Measuring the Probabilistic Powerdomain

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Abstract. In this paper we initiate the study of measurements on the probabilistic powerdomain. We show how measurements on an underlying domain naturally extend to its probabilistic powerdomain, so that the kernel of the extension consists of exactly those normalized measures on the kernel of the measurement on the underlying domain. This result is combined with now-standard results from the theory of measurements to obtain a new proof that the fixed point associated with a weakly hyperbolic IFS with probabilities is the unique invariant measure whose support is the attractor of the underlying IFS.

1 Introduction

A relatively recent discovery [14] in domain theory is that most domains come equipped with a natural measurement: a Scott continuous map into the non-negative reals which encodes the Scott topology. The existence of measurements was exploited by Martin [13–16] to study the space of maximal elements of a domain, and to formulate various fixed point theorems for domains, including fixed point theorems for non-monotonic maps.

The theory of measurements meshes particularly fruitfully with the idea of domains as models of classical spaces. Here we say that a domain D is a model of a topological space X if the set of maximal elements of D equipped with the relative Scott topology is homeomorphic to X. Under quite mild conditions on D, the set of normalized Borel measures on X, equipped with the weak topology, can be embedded into the set of maximal elements of the probabilistic powerdomain $\mathbf{P}D$. This construction was utilized by Edalat [3, 4] to provide new results on the existence of attractors for iterated function systems, and to define a generalization of the Riemann integral to functions on metric spaces.

In this paper we show that each measurement $m\colon D \to [0,1]$ satisfying a suitable condition has a natural extension to a measurement $M\colon \mathbf{P}D \to [0,1]$. Moreover we show that the kernel of M, equipped with the relative Scott topology, is homeomorphic to the space of Borel measures on the kernel of m equipped

[1,2]The support of the US Office of Naval Research is gratefully acknowledged.
[2] The support of the National Science Foundation is gratefully acknowledged.

with the weak topology. These results can be used to derive facts about domains in general which are *independent* of measurement: for example, if D is an ω-continuous model of a regular space X, then the set of normalized Borel measures on X, equipped with the weak topology, can be embedded into the set of maximal elements of $\mathbf{P}D$. They can also be used to derive results which are *independent* of domain theory altogether, such as a new proof that the fixed point associated with a weakly hyperbolic iterated function system with probabilities is the unique measure whose support is the attractor of the underlying iterated function system.

2 Background

2.1 Domain Theory

A *poset* $(P, \sqsubseteq)$ is a set P endowed with a partial order $\sqsubseteq$. The least element of P (if it exists) is denoted $\perp$, and the set of maximal elements of P is written $\mathsf{max}\, P$. Given $A \subseteq P$, we write $\uparrow A$ for the set $\{x \in P \mid (\exists a \in A)\, a \sqsubseteq x\}$; similarly, $\downarrow A$ denotes $\{x \in P \mid (\exists a \in A)\, x \sqsubseteq a\}$. A function $f \colon P \to Q$ between posets P and Q is *monotone* if $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ for all $x, y \in P$. A subset $A \subseteq P$ is *directed* if each finite subset $F \subseteq A$ has an upper bound in A. Note that since $F = \emptyset$ is a possibility, a directed subset must be non-empty. A *(directed) complete partial order (dcpo)* is a poset P in which each directed set $A \subseteq P$ has a least upper bound, denoted $\sqcup A$.

If D is a dcpo, and $x, y \in D$, then we write $x \ll y$ if for each directed subset $A \subseteq D$, if $y \sqsubseteq \sqcup A$, then $\uparrow x \cap A \neq \emptyset$. We then say *$x$ is way-below y*. Let $\mathord{\downarrow\!\!\downarrow} y = \{x \in D \mid x \ll y\}$; we say that D is *continuous* if it has a *basis*, i.e., a subset $B \subseteq D$ such that for each $y \in D$, $\mathord{\downarrow\!\!\downarrow} y \cap B$ is directed with supremum y. If D has a countable basis then we say D is *ω-continuous*. The way-below relation on a continuous dcpo has the *interpolation property*: if $x \ll y$ then there exists a basis element z such that $x \ll z \ll y$.

A subset U of a dcpo D is *Scott-open* if it is an upper set (i.e., $U = \uparrow U$) and for each directed set $A \subseteq D$, if $\sqcup A \in U$ then $A \cap U \neq \emptyset$. The collection ΣD of all Scott-open subsets of D is called the *Scott topology* on D. If D is continuous, then the Scott topology on D is locally compact, and the sets $\mathord{\uparrow\!\!\uparrow} x$ where $x \in D$ form a basis for the topology. If $S \subseteq D$, we write $\mathrm{Cl}(S)$ for the closure of S with respect to the Scott topology. Given dcpos D and E, a function $f \colon D \to E$ is continuous with respect the Scott topologies on D and E iff it is monotone and preserves directed suprema: for each directed $A \subseteq D$, $f(\sqcup A) = \sqcup f(A)$.

A *domain* is a continuous dcpo.

2.2 Valuations and the Probabilistic Powerdomain

We briefly recall some basic definitions and results about valuations and the probabilistic powerdomain.

Definition 1. *Let X be a topological space. A* continuous valuation *on X is a mapping $\nu \colon (\Omega X, \subseteq) \to ([0,1], \leqslant)$ satisfying:*

1. *Strictness:* $\nu(\emptyset) = 0$.
2. *Monotonicity:* $U \subseteq V \Rightarrow \nu(U) \leqslant \nu(V)$.
3. *Modularity: for all* $U, V \in \Omega X$, $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$.
4. *Continuity: for every directed family* $\{U_i\}_{i\in I}$, $\nu(\bigcup_{i\in I} U_i) = \sup_{i\in I} \nu(U_i)$.

Each element $x \in X$ gives rise to a valuation defined by

$$\delta_x(U) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

A *simple valuation* has the form $\sum_{a\in A} r_a \delta_a$ where A is a finite subset of X, $r_a \geqslant 0$, and $\sum_{a\in A} r_a \leqslant 1$. A valuation ν is *normalized* if $\nu(X) = 1$.

For the most part we will consider continuous valuations defined on the Scott topology ΣD of a dcpo D. The set of all such valuations, ordered by $\sigma \sqsubseteq \nu$ if and only if $\sigma(U) \leqslant \nu(U)$ for all $U \in \Sigma D$, forms a dcpo $\mathbf{P}D$: the probabilistic powerdomain of D. Our main reference for the probabilistic powerdomain is the thesis of Jones [10] from which the following result is taken.

Theorem 1 (Jones [10]). *If D is a continuous domain then $\mathbf{P}D$ is continuous with a basis $\mathcal{B} = \{\Sigma_{i=1}^n r_i \delta_{p_i} \mid p_i \in B\}$, where $B \subseteq D$ is a basis for D.*

Proof. (Sketch) Define a dissection of D to be a disjoint family of crescents $\mathcal{D} = \{C_i\}_{i\in I}$, where $C_i = \mathord{\Uparrow} x_i \setminus U_i$ for some $x_i \in \mathcal{B}$ and $U_i \in \Sigma D$. Given $\nu \in \mathbf{P}D$ and $0 < r < 1$ define

$$\nu_{\mathcal{D},r} = \sum_{i\in I} r\nu(C_i)\delta_{x_i}.$$

The substantial part of the proof, which is elided here, is to show that the set of $\nu_{\mathcal{D},r}$ for all $\mathcal{D}$ and r is directed with join ν. $\square$

Next we recall a characterization of convergence in the Scott topology on $\mathbf{P}D$.

Theorem 2 (Edalat [5]). *Suppose D is a continuous dcpo, then a net $\langle \nu_i \rangle_{i\in I}$ in $\mathbf{P}D$ converges to ν in the Scott topology iff*

$$\liminf_{i\in I} \nu_i(U) \geqslant \nu(U)$$

for all Scott open subsets $U \subseteq D$.

Obviously, valuations bear a close resemblance to measures. Lawson [12] showed that any valuation on an ω-continuous dcpo D extends uniquely to a measure on the Borel σ-algebra generated by the Scott topology (equivalently by the Lawson topology) on D. This result was generalized to continuous dcpos by Alvarez-Manilla, Edalat and Saheb-Djahromi [2]. Both these results depend heavily on the axiom of choice. In this paper, we avoid using either theorem. We do use the elementary result that each valuation on a dcpo D extends uniquely to a finitely additive set function on the field $\mathcal{F}D$ generated by ΣD. Each member R of this field can be written as a finite, disjoint union of crescents, i.e.,

$R = \bigcup_{i=1}^{n} U_i \setminus V_i$ for $U_i, V_i \in \Sigma D$. The extension of a valuation ν to $\mathcal{F}D$ assigns to R the value

$$\sum_{i=1}^{n} (\nu(U_i) - \nu(U_i \cap V_i)).$$

Also we recall from Heckmann [8, Section 3.2] that if $E \in \mathcal{F}D$ and $\nu \in \mathbf{P}D$ then we may define $\nu|_E \in \mathbf{P}D$ by $\nu|_E\,(O) = \nu(O \cap E)$ for all $O \in \Sigma D$.

In Section 6 we use the well-known fact that any continuous valuation on a metric space has a unique extension to a measure (cf. [1, Corollary 3.24]). But this is only used to mediate between the formulation of the main result of that section and the results of Hutchinson [9] which are stated for measures.

3 Measurement

Let $\mu : D \to E$ be a Scott continuous map between domains D and E.

Definition 2. *The* ε-approximations *of $x \in D$ are*

$$\mu_\varepsilon(x) := \{y \in D : y \sqsubseteq x \;\&\; \varepsilon \ll \mu y\},$$

and we say that μ measures *$x \in D$ if for all open $U \subseteq D$, we have*

$$x \in U \Rightarrow (\exists \varepsilon)\, x \in \mu_\varepsilon(x) \subseteq U.$$

The map μ measures *$X \subseteq D$ if it measures each $x \in X$.*

One of the crucial insights of [13] is that measuring "information content" amounts to measuring partiality, and that the definition above seems to provide a minimal mathematical account of what it means to measure the partiality of the objects in X. That is, if μ measures X, then we can say that μx is the amount of partiality in $x \in X$ (and then by continuity of μ, we can think of it as measuring partiality of nearby approximations of x); otherwise, it is just a mapping on a domain.

Granted this, a second and distinct question arises, "What elements should we expect μ to measure the content of?" Again, minimally, if μ is a measure of partiality, it should at least measure the objects in

$$\ker \mu := \{x \in D : \mu x \in \max E\}.$$

Why? Because from the viewpoint of μ, each $x \in \ker \mu$ has no partiality, i.e., is total or *ideal.* In a certain sense, then, these are the objects that μ should have the least difficulty measuring.

Definition 3. *A* measurement *is a continuous map $\mu : D \to E$ which measures* $\ker \mu$.

In accord with intuition, one then proves that $\ker \mu \subseteq \max D$ for a measurement μ. The property of "measuring a set" is also expressed by saying that μ *induces the Scott topology near* $\ker \mu$.

In the typical case that E is the dcpo $[0,\infty)^*$ of non-negative reals in their opposite order, μ is a measurement iff for any Scott open U and any ideal element $x \in \ker \mu$,

$$x \in U \Rightarrow (\exists\, \varepsilon > 0)\, \{y \in D : y \sqsubseteq x \text{ and } |\mu x - \mu y| < \varepsilon\} \subseteq U. \tag{1}$$

In words, what it means for μ to *measure* x is that any observation U about x is also an observation about y, where y is close to x, and fundamentally, "close" is specified simultaneously by the order and the map μ.

Example 1. The following examples of measurements are all pertinent to this paper. The first two illustrate the idea that natural models of metric spaces yield canonical measurements into $[0,\infty)^*$.

(i) If $\langle X, d\rangle$ is a locally compact metric space, then its *upper space*

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \text{ is compact}\}$$

ordered by reverse inclusion is a continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{U}X$ is $\bigcap S$, and the way-below relation is given by $A \ll B$ iff $B \subseteq \operatorname{int} A$. Given $K \in \mathbf{U}X$, defining the diameter of K by

$$|K| = \sup\{d(x,y) : x, y \in K\},$$

it is readily verified that $K \mapsto |K|$ is a measurement on $\mathbf{U}X$ whose kernel is $\max \mathbf{U}X = \{\{x\} : x \in X\}$.

(ii) Given a metric space $\langle X, d\rangle$, the *formal ball model* [6] $\mathbf{B}X = X \times [0,\infty)$ is a poset ordered by

$$(x,r) \sqsubseteq (y,s) \text{ iff } d(x,y) \leqslant r - s.$$

The way-below relation is characterized by

$$(x,r) \ll (y,s) \text{ iff } d(x,y) < r - s.$$

The poset $\mathbf{B}X$ is a continuous dcpo iff the metric d is complete. Moreover $\mathbf{B}X$ has a countable basis iff X is separable. A natural measurement π on $\mathbf{B}X$ is given by $\pi(x,r) = r$. Then $\ker \pi = \max \mathbf{B}X = \{(x,0) : x \in X\}$.

(iii) Let $X = \{x_n\}_{n\in\mathbb{N}}$ be a countably infinite set, and $(\mathbb{P}X, \subseteq)$ the lattice of subsets of X ordered by inclusion. Observe that $S \ll T$ in $\mathbf{P}X$ iff S is a finite subset of T. We can define a measurement $|\cdot| : \mathbb{P}X \to [0,\infty)^*$ by

$$|S| = 1 - \sum_{x_n \in S} s^{-(n+1)}.$$

In the case of finite X, set $|S| = \operatorname{card}(X \setminus S)$.

One of the motivations behind the introduction of measurement in [14] was to facilitate the formulation of sharper fixed point theorems. The following is a basic example of one such result.

Theorem 3. *Let* $f\colon D \to D$ *be a monotone map on a pointed continuous dcpo* D *equipped with a measurement* $\mu\colon D \to E$. *If* $\bigsqcup \mu f^n(\bot) \in \mathsf{max}\, E$, *then*

$$x^\star = \bigsqcup_{n \geqslant 0} f^n(\bot) \in \mathsf{ker}\, \mu$$

is the unique fixed point of f. *Moreover,* $x^\star$ *is an attractor: For all* x, $f^n(x) \to x^\star$ *in the Scott topology on* D. *This convergence restricts to* $\mathsf{ker}\,\mu$ *if* f *carries* $\mathsf{ker}\,\mu$ *into* $\mathsf{ker}\,\mu$.

Martin [16] gives a necessary and sufficient condition for a measurement $\mu\colon D \to [0,\infty)^*$ to extend to a measurement $\bar{\mu}\colon \mathbf{C}D \to [0,\infty)^*$ on the convex powerdomain $\mathbf{C}D$ thereby uncovering the class of *Lebesgue measurements.*

Definition 4. *A continuous map* $\mu : D \to E$ *is a* Lebesgue measurement *if*

$$K \subseteq U \Rightarrow (\exists\, \varepsilon)(\forall x \in K)\, x \in \mu_\varepsilon(x) \subseteq U,$$

for all compact $K \subseteq \mathsf{ker}\,\mu$ *and all open* $U \subseteq D$.

Knowing that Lebesgue measurements extend to the convex powerdomain enables one to prove that any ω-continuous dcpo D with $\mathsf{max}\, D$ regular satisfies the property that the Vietoris hyperspace of $\mathsf{max}\, D$ embeds into $\mathsf{max}\, \mathbf{C}D$ (as the kernel of a measurement). Further, Edalat's domain theoretic analysis of hyperbolic iterated function systems is then shown to be a consequence of standard results from measurement. In the same setting, the *necessity* of complete metrizability becomes apparent.

Theorem 4 (Martin [13]). *A space is completely metrizable iff it is the kernel of a Lebesgue measurement* $\mu : D \to [0,\infty)^*$ *on a continuous dcpo.*

Here we seek analogous results with the probabilistic powerdomain in place of the convex powerdomain, and the weak topology on Borel measures in place of the Vietoris topology on compact sets. We now identify a condition which ensures that a measurement on D extends to a measurement on the probabilistic powerdomain $\mathbf{P}D$. First, extend the definition of μ_ε to arbitrary sets S by setting

$$\mu_\varepsilon(S) = \bigcup_{s \in S} \mu_\varepsilon(s).$$

For example, μ is Lebesgue just when there is an ε such that $K \subseteq \mu_\varepsilon(K) \subseteq U$.

Definition 5. *A continuous* $\mu\colon D \to E$ *is a* regular measurement *if*

$$x \in U \Rightarrow (\exists \varepsilon)(\exists b \ll x) \mathord{\uparrow\!\!\uparrow} b \cap \mathsf{ker}\,\mu \subseteq \mu_\varepsilon(\mathord{\uparrow\!\!\uparrow} b \cap \mathsf{ker}\,\mu) \subseteq U,$$

for every $x \in \mathsf{ker}\,\mu$ *and every open* $U \subseteq D$.

Thus, with a regular measurement, the choice of ε not only applies *at* x, the way it does for a measurement, it also applies on an open set *around* x.

Proposition 1. *Every regular measurement $\mu : D \to [0,\infty)^*$ is Lebesgue. The converse holds when $\mathsf{ker}\,\mu$ is locally compact.*

Proof. We prove the first of these, after which the second is routine. If $K \subseteq \mathsf{ker}\,\mu$ is compact with $K \subseteq U$ open, then for each $x \in K$, we use the regularity of μ to obtain $b_x \ll x$ and $\varepsilon_x > 0$ such that

$$\mathord{\Uparrow} b_x \cap \mathsf{ker}\,\mu \subseteq \mu_{\varepsilon_x}(\mathord{\Uparrow} b_x \cap \mathsf{ker}\,\mu) \subseteq U.$$

Restricting $\{\mathord{\Uparrow} b_x : x \in K\}$ to a finite subcover $\{\mathord{\Uparrow} b_i\}$, leaves a finite number of $\varepsilon_i > 0$, which assures us that $\varepsilon := \min(\varepsilon_i/2) > 0$. We have $K \subseteq \mu_\varepsilon(K) \subseteq U$. $\square$

Note that all the measurements in Example 1 are regular. In Example 2, we will see a measurement which is not regular. Until then, here is a characterization of the countably based domains which admit regular measurements.

Theorem 5. *Let D be an ω-continuous dcpo. There is a regular measurement $\mu\colon D \to [0,\infty)^*$ with $\mathsf{ker}\,\mu = \mathsf{max}\,D$ iff the space $\mathsf{max}\,D$ in its relative Scott topology is regular.*

Proof. If μ is regular, then $\mathsf{ker}\,\mu$ is Lebesgue by the last result, so completely metrizable and hence regular by Theorem 4.

Conversely, let D be an ω-continuous dcpo with $\mathsf{max}\,D$ regular and a countable basis B. From [16], we know that $\lambda\colon D \to (\mathbb{P}I, \subseteq)$ defined by

$$\lambda(x) = \{(a,b) \mid x \in \mathord{\Uparrow} a \ \vee \ x \notin \mathrm{Cl}(\mathord{\Uparrow} b)\}, \tag{2}$$

where $I = \{(a,b) \in B \times B \mid \mathrm{Cl}(\mathord{\Uparrow} b) \cap \max D \subseteq \mathord{\Uparrow} a\}$, is a Lebesgue measurement. Unsurprisingly it is also regular: Given $a, b \in B$ we have that $\mathrm{Cl}(\mathord{\Uparrow} b) \cap \mathsf{ker}\,\lambda \subseteq \mathord{\Uparrow} a$ implies that there exists $\varepsilon := \{(a,b)\}$ such that $\mathord{\Uparrow} b \cap \mathsf{ker}\,\lambda \subseteq \lambda_\varepsilon(\mathord{\Uparrow} b \cap \mathsf{ker}\,\lambda) \subseteq \mathord{\Uparrow} a$.

As usual, composing this measurement with the map in Example 1(iii) yields a regular measurement into $[0,\infty)^*$ with kernel $\mathsf{max}\,D$. $\square$

As mentioned earlier, we normally work with measurements as mappings $\mu : D \to [0,\infty)^*$. Unfortunately the choice of μ obviously conflicts with the usual notation for valuations, and since most of our work here is concerned with valuations, we opt to use the letter m for measurements.

In addition, it is also more convenient to consider measurements into the unit interval $[0,1]$ in its usual order for reasons that will become clear shortly. Thus, given a measurement $\mu : D \to [0,\infty)^*$, we simply transform it to a measurement $m : D \to [0,1]$ by $m = 1/2^\mu$. Notice that the characterization of measurement given in (1) remains valid when $E = [0,1]$.

Our main result, Theorem 7, says that a regular measurement $m\colon D \to [0,1]$ extends in a natural way to a measurement $M\colon \mathbf{P}D \to [0,1]$ on the probabilistic powerdomain of D. Furthermore it holds that $\mathsf{ker}\,M$ is homeomorphic to the set of normalized measures on $\mathsf{ker}\,m$ in the weak topology. Combining this with Theorem 5 we obtain Corollary 1. This result was first proved, in a different way, in Martin [16, Theorem 11.8].

Corollary 1. *If D is an ω-continuous dcpo with* $\max D$ *regular, then the space of normalized measures on* $\max D$ *in the weak topology embeds as a subspace of* $\max \mathbf{P}D$.

4 Comparing Valuations

One of the most elegant results about the probabilistic powerdomain is the Splitting Lemma. This bears a close relationship to a classic problem in probability theory: find a joint distribution with given marginals.

Lemma 1 (Jones [10]). *Let $\sigma = \sum_{a\in A} r_a \delta_a$ and $\nu = \sum_{b\in B} s_b \delta_b$ be simple valuations. Then $\sigma \ll \nu$ if and only if there exists a family of transport (or flow) numbers $\{t_{a,b} \mid a \in A, b \in B\} \subseteq [0,1]$ satisfying*

1. *For each $a \in A$, $\sum_{b\in B} t_{a,b} = r_a$,*
2. *For each $b \in B$, $\sum_{a\in A} t_{a,b} < s_b$, and*
3. *$t_{a,b} \neq 0$ implies $a \ll b$.*

In the remainder of this section we give a characterization of when a simple valuation lies way-below an arbitrary continuous valuation.

Proposition 2 (Kirch [11]). *If ν is a continuous valuation on D, then $\sigma = \sum_{a\in A} r_a \delta_a \ll \nu$ if and only if $\forall S \subseteq A$, $\sum_{a\in S} r_a < \nu(\Uparrow S)$.*

Definition 6. *Fix a finite subset $A \subseteq D$, and for each $S \subseteq A$ define*

$$(\!|A, S|\!) = \bigcap_{a\in S} \Uparrow a \setminus \bigcup_{a'\in A\setminus S} \Uparrow a'.$$

Observe that $\{(\!|A, S|\!)\}_{S\subseteq A}$ is a family of crescents partitioning D.

Proposition 3. *Let ν be a continuous valuation on D, $\sum_{a\in A} r_a \delta_a$ a simple valuation on D, and $\{E_i\}_{i\in I} \subseteq \mathcal{F}D$ a finite partition of D refining $\{(\!|A, S|\!)\}_{S\subseteq A}$. Then $\sum_{a\in A} r_a \delta_a \ll \nu$ iff there exists a relation $R \subseteq A \times I$ such that*

(i) $(a, i) \in R$ implies $E_i \subseteq \Uparrow a$.
(ii) For all $S \subseteq A$, $\sum_{a\in S} r_a < \sum_{i\in R(S)} \nu(E_i)$.

Proof. ($\Rightarrow$) Suppose $\sum_{a\in A} r_a \delta_a \ll \nu$. Define R by $R(a, i)$ just in case $E_i \subseteq \Uparrow a$. Then, given $S \subseteq A$, by Proposition 2,

$$\sum_{a\in S} r_a < \nu(\Uparrow S) = \sum_{i\in R(S)} \nu(E_i).$$

($\Leftarrow$) Given a relation R satisfying conditions (i) and (ii) above, then for all $S \subseteq A$ we have

$$\sum_{a\in S} r_a < \sum_{i\in R(S)} \nu(E_i) \leqslant \nu(\Uparrow S).$$

Thus $\sum_{a\in A} r_a \delta_a \ll \nu$ by Proposition 2. ◻

Next we give an alternate characterization of the way-below relation on $\mathbf{P}D$. This is a slight generalization of the Splitting Lemma, and should be seen as dual to Proposition 3.

Proposition 4. *Suppose $\sum_{a \in A} r_a \delta_a$ and ν are continuous valuations on D and $\{E_i\}_{i \in I} \subseteq \mathcal{F}D$ is a partition of D refining $\{(\!|A, S|\!)\}_{S \subseteq A}$. Then $\sum_{a \in A} r_a \delta_a \ll \nu$ iff there exists a family of 'transport numbers' $\{t_{a,i}\}_{a \in A, i \in I}$ where*

1. *For each $a \in A$, $\sum_{i \in I} t_{a,i} = r_a$*
2. *For each $i \in I$, $\sum_{a \in A} t_{a,i} < \nu(E_i)$*
3. *$t_{a,i} > 0$ implies $E_i \subseteq \Uparrow a$.*

Proof. ($\Leftarrow$) Given the existence of a family of transport numbers $\{t_{a,i}\}$, define $R \subseteq A \times I$ by $R(a,i)$ iff $t_{a,i} > 0$. Then R satisfies (i) and (ii) in Proposition 3.

($\Rightarrow$) By Proposition 3 there exists a relation $R \subseteq A \times I$ satisfying conditions (i) and (ii) thereof. The proof that such a relation yields transport numbers as required uses the max-flow min-cut theorem from graph theory. The basic idea is due to Jones [10], but we refer the reader to the formulation of Heckmann [8, Lemma 2.7] which is general enough to apply to the present setting. $\square$

For our main results, we can equally-well use Proposition 3 or (the dual form) Proposition 4 to characterize the way-below relation. Next we define an operation $\star$ for composing splittings with a common index set by 'projecting out that index.' Suppose $s = \{s_{i,j}\}_{i \in I, j \in J}$ and $t = \{t_{j,k}\}_{j \in J, k \in K}$ are families of non-negative real numbers where I, J and K are finite. Assuming that $\sum_{k \in K} t_{j,k} > 0$ for each $j \in J$ we define $t \star s$ to be an $I \times K$-indexed family where

$$(t \star s)_{i,k} = \sum_{j \in J} s_{i,j} \left(\frac{t_{j,k}}{\sum_{k' \in K} t_{j,k'}} \right).$$

Proposition 5. *Let s and t be as above. Then for each $i \in I$,*

$$\sum_{k \in K} (t \star s)_{i,k} = \sum_{j \in J} s_{i,j}. \tag{3}$$

Furthermore, if $\sum_{i \in I} s_{i,j} < \sum_{k \in K} t_{j,k}$ for each $j \in J$, it follows that

$$\sum_{i \in I} (t \star s)_{i,k} < \sum_{j \in J} t_{j,k} \tag{4}$$

for each $k \in K$.

Proof. Simple algebra. $\square$

5 Measuring the Probabilistic Powerdomain

Definition 7. *If $m \colon D \to [0,1]$ is a measurement on a continuous dcpo D, then we define $M \colon \mathbf{P}D \to [0,1]$ by $M(\nu) = \int m d\nu$, where the integral is that defined by Jones [10].*

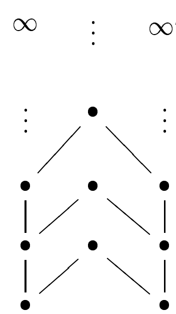
The Scott continuity of M follows directly from the continuity of the integral. In particular, we have that

$$M(\nu) = \sup\{\sum_{a\in A} r_a m(a) \mid \sum_{a\in A} r_a \delta_a \ll \nu\}.$$

We are now in a position to motivate regularity of measurements. Consider the following example where M, as defined above, fails to be a measurement.

Example 2. Let P be the dcpo obtained by adding a top element ∞ to the naturals in their usual order. Let $P' = \{n' \mid n' \in \mathbb{N}\} \cup \{\infty'\}$ be a disjoint copy of P. Finally write D for the dcpo consisting of the disjoint union of P and P' together with a copy of the naturals in the discrete order $\{n'' \mid n \in \mathbb{N}\}$, with $n, n' \sqsubseteq n''$ for all $n \in \mathbb{N}$. (See the diagram below.)

Define a measurement $m\colon D \to [0,1]$ by $m(\infty) = m(\infty') = m(n'') = 0$ for all $n \in \mathbb{N}$, and $m(n) = m(n') = 2^{-n}$ for all $n \in \mathbb{N}$. Now the valuation $\nu = \sum_{n''\in\mathbb{N}} 2^{-(n+1)}\delta_{n''}$ is in $\mathsf{ker}\, M$, and $\delta_0 \ll \nu$. Furthermore, defining $\rho_N = \sum_{n''\leqslant N} 2^{-(n+1)}\delta_{n''} + \sum_{n>N} 2^{-(n+1)}\delta_{n'}$ we have that $\rho_N \sqsubseteq \nu$ but not $\delta_0 \ll \rho_N$. However by choosing N large enough we can make $M(\rho_N)$ arbitrarily close to 1. Thus M is not a measurement on $\mathbf{P}D$, cf. Definition 3.



(5)

Assume $m\colon D \to [0,1]$ is a measurement, and M the extension to the powerdomain as in Definition 7. The next few propositions describe the kernel of M. It is worth remarking that in proving Proposition 6 we do not assume that valuations on continuous dcpos extend to measures.

Proposition 6. *Let $\nu \in \mathsf{ker}\, M$, i.e., $\int m d\nu = 1$. Then for a crescent $E = U \setminus V$, where $U, V \in \Sigma D$, we have that $\nu(E) > 0$ implies $E \cap \mathsf{ker}\, m \neq \emptyset$.*

Proof. We define a decreasing sequence of crescents $\langle E_n \mid n \in \mathbb{N}\rangle$ with $\nu(E_n) > 0$ for all $n \in \mathbb{N}$. First, $E_0 = E$. Next, assuming E_n is defined, let $\rho = \frac{1}{\nu(E_n)}\nu\,|_{E_n}$. Since

$$\nu = \nu\,|_{E_n} + \nu\,|_{E_n^c},$$

the inequality $M(\nu\,|_{E_n^c}) \leqslant \nu(E_n^c)$ forces $M(\nu\,|_{E_n}) = \nu(E_n)$, whence $M(\rho) = 1$. By the proof of Theorem 1 there is a dissection $\mathcal{D}$ of E_n and $0 < r < 1$ such that $M(\rho_{\mathcal{D},r}) > 1 - 1/n$. In particular, there exists $x_n \in E_n$, namely one of the mass points of $\rho_{\mathcal{D},r}$, such that $m(x_n) \geqslant 1 - 1/n$ and $\nu(E_n \cap \Uparrow x_n) > 0$. Now set $E_{n+1} = E_n \cap \Uparrow x_n$.

The proposition now follows since $\langle x_n \mid n \in \mathbb{N} \rangle$ is an increasing sequence in E, and so $\bigsqcup x_n \in E \cap \mathsf{ker}\, m$. $\square$

Proposition 7. *Let $\nu \in \mathsf{ker}\, M$. If $U_1, U_2 \in \Sigma D$ with $U_1 \cap \mathsf{ker}\, m = U_2 \cap \mathsf{ker}\, m$, then $\nu(U_1) = \nu(U_2)$.*

Proof. Since neither of the crescents $U_1 \setminus U_2$ and $U_2 \setminus U_1$ meets $\mathsf{ker}\, m$ it follows that

$$\begin{aligned} \nu(U_1) &= \nu(U_1 \cap U_2) + \nu(U_1 \setminus U_2) \\ &= \nu(U_1 \cap U_2) \ \ \text{(by Proposition 6)} \\ &= \nu(U_1 \cap U_2) + \nu(U_2 \setminus U_1) \ \ \text{(by Proposition 6)} \\ &= \nu(U_2). \end{aligned}$$

Theorem 6. *The set of normalized valuations on $\mathsf{ker}\, m$ is in bijection with $\mathsf{ker}\, M$. Furthermore, if continuous valuations on $\mathsf{ker}\, m$ are in one-to-one correspondence with Borel measures on $\mathsf{ker}\, m$, then the space of these measures in the weak topology is homeomorphic to $\mathsf{ker}\, M$ in the relative Scott topology.*

Proof. Suppose ν is a valuation on $\mathsf{ker}\, m$ with total mass 1. Then we easily see that $\nu^* : \Sigma D \to [0,1]$ defined by $\nu^*(O) = \nu(O \cap \mathsf{ker}\, m)$ is a valuation on ΣD. For all positive integers n, since

$$\nu^*\{x : m(x) > 1 - 1/n\} = \nu(\mathsf{ker}\, m) = 1,$$

$M(\nu^*) \geqslant 1 - 1/n$. Thus $\nu^* \in \mathsf{ker}\, M$.

Conversely, suppose $\nu \in \mathsf{ker}\, M$. We define a valuation ν_* on the open sets of $\mathsf{ker}\, m$ as follows. For an open set $O \subseteq \mathsf{ker}\, m$ we define $\nu_*(O) = \nu(O^\dagger)$ where $O^\dagger$ is the greatest Scott open subset of D such that $O^\dagger \cap \mathsf{ker}\, m = O$. Now for all open subsets O_1, O_2 of $\mathsf{ker}\, m$,

$$\begin{aligned} \nu_*(O_1 \cup O_2) + \nu_*(O_1 \cap O_2) &= \nu((O_1 \cup O_2)^\dagger) + \nu((O_1 \cap O_2)^\dagger) \\ &= \nu(O_1^\dagger \cup O_2^\dagger) + \nu(O_1^\dagger \cap O_2^\dagger) \quad \text{(by Proposition 7)} \\ &= \nu(O_1^\dagger) + \nu(O_2^\dagger) \quad \text{(by modularity)} \\ &= \nu_*(O_1) + \nu_*(O_2). \end{aligned}$$

Thus ν_* is modular. By similar reasoning it also follows that ν_* is Scott continuous. One easily sees that the maps $\nu \mapsto \nu^*$ and $\nu \mapsto \nu_*$ are inverse.

Suppose that continuous valuations on $\mathsf{ker}\, m$ are in one-to-one correspondence with Borel measures on $\mathsf{ker}\, m$. Recall that a net $\langle \nu_i \rangle_{i \in I}$ of normalized Borel measures on a Hausdorff space X converges to ν in the weak topology iff $\liminf_{i \in I} \nu_i(O) \geqslant \nu(O)$ for all open $O \subseteq X$. Using Theorem 2 it is routine to show that the bijection above is a homeomorphism. $\square$

Corollary 2. *If m is a regular measurement and D is ω-continuous, then the space of normalized Borel measures on $\mathsf{ker}\, m$ in the weak topology is homeomorphic to $\mathsf{ker}\, M$ in the relative Scott topology.*

Proof. By Theorem 4, $\mathsf{ker}\, m$ is a separable metric space. In this case, as we remarked earlier, continuous valuations and Borel measures are in one-to-one correspondence. $\square$

Henceforth we assume that m is a regular measurement.

Proposition 8. *Given $\nu \in \mathsf{ker}\, M$, $U \subseteq D$ Scott open and $\varepsilon > 0$, there exists a Scott open set $V \subseteq U$ and $\delta > 0$ such that $\nu(U \setminus V) < \varepsilon$ and $m_\delta(V \cap \mathsf{ker}\, m) \subseteq U$.*

Proof. Since m is regular we can write $U \cap \mathsf{ker}\, m$ as the directed union

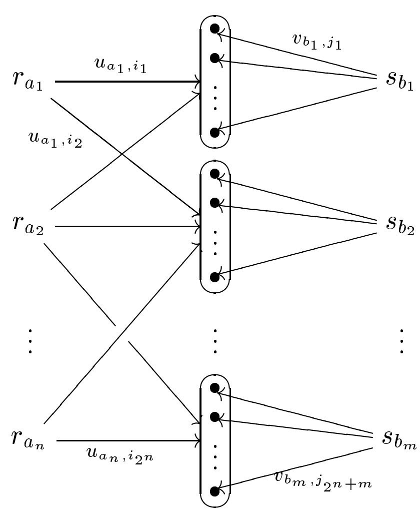
$$\bigcup \{V \cap \mathsf{ker}\, m \mid V \in \Sigma D, (\exists \delta < 1) m_\delta(V \cap \mathsf{ker}\, m) \subseteq U\}.$$

The valuation ν_* is continuous, thus there exists a Scott open set $V \subseteq U$ and $\delta > 0$ such that $\nu_*((U \setminus V) \cap \mathsf{ker}\, m) < \varepsilon$ and $m_\delta(V \cap \mathsf{ker}\, m) \subseteq U$. But $\nu = (\nu_*)^*$ satisfies $\nu(U \setminus V) < \varepsilon$. $\square$

We are now in a position to prove the main result of the paper, namely that M as given in Definition 7 is a measurement on $\mathbf{P}D$. Most of the work in the proof is contained in the following technical lemma.

Lemma 2. *Let $\nu \in \mathsf{ker}\, M$ and $\sigma = \sum_{a \in A} r_a \delta_a \ll \nu$. Then there exists $\varepsilon > 0$ such that whenever $\rho = \sum_{b \in B} s_b \delta_b \sqsubseteq \nu$ and $|M(\rho) - M(\nu)| < \varepsilon$, then $\sigma \ll \rho$.*

Proof. Applying Proposition 4 with the partition $\{E_i\}_{i \in I}$, where $I = \mathbb{P}A$ and $E_i = (\!|A, i|\!)$, we obtain a splitting $u = \{u_{a,i}\}$ between σ and ν. Now, given $\rho = \sum_{b \in B} s_b \delta_b \leqslant \nu$, we apply Proposition 4 once again, with the partition $\{F_j\}_{j \in J}$, where $J = \mathbb{P}(A \cup B)$ and $F_j = (\!|A \cup B, j|\!)$, we obtain a splitting $v = \{v_{b,j}\}$ between ρ and ν. Notice that the partition $\{F_j\}$ refines $\{E_i\}$. We write $j \equiv i$ whenever $j \cap A = i$, so $E_i = \bigcup_{j \equiv i} F_j$. We illustrate the splittings u and v in the following flow diagram.



(6)

Our strategy is to obtain a splitting between σ and ρ, in the sense of Lemma 1, by combining u and (a modification of) v using Proposition 5.

First we choose $\varepsilon_1 > 0$ to be a lower bound on the unfulfilled demand at each of the circled groups of nodes in the centre of the diagram above, i.e.,

$$\varepsilon_1 = \min_{i \in I} \left(\nu(E_i) - \sum_{a \in A} u_{a,i} \right) . \tag{7}$$

By Proposition 8, for each $i \in I$ there exists $\varepsilon_i > 0$ and a crescent $G_i \subseteq E_i$ such that $\nu(E_i \setminus G_i) < \varepsilon_1/3$ and $m_{\varepsilon_1}(G_i \cap \ker m) \subseteq E_i$. We now set $\varepsilon_2 = \min_{i \in I} \varepsilon_i$ and $\varepsilon = \varepsilon_1 \varepsilon_2/3$. Notice that the value of ε does not depend on ρ.

Next we amalgamate the flows on the right which go into the same circled group of nodes. In fact we also discard a flow number $v_{b,j}$ if the measurement of the source node b is too low or the target crescent F_j does not meet any G_i. Formally we let $\mathrm{Ok}_B = \{b \in B : 1 - m(b) < \varepsilon_2\}$ and $\mathrm{Ok}_i = \{j \in J : j \equiv i \ \wedge \ \nu(G_i \cap F_j) \neq 0\}$. We define a $B \times I$-indexed family of transport numbers $w_{b,i}$ by $w_{b,i} = 0$ if $b \notin \mathrm{Ok}_B$, otherwise

$$w_{b,i} = \sum_{j \in \mathrm{Ok}_i} v_{b,j}.$$

We claim that $w \star u$ defines a splitting between σ and ρ in the sense of Lemma 1. We verify condition (iii) of the lemma as follows.

$$\begin{aligned}
(w \star u)_{a,b} > 0 &\Rightarrow (\exists i)(u_{a,i} > 0 \ \wedge \ w_{b,i} > 0) \\
&\Rightarrow (\exists i)(\exists j)(u_{a,i} > 0 \ \wedge \ v_{b,j} > 0 \ \wedge \ b \in \mathrm{Ok}_B \ \wedge \ j \in \mathrm{Ok}_i) \\
&\Rightarrow (\exists i)(\exists j)(a \in i \ \wedge \ b \in j \ \wedge \ b \in \mathrm{Ok}_B \ \wedge \ j \in \mathrm{Ok}_i)
\end{aligned}$$

Now $j \in \mathrm{Ok}_i$ implies that $\nu(G_i \cap F_j) \neq 0$. Thus, by Proposition 6, there exists $z \in G_i \cap F_j \cap \ker m$. Since $b \sqsubseteq z$, we have $b \in m_{\varepsilon_2}(G_i \cap \ker m) \subseteq \Uparrow a$, i.e., $a \ll b$.

We wish to apply Proposition 5 to complete the proof that $w \star u$ defines a splitting between σ and ρ. To do this we need some estimates (given in (8) and (9) below) of the mass we 'threw away' in going from v to w. Firstly, from

$$\sum_{b \in B} \sum_{j \in J} v_{b,j}(1 - m(b)) = \sum_{b \in B} s_b(1 - m(b)) \leqslant 1 - M(\rho) < \varepsilon$$

it follows that

$$\sum_{b \notin \mathrm{Ok}_B} \sum_{j \in J} v_{b,j} \leqslant \sum_{b \notin \mathrm{Ok}_B} \sum_{j \in J} \frac{v_{b,j}(1 - m(b))}{\varepsilon_2} < \frac{\varepsilon}{\varepsilon_2} = \frac{\varepsilon_1}{3}. \tag{8}$$

Also, from the definition of G_i we have that for each fixed $i \in I$,

$$\nu(\bigcup_{j \equiv i, j \notin \mathrm{Ok}_i} F_j) \leqslant \nu(E_i \setminus G_i) < \frac{\varepsilon_1}{3},$$

and so, since $\sum_{b\in B} v_{b,j} \leqslant \nu(F_j)$,

$$\sum_{j\equiv i, j\notin \mathrm{Ok}_i} \sum_{b\in B} v_{b,j} < \frac{\varepsilon_1}{3}. \tag{9}$$

Combining (8), (9) and the definition of $w_{b,i}$ we get that for each $i \in I$

$$\left(\sum_{b\in B}\sum_{j\equiv i} v_{b,j}\right) - \sum_{b\in B} w_{b,i} < \frac{2\varepsilon_1}{3}. \tag{10}$$

Now the total mass of a valuation is no bigger than its measurement, thus

$$\sum_{j\in J}\left(\nu(F_j) - \sum_{b\in B} v_{b,j}\right) \leqslant 1 - M(\rho) < \varepsilon \leqslant \frac{\epsilon_1}{3}. \tag{11}$$

Each term in the summation over $j \in J$ is positive. Thus, for each $i \in I$, taking the partial sum in (11) over those $j \in J$ with $j \equiv i$, we get

$$\nu(E_i) - \sum_{j\equiv i}\sum_{b\in B} v_{b,j} < \frac{\varepsilon_1}{3}.$$

Adding this inequality to (10) we get

$$\nu(E_i) - \sum_{b\in B} w_{b,i} < \varepsilon_1.$$

From the definition of ε_1 it follows that for each $i \in I$,

$$\sum_{a\in A} u_{a,i} < \sum_{b\in B} w_{b,i}.$$

Thus we may apply Proposition 5 to deduce that $\sum_{b\in B}(w\star u)_{a,b} = \sum_{i\in I} u_{a,i} = r_a$ and $\sum_{a\in A}(w\star u)_{a,b} < \sum_{i\in I} w_{b,i} \leqslant s_b$. □

Having proved Lemma 2, the result that M is a measurement now follows from general domain theory.

Theorem 7. *Let $\nu \in \ker M$ and $\sigma \ll \nu$. Then there exists $\varepsilon > 0$ such that whenever $\rho \sqsubseteq \nu$ and $|M(\rho) - M(\nu)| < \varepsilon$, then $\sigma \ll \rho$.*

Proof. By the interpolation property of $\ll$ there exists a simple valuation σ' with $\sigma \ll \sigma' \ll \nu$. By Lemma 2 there exists $\varepsilon > 0$ such that whenever $\rho' \sqsubseteq \nu$ is simple and $M(\rho') > 1 - \varepsilon$, then $\sigma' \ll \rho'$. But if $\rho \sqsubseteq \nu$ is an arbitrary valuation with $M(\rho) > 1 - \varepsilon$, then there is a simple valuation $\rho' \ll \rho$ with $M(\rho') > 1 - \varepsilon$. Thus $\sigma \ll \sigma' \ll \rho' \ll \rho$. □

6 Iterated Function Systems

Definition 8. *An* iterated function system *(IFS) on a complete metric space X is a collection of continuous maps $f_i\colon X \to X$ indexed over a finite set I. Such an IFS is denoted $\langle X, \{f_i\}_{i\in I}\rangle$. If each map f_i is contracting then the IFS is said to be* hyperbolic.

A hyperbolic IFS induces a contraction F on the complete metric space of non-empty compact subsets of X equipped with the Hausdorff metric. F is defined by

$$F(K) = \bigcup_{i\in I} f_i(K).$$

By Banach's contraction mapping theorem, F has a unique fixed point: the *attractor* of the IFS. An alternate domain-theoretic proof this result, due to Hayashi [7], involves considering F as a continuous selfmap of $\mathbf{U}X$ and deducing that the least fixed point of F is maximal in $\mathbf{U}X$, and therefore is a unique fixed point. Many different fractal sets arise as, or can be approximated by, attractors of IFSs.

Definition 9. *A* weighted IFS $\langle X, \{(f_i, p_i)\}_{i\in I}\rangle$ *consists of an IFS* $\langle X, \{f_i\}_{i\in I}\rangle$ *and a family of weights* $0 < p_i < 1$, *where* $\sum_{i\in I} p_i = 1$. *These data induce a so-called Markov operator* $G\colon \mathcal{M}X \to \mathcal{M}X$ *on the set* $\mathcal{M}X$ *of normalized Borel measures on* X, *given by*

$$G(\nu)(B) = \sum_{i\in I} p_i \nu(f_i^{-1}(B)) \tag{12}$$

for each Borel subset $B \subseteq X$.

The space $\mathcal{M}X$ equipped with the weak topology can be metrized by the *Hutchinson metric* [9]. Furthermore, if a weighted IFS is hyperbolic then the map G is contracting with respect to the Hutchinson metric. In this case the unique fixed point of G, obtained by the contraction mapping theorem, defines a normalized measure called an *invariant measure* for the IFS. The support of the invariant measure is the attractor of the underlying IFS. This construction is an important method of defining fractal measures. Next we outline a domain-theoretic construction, due to Edalat [3], of invariant measures for so-called weakly hyperbolic IFSs on compact metric spaces.

Edalat's approach involves embedding the set of measures on a compact metric space X in the domain $\mathbf{PU}X$ of valuations on the upper space of X. Recall from Section 3 that $\mathbf{U}X$ admits a natural measurement $m\colon \mathbf{U}X \to [0,1]$, where $m(K) = 2^{-|K|}$; in turn this yields a measurement M on $\mathbf{PU}X$. Next, a weighted IFS $\langle X, \{(f_i, p_i)\}_{i\in I}\rangle$ induces a continuous map $T\colon \mathbf{PU}X \to \mathbf{PU}X$ – the domain theoretic analogue of the Markov operator – defined by

$$T(\nu)(O) = \sum_{i\in I} p_i \nu((\mathbf{U}f_i)^{-1}(O)) \tag{13}$$

where $\mathbf{U}f_i : \mathbf{U}X \to \mathbf{U}X$ is the map $K \mapsto f_i(K)$.

Applying T to δ_X, the point valuation concentrated at X, one obtains $T(\delta_X) = \sum_{i\in I} p_i \delta_{f_i(X)}$. Iterating, it follows that

$$T^n(\delta_X) = \sum_{i_1,\dots,i_n \in I} p_{i_1} \dots p_{i_n} \delta_{f_{i_1}\dots f_{i_n}(X)}. \tag{14}$$

Thus $M(T^n(\delta_X))$, the measurement of the n-th iterate, equals

$$\sum_{i_1,\dots,i_n \in I} p_{i_1} ... p_{i_n} m(f_{i_1} \dots f_{i_n}(X)).$$

A sufficient condition ensuring that $M(T^n(\delta_X)) \to 1$ as $n \to \infty$ is to require that for all $\varepsilon > 0$, there exists $n \geqslant 0$ such that $|f_{i_1} \dots f_{i_n}(X)| < \varepsilon$ for all sequences $i_1 i_2 ... i_n \in I^n$. In fact, by König's lemma, it is sufficient that for each infinite sequence $i_1 i_2 ... \in I^\omega$, $|f_{i_1} \dots f_{i_n}(X)| \to 0$ an $n \to \infty$. Edalat calls an IFS satisfying the latter condition *weakly hyperbolic*. It is clearly the case that every hyperbolic IFS is weakly hyperbolic.

Theorem 8 (Edalat [3]). *A weakly hyperbolic weighted IFS $\langle X, \{(f_i, p_i)\}_{i\in I}\rangle$ on a compact metric space X has a unique invariant measure which is moreover an attractor for the Markov operator (12).*

Proof. Every continuous valuation on a compact metric space extends to a Borel measure, and conversely every Borel measure restricts to a continuous valuation. Thus, to prove the existence of a unique invariant measure, it suffices to prove that there is a unique continuous valuation ν on X such that $\nu(O) = \sum_{i\in I} p_i \nu(f_i^{-1}(O))$ for all open $O \subseteq X$.

Let D be the sub-dcpo of $\mathbf{PU}X$ consisting of valuations with mass 1. Then D is pointed and continuous, and T restricts to a monotone map $D \to D$. Thus we may apply Theorem 3 to deduce that T has a unique fixed point on D, and this point lies in $\mathsf{ker}\, M$.

By an obvious identification of X with $\mathsf{ker}\, m$ we may regard the Markov operator G, defined in (12), as a selfmap of the set of valuations on $\mathsf{ker}\, m$. Next we show that G so regarded agrees with T. Formally, if $O \subseteq \mathbf{U}X$ is Scott open, then using the notation of Theorem 6, we have

$$\begin{aligned} G(\nu_*)^*(O) &= G(\nu_*)(O \cap \mathsf{ker}\, m) \\ &= \sum_{i\in I} p_i \nu_*(f_i^{-1}(O \cap \mathsf{ker}\, m)) \\ &= \sum_{i\in I} p_i \nu((\mathbf{U}f_i)^{-1}(O)) \quad (\text{as } f_i^{-1}(O \cap \mathsf{ker}\, m) = (\mathbf{U}f_i)^{-1}(O) \cap \mathsf{ker}\, m) \\ &= T(\nu)(O) \end{aligned}$$

Since $T = G$ on $\mathsf{ker}\, m$ we know that the unique fixed point of T is a unique invariant measure. Furthermore, it also follows that T takes $\mathsf{ker}\, M$ into $\mathsf{ker}\, M$, so, by Theorem 3, the fixed point of T is an attractor for T in the relative Scott topology on $\mathsf{ker}\, M$. But $\mathsf{ker}\, M \simeq \mathcal{M}X$, so the invariant measure for G is also an attractor in the weak topology. □

The construction of the unique invariant measure here is essentially the same as in Edalat [3]. However it is justified in a different way. Edalat deduces that the least fixed point of T is a unique fixed point by proving that it is maximal. This observation depends on a characterization of the maximal elements of $\mathbf{PU}X$ in terms of their supports. This last requires some more measure-theoretic machinery than we have used here: in particular he uses the result of Lawson [12] on extending continuous valuations on ω-continuous dcpos to Borel measures over the Lawson topology.

7 Future Work

By way of conclusion here are some questions we would like to know the answers to.

- Is there a Lebesgue measurement which is not regular? Possibly the domain of partial functions on the naturals admits such a measurement since the maximal elements of this domain are not locally compact in the relative Scott topology.
- We know that if m is a regular measurement, then M is a measurement. Does the converse hold?
- Are the following equivalent for a countably based domain D?
  (i) Every Lebesgue measurement μ with $\mathsf{ker}\,\mu = \mathsf{max}\,D$ is regular.
  (ii) The space $\mathsf{max}\,D$ is locally compact, regular and second countable.
- An interesting problem is to characterize the maximal elements of the probabilistic powerdomain. In particular, if $\mathsf{ker}\,m = \mathsf{max}\,D$, do we have $\mathsf{ker}\,M = \mathsf{max}\,\mathbf{P}D$?

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