in the above definition, the functions $f\colon k \subseteq |e| \twoheadrightarrow |d|$ are not required to be monotone. This is a reflection of commutativity of the operation of concatenation operation.

— We call these preorders and their associated partial orders *commutative* because they define partial orders on $P_{\mathbb{N}}$ relative to which concatenation is a commutative monoid operation. These three orders are inspired by the work Varacca (Varacca 2003), who in turn was inspired by the results of Hennessy and Plotkin (Hennessy and Plotkin 1979).

Lemma 3.3. Let P be a poset. Then

$$\equiv_L = \equiv_U = \equiv_C = \{(p, q) \mid |p| = |q| \ \& \ p \equiv_{|p|} q\}.$$

Proof. If $p \sqsubseteq_L q \sqsubseteq_L p$, then $|p| \leq |q| \leq |p|$, so they are equal, and the hypothesized surjections $f\colon |q| \twoheadrightarrow |p|$ and $f'\colon |q| \twoheadrightarrow |p|$ are in fact bijections, and hence permutations. Hence, $p_{f(i)} = q_i$ for all $i \leq |q|$. A similar analysis applies to the other cases. $\square$

Notation We let $\equiv$ denote the equivalence relations $\equiv_L = \equiv_U = \equiv_C$.

Remark 3.4. We recall that for a continuous poset P, a *round ideal* of P is a directed lower set $I \subseteq P$ satisfying $x \in I \;\Rightarrow\; (\exists y \in I)\; x \ll y$. The *round ideal completion* of P is formed by taking the family $\mathrm{RId}(P) = \{I \subseteq P \mid I \textit{ round ideal}\}$ in the containment order; it is a standard result of domain theory that this family is a domain. This construction also can be realized topologically as the sobrification of P in the Scott topology. So, we also can denote the round-ideal completion of P by $\mathrm{Sob}(P)$.

Theorem 3.2. Let P be a dcpo.

(i) $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CU})$ and $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CC})$ are dcpos.

(ii) If P is continuous, then $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CL})$, $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CU})$ and $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CC})$ are continuous posets. Hence, if P is a domain, then so are $(P_{\mathbb{N}}/\equiv_U, \sqsubseteq_{CU})$ and $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CC})$.

(iii) If P is a continuous poset, then $\mathrm{Sob}(P_{\mathbb{N}}, \sqsubseteq_{CL})$, $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CU})$ and $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CC})$ each define the object level of a left adjoint of the forgetful functor from the category of commutative ordered monoid domains satisfying the appropriate inequation and Scott continuous monoid morphisms.

Proof. *(i):* To begin, note that $\sqsubseteq_{CL}, \sqsubseteq_{CU}$ and $\sqsubseteq_{CC}$ all yield $\sqsubseteq_n$ when restricted to P^n for any $n \in \mathbb{N}$. Since directed sets in $(P_{\mathbb{N}}, \sqsubseteq_{CC})$ are within $P^n/{\equiv_n}$ for some n, and since P^n is a dcpo, it follows that $(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CC})$ is one as well.

Now, suppose that $A \subseteq P_{\mathbb{N}}/\equiv$ is $\sqsubseteq_{CU}$-directed. Then the definition of the order implies there is some n with $A \cap P^n/{\equiv}$ cofinal in A. Since $P^n/{\equiv_n}$ is a dcpo, A_0 has a supremum in $P^n/{\equiv_n}$, and this is the supremum of A. Hence $(P_{\mathbb{N}}, \sqsubseteq_{CU})$ is a dcpo if P is one.

(ii): Suppose $A \subseteq P_{\mathbb{N}}$ is $\sqsubseteq_{CL}$-directed with a least upper bound, $\sqcup A$. Then $|[a]| \leq |\sqcup A|$ for each $[a] \in A$. Since $|\;\;|$ is clearly monotone, there is some $n \in \mathbb{N}$ and some $[a_0] \in A$ with $|[a]| = n$ for $[a_0] \sqsubseteq_{CL} [a]$. And since $P^n/{\equiv}$ is a dcpo, it follows that $\sqcup A \in P^n/{\equiv}$.

Now, suppose that $d, e \in P^n$ and that $[d] \ll [e]$ in $P^n/{\equiv}$ and let $A \subseteq P_{\mathbb{N}}$ is $\sqsubseteq_{CL}$-directed with $[e] \sqsubseteq_{CL} \sqcup A$. Then $|[e]| \leq |\sqcup A|$, and there is some $k \subseteq |\sqcup A|$, some $f\colon k \twoheadrightarrow |[e]|$ with $e_{f(i)} \sqsubseteq (\sqcup A)_i$ for $i \in k$. We can assume that $|[a]| = |\sqcup A|$ for each $[a] \in A$, and we know there is a directed set $B \subseteq \bigcup\{\phi(a) \mid [a] \in A\}$ with $[\sqcup B] = \sqcup A$. Then $d_{f(i)} \ll (\sqcup B)_i$ for each $i \in k$, and so there is some $b \in B$ with $d_{f(i)} \sqsubseteq b_i$ for each $i \in k$. It follows that $[d] \sqsubseteq_{CL} [b]$, and so $[d] \ll [e]$ in $P_{\mathbb{N}}$. Now $\twoheaddownarrow_{P^n/\equiv}[e]$ is directed and satisfies $\sqcup\twoheaddownarrow_{P^n/\equiv}[e] = [e]$ and since this is a subset of $\twoheaddownarrow_{P_{\mathbb{N}}}[e]$, it follows that $\twoheaddownarrow_{P_{\mathbb{N}}}[e]$ is directed and satisfies $[e] = \sqcup\twoheaddownarrow_{P_{\mathbb{N}}}[e]$. Since $[e] \in P_{\mathbb{N}}$ is arbitrary, it follows that $(P_{\mathbb{N}}, \sqsubseteq_{CL})$ is a continuous poset.

A similar argument applies to $(P_{\mathbb{N}}, \sqsubseteq_{CU})$, and since $(P_{\mathbb{N}}, \sqsubseteq_{CC})$ is a disjoint sum of continuous posets, it is a continuous poset.

The fact that $(P_{\mathbb{N}}, \sqsubseteq_{CC})$ and $(P_{\mathbb{N}}, \sqsubseteq_{CU})$ are domains if P is one now follows from (i).

(iii): The arguments here are analogous to those given in the proof of Theorem 3.1. $\square$

3.3. *Making ϵ the least element*

So far we have not mentioned cpos in the context of monoids over dcpos. If P is a cpo, then each component $P^n/{\equiv_n}$ has a least element, the tuple $[\perp]$ which has every entry $\perp$.

But $P_{\mathbb{N}}$ has no least element. An obvious way to create one is to identify the elements $[\bot] \in P^N/\equiv$ for all n – this works, and it is what is called the *coalesced sum* of the cpos $P^n/\equiv_n$. But we take another approach, which is to note that $P^0 = \{\epsilon\}$ has only one element, and we can refine the order on $P_{\mathbb{N}}$ so that this is the least element, and this has the effect of making $\bot_{P_{\mathbb{N}}}$ the identity for the monoid structure on $P_{\mathbb{N}}$. The fact that this defines a monad is the content of the following.

Proposition 3.5. Let P be a continuous poset. We define $P_{\mathbb{N}\,L}$ to be the domain $Sob(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CL})$ with

$$x \sqsubseteq y \quad \text{iff} \quad x = \epsilon \text{ or } x \sqsubseteq_{CL} y.$$

Then $P_{\mathbb{N}\,L}$ is a commutative domain monoid satisfying $x, y \sqsubseteq x * y$ and $\epsilon \sqsubseteq x$ for all $x, y \in P_{\mathbb{N}\,L}$. In fact, this defines the object level of a left adjoint to the forgetful functor from the category of commutative domain monoids satisfying these laws.

Proof. The element ϵ is both minimal and maximal in $Sob(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CL})$, from which it follows that $Sob(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CL}) \setminus \{\epsilon\}$ is a subdomain and also a Scott-closed subset of $Sob(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CL})$. The structure we have defined is the lift of $Sob(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CL}) \setminus \{\epsilon\}$ (cf. (Abramsky and Jung 1994)), which is again a domain, and in which we have extended the semigroup operation to make the least element an identity. A similar argument applies to the case of $P_{\mathbb{N}\,C}$. □

Remark 3.5.

— We could also define $P_{\mathbb{N}\,C}$ to be the domain $Sob(P_{\mathbb{N}}/\equiv, \sqsubseteq_{CC})$ with

$$x \sqsubseteq y \quad \text{iff} \quad x = \epsilon \text{ or } x \sqsubseteq_{CC} y.$$

Then $P_{\mathbb{N}\,C}$ is a commutative domain monoid satisfying $\epsilon \sqsubseteq x$ for all $x, y \in P_{\mathbb{N},C}$. But this also implies that $y = \epsilon * y \sqsubseteq x * y$, so we have the same theory as for $P_{\mathbb{N}\,L}$.

— The above construct fails in the case of the upper order because $\epsilon \sqsubseteq x$ and $x * y \sqsubseteq x, y$ would imply $x = x * \epsilon \sqsubseteq x, \epsilon$, collapsing the order. On the other hand, we could achieve a result if we were to define ϵ to be the largest element of the construction.

4. Indexed Valuations Over Domains

We now review Varacca's constructions from (Varacca 2003). Varacca was motived by the fact that there is no distributive law for the probabilistic power domain over any of the power domains for nondeterminism, which implies the composition of the probabilistic power domain monad and any of the monads for nondeterminism would not be a monad, so some law of one of the components would be broken by such a composition. However, he found that by weakening one of the laws of the probabilistic power domain—namely, the law

$$pA + (1-p)A = A \tag{4}$$

he could find monads that do satisfy a distributive law with the analogous power domain. We focus on his construction of the so-called *Hoare indexed valuations* over a domain, because this fits within our theory, and it also is the construction he exploits most extensively in his work. We show how to reconstruct this family using our theory of commutative monoid domains, and in the process we discover a remarkable construction relating two monads over domains.

4.1. *Hoare Indexed Valuations*

To begin, we recall Varacca's construction. First, an *indexed valuation* over the poset P is a tuple $x \in (\overline{\mathbb{R}_{\geq 0}} \times P)^n$ where $\pi_{\overline{\mathbb{R}_{\geq 0}}}(x) \geq 0$ is an extended, non-negative real number and $\pi_P(x) \in P$ for each $i \leq n$. If x is an indexed valuation, we let $|x| = n$ if $x \in (\overline{\mathbb{R}_{\geq 0}} \times P)^n$. Two indexed valuations x and y satisfy $x \simeq_1 y$ if $|x| = |y|$ and there is a permutation $\phi \in S(|x|)$ with $\pi_{\overline{\mathbb{R}_{\geq 0}}}(x)_{\phi(i)} = \pi_{\overline{\mathbb{R}_{\geq 0}}}(y)_i$ and $\pi_P(p)_{\phi(i)} = \pi_P(q)_i$ for each $i \leq |x|$. If we let $\overline{x}$ denote the subtuple of x consisting of only those pairs $(\pi_{\overline{\mathbb{R}_{\geq 0}}}(x)_i, \pi_P(x)_i)$ for which $\pi_{\overline{\mathbb{R}_{\geq 0}}}(x)_i \neq 0$, then $x \simeq_2 y$ if $\overline{x} \simeq_1 \overline{y}$. Varacca then identifies indexed valuations modulo the equivalence relation $\simeq$ generated by $\simeq_1 \cup \simeq_2$, so we let $\langle x\rangle$ denote the $\simeq$-equivalence class of $x \in \bigcup_{n\geq 0}(\overline{\mathbb{R}_{\geq 0}} \times P)^n$.

Next, for a domain P, Varacca defines a relation on $(\bigcup_{n\geq 0}(\overline{\mathbb{R}_{\geq 0}} \times P)^n)/{\simeq}$ by

$$\begin{aligned} \langle x\rangle \ll_L \langle y\rangle \qquad \text{iff} \qquad & (\exists f : k \subseteq |y| \twoheadrightarrow |x|) \\ & \left(\pi_{\overline{\mathbb{R}_{\geq 0}}}(x)_i = 0\right) \vee \left(\pi_{\overline{\mathbb{R}_{\geq 0}}}(x)_i < \sum_{f(j)=i} \pi_{\overline{\mathbb{R}_{\geq 0}}}(y)_j\right) \qquad (5) \\ & \& \ \pi_P(x)_{f(j)} \ll_P \pi_P(y)_j \ (\forall j \in k). \end{aligned}$$

Remark 4.1. Note that although the relation $\ll_L$ is defined on $(\bigcup_{n\geq 0}(\overline{\mathbb{R}_{\geq 0}} \times P)^n)/{\simeq}$, it actually involves representatives of the equivalence classes in this family. As Varacca points out, it should be read as saying, "*$\langle x\rangle \ll_L \langle y\rangle$ iff there are representatives of $\langle x\rangle$ and of $\langle y\rangle$" satisfying the condition (5).* Like Varacca, we have overloaded notation here by assuming that the representatives are x and y themselves. But regardless, the definition of (5) requires dealing with representatives of these equivalences classes, rather than with the equivalence classes themselves. We believe avoiding this is the main contribution of our approach to Varacca's construction.

Varacca's main result for the family of Hoare indexed valuations is the following:

Theorem 4.1 (Varacca 2003).

(i) If P is a continuous poset, then the family $(\bigcup_{n\geq 0}(\overline{\mathbb{R}_{\geq 0}} \times P)^n)/{\simeq}$ endowed with the relation $\ll_L$ as defined in (5) is an abstract basis. The family $IV_L(P)$, the domain of *lower indexed valuations*, is the round ideal completion of the lower indexed valuations, and it satisfies the following family of inequations:

(1)	$A \oplus B = B \oplus A$	(2)	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$
(3)	$A \oplus \underline{0} = A$	(4)	$0A = \underline{0}$
(5)	$1A = A$	(6)	$p(A \oplus B) = pA \oplus pB$
(7)	$p(qA) = (pq)A$	(HV)	$(p+q)A \sqsubseteq_L pA \oplus qA$,

where $p, q \in \mathbb{R}_+,\ A, B \in IV_L(P)$ and $\underline{0}$ denotes the equivalence class of ϵ, the empty word over $\overline{\mathbb{R}_{\geq 0}} \times P$.

(ii) The family of lower indexed valuations IV_L defines the object level of a functor which is monadic over Dom; the lower power domain monad satisfies a distributive law with respect to the lower indexed valuations monad. $\square$

A corollary of this result is that the composition $\mathcal{P}_L \circ IV_L$ defines a monad on Dom whose algebras satisfy the laws listed in Theorem 4.1 and the laws of the lower power domain:

(1)	$X * Y = Y * X$	(2)	$X * X = X$
(3)	$X * (Y * Z) = (X * Y) * Z$	(4)	$X, Y \sqsubseteq X * Y$

In other words, $\mathcal{P}_L(IV(P))$ is the initial sup-semilattice algebra over P that also satisfies the laws listed in Theorem 4.1.

4.2. *A special structure on* $\overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}$

To construct Varacca's lower indexed valuations using our approach, we begin with $(\overline{\mathbb{R}_{\geq 0}}, \leq)$ which is a commutative monoid satisfying $0 \leq r, s \leq r + s$. Recalling that $\overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}$ is $Sob(\overline{\mathbb{R}_{+}}_{\mathbb{N}}, \sqsubseteq_{LC})$ with ϵ made the least element, we have $\overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}$ is the initial such monoid over $\overline{\mathbb{R}_{+}}$ by Proposition 3.5. Since the identity $1_{\overline{\mathbb{R}_{+}}} \colon \overline{\mathbb{R}_{+}} \to \overline{\mathbb{R}_{\geq 0}}$ is continuous, it has a continuous monoid extension $\widehat{1_{\overline{\mathbb{R}_{+}}}} \colon \overline{\mathbb{R}_{+}}_{\mathbb{N}\, L} \to \overline{\mathbb{R}_{\geq 0}}$ by $\widehat{1_{\overline{\mathbb{R}_{+}}}}([r]) = \sum_{i \leq |r|} r_i$. We use this morphism to refine the order $\sqsubseteq_{L\,0}$ on $\overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}$ by

$$[r] \sqsubseteq_{+} [s] \ \text{ iff } \ [r] = [\epsilon] \ \vee \ (\exists f \colon k \subseteq |s| \twoheadrightarrow |r|)\ r_i \leq \sum_{f(j)=i} s_j \ \ \forall j \in k.$$

Lemma 4.1. $(\overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}, \sqsubseteq_{+})$ is a commutative monoid with $[\epsilon] \sqsubseteq_{+} [r], [s_j] \sqsubseteq_{+} [r] * [s]$ and a continuous poset with

$$[r] \ll_{+} [s] \ \text{ iff } \ [r] = [\epsilon] \ \vee \ (\exists f \colon k \subseteq |s| \twoheadrightarrow |r|)\ r_i < \sum_{f(j)=i} s_j \ \ \forall j \in k.$$

Proof. It's routine to check that $\sqsubseteq_{+}$ is a partial order, and it's important to note that $\sqsubseteq_{+} \cap (\overline{\mathbb{R}_{+}}^{m}/\!\equiv \times \overline{\mathbb{R}_{+}}^{m}/\!\equiv)$ is the quotient of the usual product order on $\overline{\mathbb{R}_{+}}^{m}$ for $m > 0$.

Now, to see that $\sqsubseteq_{+}$ makes $\overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}$ into a continuous poset, we proceed as in the proof of Theorem 3.2 to see how directed suprema are calculated. Indeed, it's clear that $[r] \sqsubseteq_{+} [s]$ implies $|r| \leq |s|$, so if $A \subseteq \overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}$ is directed and bounded, then there is some m_0 for which $A_0 \equiv A \cap \overline{\mathbb{R}_{+}}^{m_0}$ is cofinal in A. Then A_0 has a supremum in $\overline{\mathbb{R}_{+}}^{m_0}/\equiv$ by Proposition 3.3 and our comment above that the restriction of $\sqsubseteq_{+}$ to $\overline{\mathbb{R}_{+}}^{m_0}/\equiv$ is the quotient of the product order. The cofinality of A_0 in A implies this also is the supremum of A in $\overline{\mathbb{R}_{+}}_{\mathbb{N}\, L}$.

Next, we note that if $[r] \sqsubseteq_{+} \sqcup A = [x]$ for some directed set A, then, assuming $[r] \neq \underline{0}$, there is some $f \colon k \subseteq |x| \twoheadrightarrow |r|$ with $r_i \leq \sum_{f(j)=i} x_j$ for $j \in k$. Since $\sqsubseteq_{+} |_{\overline{\mathbb{R}_{+}}^{n}}$ is the quotient of the product order and since $+$ is continuous on $\overline{\mathbb{R}_{+}}$, if $[r'] \ll [r]$ in $\overline{\mathbb{R}_{+}}^{|r|}$, then there is some $[x'] \ll [x]$ in $\overline{\mathbb{R}_{+}}^{|x|}$and $r'_i < \sum_{f(j)=i} x'_j$. Then $[x] = \sqcup A$ implies there is some $a_0 \in A$ with $[x'] \leq a$ for $a_0 \leq a$, so $[r'] \sqsubseteq_{+} a_0$. It follows that the way-below relation on $\overline{\mathbb{R}_{+}}_{\mathbb{N}}/\equiv$ is generated by the way-below relations on $\overline{\mathbb{R}_{+}}^{n}/\equiv$ so that $\ll_{+}$ is given by

$$[r] \ll_+ [s] \;\text{ iff }\; [r] = [\epsilon] \;\vee\; (\exists f\colon k \subseteq |s| \twoheadrightarrow |r|)\; r_i < \sum_{f(j)=i} s_j \;\; \forall j \in k. \tag{6}$$

Thus, $\sqsubseteq_+$ defines a continuous partial order on $\overline{\mathbb{R}_+}_{\mathbb{N}\,L}$, so $(\overline{\mathbb{R}_+}_{\mathbb{N}\,L}, \sqsubseteq_+)$ is a domain.

It also is clear that $*\colon \overline{\mathbb{R}_+}_{\mathbb{N}\,L} \times \overline{\mathbb{R}_+}_{\mathbb{N}\,L} \to \overline{\mathbb{R}_+}_{\mathbb{N}\,L}$ is commutative and continuous, and that $[\epsilon] \sqsubseteq_+ [r], [s] \sqsubseteq_+ [r] * [s]$ and $[\epsilon] * [r] = [r]$ hold. $\square$

Theorem 4.2.

(i) The identity map $\mathrm{Id}\colon \mathit{Sob}(\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv, \sqsubseteq_{CL}) \to (\overline{\mathbb{R}_+}_{\mathbb{N}\,L}, \sqsubseteq_+)$ is Scott continuous, but is not an order isomorphism.

(ii) The mapping $\widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}}\colon (\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv, \sqsubseteq_+) \to \overline{\mathbb{R}_+}$ by $\widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}}([r]) = \sum_{i \le |r|} r_i$ is a projection whose associated embedding is the unit $\eta_{\overline{\mathbb{R}_+}}\colon \overline{\mathbb{R}_+} \to \overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv$.

Proof. These follow from the results derived in the proof of Lemma 4.1. In particular, $\equiv_L = \sqsubseteq_L \cap \sqsupseteq_L = \cup_n \equiv_n$, and $\equiv_+ = \sqsubseteq_+ \cap \sqsupseteq_+ = \cup_n \equiv_n$, the identity map $\mathrm{Id}\colon \mathit{Sob}(\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv, \sqsubseteq_{CL}) \to (\overline{\mathbb{R}_+}_{\mathbb{N}\,L}, \sqsubseteq_+)$ is well-defined. The above proof characterizing $\ll_+$ and the proof of Theorem 3.2, where the characterization of $\ll_{CL}$ was given show Id preserves this relation. The fact that the identity map is Scott continuous now is clear. For the claim that the identity map is not an order isomorphism, we note that, e.g., $[1] \sqsubseteq_+ [1/2, 1/2]$, but $[1] \not\sqsubseteq_{CL} [1/2, 1/2]$. This concludes the proof of (i).

For (ii), it is routine to show that $\widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}}$ is a Scott-continuous monoid morphism of $\mathit{Sob}(\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv, \sqsubseteq_{CL})$ to $\overline{\mathbb{R}_+}_{\mathbb{N}\,L}$ that satisfies $\widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}} \circ \eta_{\overline{\mathbb{R}_+}} = 1_{\overline{\mathbb{R}_+}}$. On the other hand, the definition of $\sqsubseteq_+$ implies that $\widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}}([r]) \sqsubseteq_+ [r]$, so $\eta_{\overline{\mathbb{R}_+}} \circ \widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}} \sqsubseteq 1_{\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv}$. $\square$

Remark 4.2.

— It may seem surprising that the quotient map is continuous but not an order isomorphism from $\mathit{Sob}(\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv, \sqsubseteq_{CL})$ to $\overline{\mathbb{R}_+}_{\mathbb{N}\,L}$. But the same phenomenon occurs in a much simpler setting: just consider the flat natural numbers $\mathbb{N}^\flat$ with a top element adjoined, and the identity map onto the ideal completion of $(\mathbb{N}, \le)$.

— The property that distinguishes $\sqsubseteq_+$ from $\sqsubseteq_{CL}$ on $\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv$ is part (ii) above, namely, that $\widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}}$ and $\eta_{\overline{\mathbb{R}_+}}$ form an embedding–projection pair with respect to this order. This is not true of $\sqsubseteq_{CL}$, even though $\widehat{\mathrm{Id}_{\overline{\mathbb{R}_+}}}$ is the (grounding of the) counit of the adjunction defined by $\overline{\mathbb{R}_+} \mapsto (\overline{\mathbb{R}_+}_{\mathbb{N}}/\equiv, \sqsubseteq_{CL})$. The point here is that it is just the order $\sqsubseteq_{CL}$ that needs to be refined to $\sqsubseteq_+$, without changing the mappings, for the unit and counit to form an e–p pair.

4.3. *Reconstructing Varacca's Hoare Indexed Valuations*

Using Theorem 4.2, we can now describe Varacca's Hoare Indexed Valuations $IV_L(P)$ for a continuous poset P. We begin with a definition.

Definition 4.1. Let P be a continuous poset. Then we define an order $\sqsubseteq_+$ on $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ by

$$[x] \sqsubseteq_+ [y] \quad \text{iff} \quad [x] = [\epsilon] \ \vee \ (\exists f\colon k \subseteq |y| \twoheadrightarrow |x|) \text{ with}$$
$$\pi_{\overline{\mathbb{R}_+}}(x)_i \leq \sum_{f(j)=i} \pi_{\overline{\mathbb{R}_+}}(y)_j \ \& \ \pi_P(x)_{f(j)} \sqsubseteq_P \pi_P(y)_j \ \forall j \in k.$$

Lemma 4.2. Let P be a continuous poset. Then $((\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}, \sqsubseteq_+)$ is a domain for which the way below relation is given by

$$[x] \ll_+ [y] \quad \text{iff} \quad [x] = [\epsilon] \ \vee \ (\exists f\colon k \subseteq |y| \twoheadrightarrow |x|) \text{ with} \tag{7}$$
$$\pi_{\overline{\mathbb{R}_+}}(x)_i < \sum_{f(j)=i} \pi_{\overline{\mathbb{R}_+}}(y)_j \ \& \ \pi_P(x)_{f(j)} \ll_P \pi_P(y)_j \ \forall j \in k.$$

Proof. This follows from the characterization of the way-below relation on $\overline{\mathbb{R}_+}_{\mathbb{N}\,L}$ given in Equation 6 and that of the way-below relation on $Sob(P_{\mathbb{N}}, \sqsubseteq_{CL})$ given in the proof of Proposition 3.4 and Theorem 3.2. □

Theorem 4.3. Let P be a continuous poset. Then $((\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}, \sqsubseteq_+)$ is an initial continuous algebra satisfying the laws of Theorem 4.1(i). It also is isomorphic to $IV_L(P)$.

Proof. We offer two proofs of these claims. The first begins by showing the second claim, and then relies on Varacca's work to deduce that $((\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}, \sqsubseteq_+)$ is an initial continuous algebra of the indicated type. The second is a direct verification that $((\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}, \sqsubseteq_+)$ satisfies the indicated laws and that it is an initial such algebra, and then the proof that it is isomorphic to $IV_L(P)$ follows since $IV_L(P)$ also is initial. The latter approach also is useful for the result that follows this one.

For the first proof, an abstract basis for $IV_L(P)$ consists of tuples $x \in \bigcup_m (\overline{\mathbb{R}_{\geq 0}} \times P)^m$, where a tuple x is identified with the subtuple x' whose real components are non-zero. So, the identity map takes $[x]$ to $[x'] \in (\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$, and sends $[0]$ to $[\epsilon]$. The mapping is an injection of the abstract basis for $IV_L(P)$ into $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$. Moreover, the way-below relation $\ll$ on the abstract basis for $IV_L(P)$ is the same as the way-below relation $\ll_+$ we defined in Equation 7 above on $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$. Since the mapping is an isomorphism of abstract bases, it extends to an isomorphism of their sobrifications. The rest of the theorem now follows from Theorem 4.1.

For the second proof, one first verifies that $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ satisfies the laws of Theorem 4.1(i). Most of the laws are straightforward to verify, once the operations are defined. To begin, we let $*$ denote addition in $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$, and we let $[\epsilon] = \underline{0}$. The action of $\mathbb{R}_+$ is given by $\pi_{\overline{\mathbb{R}_+}}(r \cdot x) = r\pi_{\overline{\mathbb{R}_+}}(x)$ and $\pi_P(r \cdot x) = \pi_P(x)$, for $r \in \mathbb{R}_+$ and $x \in (\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$. Given these definitions, we first observe that these operations are continuous by our construction method (in particular, scalar multiplication is monotone because we defined $[\epsilon]$ to be the least element). Also, the laws (1) and (2) are satisfied because $*$ is a commutative and associative operation, and (3) follows from the definition of $[\epsilon]$ as the identity for $*$. The law (4) is obvious, as is (5), while (6) and (7) follow from our definition above of scalar multiplication. Finally, (HV) follows from the construction of $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$.

Now that the laws are verified, it is straightforward to show that $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ is initial:

indeed, if $f\colon P \to S$ is Scott-continuous, and S satisfies the laws of Theorem 4.1(i), then we define $\widehat{f}\colon (\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L} \to S$ to be the continuous extension of the map that sends $[\epsilon]$ to $\underline{0}_S$, and that satisfies $\widehat{f}(x) = \pi_{\overline{\mathbb{R}_{\geq 0}}}(x) \cdot_S f(\pi_P(x))$. The unit of the adjunction sends $x \in P$ to $\eta_P(x)$ where $|x| = 1$, $\pi_{\overline{\mathbb{R}_+}}(\eta_P(x)) = 1$ and $\pi_P(\eta_P(x)) = x$. It's routine to show $\widehat{f} \circ \eta_P = f$, and that $\widehat{f}$ is the unique such map.

Since $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ is initial for the laws of Theorem 4.1(i), it is isomorphic to $IV_L(P)$, since the latter is initial as well. □

Notation Since $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ is initial for the laws of Theorem 4.1(i), it defines the object level of a left adjoint to the forgetful functor from continuous algebras satisfying those laws. We denote this functor by $\mathcal{P}_{\mathbb{N}\,L}$, so $\mathcal{P}_{\mathbb{N}\,L}(P) = (\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ and, given $f\colon P \to Q$, we define $\mathcal{P}_{\mathbb{N}\,L}(f)\colon (\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L} \to (\overline{\mathbb{R}_+} \times Q)_{\mathbb{N}\,L}$ by

— $|\mathcal{P}_{\mathbb{N}\,L}(f)(x)| = |x|$,
— $\pi_{\overline{\mathbb{R}_+}}(\mathcal{P}_{\mathbb{N}\,L}(f)(x)_i) = \pi_{\overline{\mathbb{R}_+}}(x_i)$, and
— $\pi_Q(\mathcal{P}_{\mathbb{N}\,L}(f)(x)_i) = f(x_i)$.

Corollary 4.1. If P is a continuous poset, then the nondeterminism monad $\mathcal{P}_L$ lifts to a monad on the family of Hoare indexed valuations over P.

Proof. We can appeal to Varacca's work to prove this, since we have already shown that $((\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}, \sqsubseteq_+)$ is isomorphic to $IV_L(P)$. For example, Theorem 4.4.4 of (Varacca 2003) uses Beck's Theorem 2.1 and exhibits the distributive law of IV_L over $\mathcal{P}_L$ to prove the result. Alternatively, (Varacca 2003, Theorem 4.4.2) gives a direct proof that $\mathcal{P}_L(IV_L(P))$ is a nondeterministic algebra that satisfies the laws enumerated in Theorem 4.1(i), and again, since $\mathcal{P}_L((\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}, \sqsubseteq_+)) \simeq \mathcal{P}_L(IV_L(P))$, we conclude our result.

There is another approach available. Since $\mathcal{P}_L$ and $\mathcal{P}_{\mathbb{N}\,L}$ are left adjoints, and left adjoints compose, we only need to show that, if (S, h) is an $\mathcal{P}_{\mathbb{N}\,L}$-algebra, then $(P_L(S), H)$ also is an $\mathcal{P}_{\mathbb{N}\,L}$-algebra for some mapping $H\colon (\overline{\mathbb{R}_+} \times \mathcal{P}_L(S))_{\mathbb{N}\,L} \to \mathcal{P}_L(S)$. Further, it is sufficient to define H on a dense subset of $(\overline{\mathbb{R}_+} \times \mathcal{P}_L(S))_{\mathbb{N}\,L}$ so that it is Scott-continous and satisfies the expected laws:

(i) $H \circ \eta = 1_{\mathcal{P}_L(S)}$, and
(ii)$H \circ \mu = H \circ \mathcal{P}_{\mathbb{N}\,L}H\colon \mathcal{P}^2_{\mathbb{N}\,L}\mathcal{P}_L(S) \to \mathcal{P}_L(S)$,

where η is the unit of the $\mathcal{P}_{\mathbb{N}\,L}$ monad, and μ is its multiplication.

Now, let $h\colon (\overline{\mathbb{R}_+} \times S)_{\mathbb{N}\,L} \to S$ be the structure map for S. Then

(i) $h \circ \eta = 1_S$, and
(ii)$h \circ \mu = h \circ \mathcal{P}_{\mathbb{N}\,L}h\colon \mathcal{P}^2_{\mathbb{N}\,L}S \to S$,

where η is the unit of the $\mathcal{P}_{\mathbb{N}\,L}$ monad, and μ is its multiplication.

We know the structure of $(\overline{\mathbb{R}_+} \times \mathcal{P}_L(S))_{\mathbb{N}\,L}$ to be $Sob((\overline{\mathbb{R}_+} \times \mathcal{P}_L(S))_{\mathbb{N}}, \sqsubseteq_+)$ with $[\epsilon]$ the least element. So a dense subset of this is $\bigcup_{n\geq 0} \big((\overline{\mathbb{R}_+} \times \mathcal{P}_L(S))^n/{\equiv}\big)$, where $[\epsilon]$ is the least element. Further, a dense subset of $\mathcal{P}_L(S)$ is $\{\downarrow F \mid \emptyset \neq F \subseteq S \text{ finite}\}$ under usual containment. So, it is sufficient to define

$$H\colon \bigcup_{n\geq 0} \big((\overline{\mathbb{R}_+} \times \{\downarrow F \mid \emptyset \neq F \subseteq S \text{ finite}\})^n/{\equiv}\big) \to \mathcal{P}_L(S).$$

Now since $\mathcal{P}_{\mathbb{N}\,L}$ forms a monad, we know that $h = \widehat{1_S} = \epsilon_{\mathcal{P}_{\mathbb{N}\,L}}$ is the counit of the adjunction. Moreover, the structure of $(\overline{\mathbb{R}_+} \times S)_{\mathbb{N}\,L}$ means h has a restriction to $(\overline{\mathbb{R}_+} \times S)^n/{\equiv}$ for each n. This implies we can define

$$H\colon (\overline{\mathbb{R}_+} \times \{\downarrow F \mid \emptyset \neq F \subseteq S \text{ finite}\})^n/{\equiv} \to\ \mathcal{P}_L(S)$$

by $H[[r_1, \downarrow F_1], \ldots, [r_n, \downarrow F_n]] = \bigcup_{i \leq n} \downarrow\{r_i \cdot_S h(x_i) \mid x_i \in F_i\}$. Then the restriction $H|_{(\overline{\mathbb{R}_+} \times \{\downarrow F | \emptyset \neq F \subseteq S \text{ finite}\})^n/\equiv}$ is continuous for each n, and the rest of the argument follows by a diagram chase, using the properties of h. For example, since $\eta(s) = [1, s]$, it follows that $H \circ \eta(\downarrow F) = H[1, \downarrow F] = \downarrow h(F) = \downarrow F$, for each $F \subseteq S$ finite. Hence the first law is fulfilled. $\square$

5. Hoare random variables

We now show how to construct the power domain of *Hoare random variables* over a domain. Recall that a *random variable* is a function $f\colon (X, \mu) \to (Y, \Sigma)$ where (X, μ) is a probability space, (Y, Σ) is a measure space, and f is a measurable function, which means $f^{-1}(A)$ is measurable in X for every $A \in \Sigma$, the specified σ-algebra of subsets of Y. Most often random variables take their values in $\mathbb{R}$, equipped with the usual Borel σ-algebra. For us, X will be a countable, discrete space, and Y will be a domain, where Σ will be the Borel σ-algebra generated by the Scott-open subsets.

Given a random variable $f\colon X \to Y$, the usual approach is to "push the probability measure μ forward" onto Y by defining $f\mu\,(A) = \mu(f^{-1}(A))$ for each measurable subset A of Y. But this defeats one of the features of random variables: there may be several points $x \in X$ which f takes to the same value $y \in Y$. Retaining this feature would allow the random variable f to make distinctions that the probability measure $f\mu$ does not. Varacca makes exactly this point in his work (Varacca 2003), a point he justifies by showing how to distinguish the random variable f from the probability measure $f\mu$ operationally.

Definition 5.1. For a domain P, we define the *Hoare power domain of random variables over* P to be the subdomain

$$\mathbb{RV}_H(P) = \{x \in (\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L} \mid \sum_{i \leq |x|} \pi_{\overline{\mathbb{R}_+}}(x) \leq 1\}$$

In order to show that $\mathbb{RV}$ is a monad, we need an enumeration of the laws that a random variable algebra should satisfy. These are adapted from the laws for probabilistic algebras first defined by Graham (Graham 1985):

Definition 5.2. A *Hoare random variable algebra* is a domain P with $\underline{0}$, a least element and with a Scott-continuous mapping $+\colon [0, 1] \times P \times P \to P$ satisfying:

— $p +_r 0 = p$ for all $0 < r \leq 1$.[‡]
— $a +_1 b = a$,

[‡] We use $a +_r b$ as infix notation for $+(r, a, b)$.

— $a +_r b = b +_{1-r} a$, and
— $(a +_r b) +_s c = a +_{rs} (b +_{\frac{s(1-r)}{1-sr}} c)$,
— $a \sqsubseteq a +_r a$,

where $r, s \in (0,1)$ and $a, b, c \in P$. We let $\mathbb{RV}_H(P)$ denote the family of Hoare random variables over P, endowed with the order inherited from $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$.

A *morphism of Hoare random variable algebras* is a Scott-continuous map $f\colon S \to T$ satisfying $f(0_S) = 0_T, f(\bot_S) = \bot_T$ and $f(s +_r s') = f(s) +_r f(s')$, for all $s, s' \in S$ and all $r \in (0,1]$.

The difference between our laws and those from Graham's characterization of probabilistic algebras are that (i) we restrict the application of the laws involving $+_r$ to cases when $0 < r < 1$ (which avoids some annoying side conditions in Graham's listing), and (ii) the law $a +_r a = a$ is replaced by the last inequation. This last is exactly the law Varacca weakened to allow a distributive law to hold.

Proposition 5.1. Let P be a domain, and for $x, y \in \mathbb{RV}_H(P)$ and $0 \leq r \leq 1$, define $x +_r y = r \cdot x \ * \ (1-r) \cdot y$. Then:

(i) $\mathbb{RV}_H(P)$ is a Hoare random variable algebra, and
(ii) $[(r,p)] = [(1,p)] +_r \underline{0}$ for all $p \in P$ and all $r \in (0,1)$, and

$$[(r_1,p_1),\ldots,(r_m,p_m)] = [(1,p_1)] +_{r_1} [(\frac{r_2}{(1-r_1)},p_2),\ldots,(\frac{r_m}{(1-r_1)},p_m)]$$

for all $[(r_1,p_1),\ldots,(r_m,p_m)] \in \mathbb{RV}_H(P)$.

Proof. Given a domain P, we can define $+\colon [0,1] \times (\overline{\mathbb{R}_+} \times P)^2_{\mathbb{N}\,L} \to (\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ by $+(r,x,y) = r \cdot x \ * \ (1-r) \cdot y$. Because $\mathbb{R}_+$ acts continuously on $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ and because $*$ is continuous, $+$ is a continuous operation. $\mathbb{RV}_H(P)$ is the subfamily of $(\overline{\mathbb{R}_+} \times P)_{\mathbb{N}\,L}$ whose real components are bounded by 1, and this family is clearly invariant under the action of $\mathbb{R}_+$, so this defines a continuous mapping $+\colon [0,1] \times \mathbb{RV}_H(P)^2 \to \mathbb{RV}_H(P)$. It now is routine to verify that the laws of Definition 5.2 are satisfied.

The results in (ii) are simple calculations. □

We now come to our main result.

Theorem 5.1.

(i) $\mathbb{RV}_H$ defines a monad on DOM.
(ii) The lower power domain monad $\mathcal{P}_L$ lifts to a monad on Hoare random variable algebras.

Proof. For the first claim, we begin by noting that $\mathbb{RV}_H(P)$ is obtained by restricting $\mathcal{P}_{\mathbb{N}\,L}(P)$ in the "real components" to ones whose sum is at most 1. This family is a Scott-closed subset of $\mathcal{P}_{\mathbb{N}\,L}(P)$. Hence $\mathbb{RV}_H(P)$ is a domain if P is one. Continuous maps $f\colon P \to Q$ extend to $\mathcal{P}_{\mathbb{N}\,L}(P)$ by $\pi_{\overline{\mathbb{R}_{\geq 0}}}(\mathcal{P}_{\mathbb{N}\,L}(f)(x)) = \pi_{\overline{\mathbb{R}_{\geq 0}}}(x)$ and the elements in $\mathbb{RV}_H(P)$ are those in $\mathcal{P}_{\mathbb{N}\,L}(P)$ whose real components sum to at most 1; it follows that $\mathbb{RV}_H(f)(P) \subseteq \mathbb{RV}_H(Q)$.

Now, we show that $\mathbb{RV}_H$ is left adjoint to the forgetful functor from Hoare random variable algebras into DOM. To begin, we let $\eta\colon P \to \mathbb{RV}(P)$ by $\eta(p) = [1, p]$ define the unit of the adjunction.

Next, let S be a Hoare random variable domain algebra, P a domain, and let $f\colon P \to S$ be a Scott continuous map. We define $\widehat{f}\colon \mathbb{RV}_H(P) \to S$ via $\widehat{f}(x)$ by induction on $|x|$.

If $x = [\epsilon]$, then $\widehat{f}([\epsilon]) = 0_S$ must hold. In case of $x = [r, p]$, we have $[r, p] = [1, p] +_r \underline{0}$ by Proposition 5.1, so we define $\widehat{f}([r, p]) = f(p) +_r 0_S$. This mapping is clearly continuous on $P/{\equiv_1} \subseteq \mathbb{RV}_H(P)$, since $P/{\equiv_1}$ inherits its Scott topology from that of $\mathbb{RV}_H(P)$. This is also the unique such function on $P/{\equiv_1}$ satisfying $\widehat{f} \circ \eta = f$.

Continuing the inductive definition of $\widehat{f}$, assume we have defined $\widehat{f}$ on $\cup_{k \leq n}(P^k/{\equiv_k})$ uniquely so that it is continuous and satisfies $\widehat{f} \circ \eta = f$. Let $x = [(r_1, p_1), \ldots, (r_{m+1}, p_{m+1})] \in P^{m+1}/{\equiv_{m+1}}$, and then define

$$\widehat{f}([(r_1, p_1), \ldots, (r_{m+1}, p_{m+1})]) = f(p_1) +_{r_1} \widehat{f}([(\frac{r_2}{1 - r_1}, p_2), \ldots, (\frac{r_{m+1}}{1 - r_1}, p_{m+1})].$$

This is well-defined by Proposition 5.1(ii), and it is the composition of continuous functions, so it is continuous. It also satisfies $\widehat{f} \circ \eta = f$ because it's restriction to $P/{\equiv_1}$ does by definition. Finally, Proposition 5.1(ii) again shows it is the unique such function.

This shows that $\mathbb{RV}_H$ is left adjoint to the forgetful functor from Hoare random variable algebras into DOM, so it defines a monad on DOM.

The second claim follows from the above and from Corollary 4.1, since the lower power domain of a Scott-closed subset A of a domain consists of Scott-closed subsets of A. $\square$

6. Summary and Future Work

We have presented a construction of ordered monoids and their commutative analogs over domains, paralleling the construction of the three power domains for nondeterminism. Our results were inspired by the results of Varacca, whose indexed valuations define monads each of which enjoys a distributive law over the appropriate power domain monad. We have also shown how to alter our construction of a lower commutative monoid over $\overline{\mathbb{R}_+} \times P$ to achieve Varacca's construction. We have shown how our approach allows to recapture Varacca's in ways that avoid having to deal with the identifications between various elements of indexed valuations. We believe this makes the proofs easier to follow. We also believe that our approach reveals more information about the internal structure of the domain of Hoare indexed valuations. In particular, our approach provides mechanism to define the continuous algebra of Hoare random variables over a continuous poset, and to prove it defines a monad on DOM. This is a direct generalization of the probabilistic power domain, and it enjoys a distributive law over the lower power domain.

There are some interesting questions yet to be explored in this area. The first one is to generalize our construction to accommodate the other indexed valuation constructions of Varacca. Of course, this is motivated by the utility of Varacca's construction, since it provides a simple method for building models supporting both nondeterminism and Varacca's version of probabilistic choice. Part of this was accomplished in (Mislove 2005), but none of the work so far on random variables over domains extends beyond the discrete case. The main stumbling block in this regard is our reliance on Rudin's Lemma 3.2.

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