Draft (22 April 2006)
27 pages

Discrete Random Variables Over Domains

Michael Mislove [1]

Department of Mathematics
Tulane University
New Orleans, LA 70118

Abstract

In this paper we initiate the study of discrete random variables over domains. Our work is inspired by work of Daniele Varacca, who devised indexed valuations as models of probabilistic computation within domain theory. Our approach relies on new results about commutative monoids defined on domains that also allow actions of the non-negative reals. Using our approach, we define two such families of *real domain monoids*, one of which allows us to recapture Varacca's construction of the *Plotkin indexed valuations* over a domain. Each of these families leads to the construction of a family of discrete random variables over domains, the second of which forms the object level of a continuous endofunctor on the categories RB (domains that are retracts of bifinite domains), and on FS (domains where the identity map is the directed supremum of deflations finitely separated from the identity). The significance of this last result lies in the fact that there is no known category of continuous domains that is closed under the probabilistic power domain, which forms the standard approach to modeling probabilistic choice over domains. The fact that RB and FS are Cartesian closed and also are closed under a power domain of discrete random variables means we can now model, e.g., the untyped lambda calculus extended with a probabilistic choice operator, implemented via random variables.

1 Introduction

Domain theory, perhaps the most widely used method for assigning denotational meanings to programming languages, has recently seen its influence broaden to other areas of computation and mathematics. It provides a broad range of constructors for modeling data types, nondeterminism, functional programming, and several other constructs needed in semantics. Domain theory also admits a number of Cartesian closed categories, the fundamental

[1] This work supported by the US National Science Foundation and the US Office of Naval Research

objects needed to model the lambda calculus. Even probabilistic computation admits a model in the theory, although truth to tell, this particular constructor has proven to be very difficult to unravel. Of particular interest is the question

Is there a Cartesian closed category of domains
that is closed under the probabilistic power domain?

There have been many attempts to resolve this, but the most we know to date is contained in [12], where it is shown that the probabilistic power domain of a finite tree is in RB, that the probabilistic power domain of a finite reversed tree is in FS, and that RB is closed under the probabilistic power domain if the probabilistic power domain of every finite poset is in RB. But, other than finite trees, the only finite posets whose probabilistic power domain is known to be in RB is the class of flat posets, whose probabilistic power domains are bounded complete (the continuous analogs of Scott domains).

We do not contribute to settling this question here, but we do provide an alternative construction—what we call the *power domain of discrete* $\times$*-random variables*, which we show defines a continuous endofunctor on the category RB, as well as on FS and on CDOM, the category of coherent domains.

Objects in RB are retracts of *bifinite domains*, those that can be expressed as bilimits of finite posets under embedding–projection pairs. This category is Cartesian closed, and it also is closed under the various power domains for nondeterminism [7]. With the addition of a mechanism to model probabilistic choice, RB provides virtually all the tools required to support semantic modeling. Furthermore, playing off results of Varacca [23,24], we show that the formation of the power domain of discrete $\times$-random variables over RB yields a monad that enjoys a distributive law with respect to each of the power domain monads, and this in turn implies that each of these power domain monads lifts to a monad on the category RB that are also algebras for the discrete $\times$-random variable power domain monad. These laws are enumerated in Definition 4.8. In short, we can now form domain-theoretic models of computation that respect the laws of discrete $\times$-random variables as well as any of the laws we choose for nondeterminism: angelic, demonic or convex choice.

The outline of the rest of the paper is as follows. In the next section we provide some background about domains and about the constructions we need. We then review briefly a construction by Varacca [24] which inspired our work, and that Varacca calls the *Plotkin indexed valuations*. In the following section, we investigate bag domains—domain-theoretic models for multisets, and we also explore the structure of bag domains that also admit an action of the nonnegative reals. We single out two such constructs, which are distinguished by how the least element acts. In the first, $\perp$ is the identity of the monoid, as well as being the image of 0 under the action ot $\mathbb{R}_{\geq 0}$. We define a functor which produces the initial such algebra over a bag domain, and this in turn is what we use to recapture Varacca's family of Plotkin indexed valuations. The second such construct is distinguished by the fact that $\perp$ acts as a multiplicative zero in the monoid, rather than as an identity, and we again define a functor which produces the initial such algebra over a bag domain.

The constructions of initial bag domain monoids admitting actions of the non-negative reals then are exploited to define two families of discrete random variables over domains. The first is based on the construction that led to recapturing Varacca's results, and because $\perp$ acts like an additive zero, we denote this family by $\mathbb{RV}_+(P)$, for a domain P. The second family is based on our second initial algebra over bag domains, and since $\perp$ acts like a multiplicative zero in this case, we denote this family by $\mathbb{RV}_\times(P)$. We also show that $\mathbb{RV}_\times$ defines an continuous endofunctor on the category of domains that leaves both RB and FS invariant. This yields two Cartesian closed categories of domains that support a model of probabilistic computation. In the final section, we discuss further work along this line, including how to construct Varacca's other families of indexed valuations. We also discuss the relationship between a random variable approach to modeling probabilistic computation and one based directly on probability distributions.

1.1 Background

We begin with some basic results about partial orders, and about domains in particular. A general reference for this material is [1] or [4].

A subset A of a partially ordered set P is *directed* if A has an upper bound for each of its finite subsets. A mapping between partially ordered sets is *Scott continuous* if it preserves the order and the suprema of those directed sets that have a supremum. A *directed complete partial order (dcpo)* is a partially ordered set in which each directed set has a least upper bound. A *cpo* is a dcpo with a least element $\perp$.

If P is a partial order and $x, y \in P$, then we say *x is way-below y ($x \ll y$)* if whenever $A \subseteq P$ is directed and has a supremum, if $y \sqsubseteq \sqcup A$, then $x \sqsubseteq a$ for some $a \in A$. A poset P is *continuous* if $\Downarrow y = \{x \in P \mid x \ll y\}$ is directed and $y = \sqcup \Downarrow y$ for each $y \in P$. A *domain* is a continuous dcpo. We let CPOS denote the category of continuous posets and Scott continuous maps, and DOM the full subcategory of domains.

An *abstract basis* is a pair $(B, \ll)$ where $\ll$ is a transitive relation on B satisfying the *interpolation property:*

$$F \ll x \ \&\ F \subseteq B \text{ finite } \Rightarrow\ (\exists y \in B)\ F \ll y \ll x.$$

By $F \ll x$ we mean $z \ll x\ \forall z \in F$. If $(B, \ll)$ is an abstract basis, then $I \subseteq B$ is a *round ideal* if I is a $\ll$-directed, $\ll$-lower set, and $x \in I \Rightarrow (\exists y \in I)\ x \ll y$. The *round-ideal completion* of an abstract basis $(B, \ll)$ is the family of round ideals, ordered by inclusion. This forms a domain, where $I \ll J$ iff $(\exists x \ll y \in B)\ I \subseteq \Downarrow x \subseteq \Downarrow y \subseteq J$. In fact, every domain P is isomorphic to the round-ideal completion of an abstract basis, namely P is isomorphic to the round ideal completion of $(P, \ll)$ under the mapping sending a point x to $\Downarrow x$, whose inverse is the mapping that sends a round ideal to its supremum.

One of the fundamental results about dcpos is that the family of Scott continuous maps between two dcpos is another dcpo in the pointwise order. Since it's easy to show that the finite product of a family of continuous posets is another such, and the one-point poset is a terminal object, a central question is under what conditions is the family of Scott continuous selfmaps $[D \to E]$ between domains also continuous, i.e., which categories of dcpos and Scott continuous maps are Cartesian closed? This is true of DCPO, the category of dcpos and Scott continuous maps, but not of DOM. Still, there are several full subcategories of DOM that are Cartesian closed. Among the notable such categories are the following: [2]

- BCD *Bounded complete domains*, in which every subset having an upper bound has a least upper bound; equivalently, every non-empty subset has a greatest lower bound.
- RB Domains which are retracts of *bifinite domains*, themselves bilimits of families of finite posets under embedding-projection maps; these are pairs of Scott continuous mappings $e\colon P \to Q$ and $p\colon Q \to P$ satisfying $p \circ e = 1_P$ and $e \circ p \leq 1_Q$.
- FS Domains D satisfying the property that the identity map is the directed supremum of Scott-cotinuous selfmaps $f\colon D \to D$ each *finitely separated* from the identity: i.e., for each f there is a finite subset $M_f \subseteq D$ with the property that, for each $x \in D$, there is some $m \in M_f$ with $f(x) \leq m \leq x$.

Actually, BCD is a full subcategory of RB, which in turn is a full subcategory of FS, and FS is a maximal ccc of domains. An interesting (some might say frustrating) open question is whether RB and FS are equal. The objects in all of these categories are *coherent*, [3] but the category CDOM of coherent domains and Scott continuous maps is not a ccc.

We also recall some facts about categories. A *monad* or *triple* on a category A is a a 3-tuple $\langle T, \mu, \eta\rangle$ where $T\colon \mathsf{A} \to \mathsf{A}$ is an endofunctor, and $\mu\colon T^2 \stackrel{\cdot}{\longrightarrow} T$ and $\eta\colon 1_{\mathsf{A}} \stackrel{\cdot}{\longrightarrow} T$ are natural transformations satisfying the laws:

$$\mu \circ T\mu = \mu \circ \mu_T \quad \text{and} \quad \mu \circ \eta_T = T = \mu \circ T\eta.$$

Equivalently, if $F\colon \mathsf{A} \to \mathsf{B}$ is left adjoint to $G\colon \mathsf{B} \to \mathsf{A}$ with unit $\eta\colon 1_{\mathsf{A}} \stackrel{\cdot}{\longrightarrow} GF$ and counit $\epsilon\colon FG \stackrel{\cdot}{\longrightarrow} 1_{\mathsf{B}}$, then $\langle GF, G\epsilon F, \eta\rangle$ forms a monad on A, and every monad arises in this fashion.

If $\langle T, \mu, \eta\rangle$ is a monad, then a *T-algebra* is a pair (a, h), where $a \in \mathsf{A}$ and $h\colon Ta \to a$ is an A-morphism satisfying $h \circ \eta_a = 1_a$ and $h \circ Th = h \circ \mu_a$.

For example, domain theory provides three models for nondeterminism, the *lower power domain* $\mathcal{P}_L$,which assigns to a domain the family of Scott-closed

[2] See [1] for details about these categories.
[3] A domain is *coherent* if its Lawson topology is compact; cf. [1]

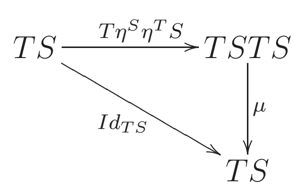
lower sets with union as the semilattice operation, the *upper power domain*, $\mathcal{P}_U$ which assigns to a domain the family of Scott-compact upper sets with union as the operation, but with reverse inclusion as the order, and the *convex power domain*, $\mathcal{P}_C$, which assigns to a (coherent) domain the family of sets that can be expressed as the intersection of a Scott-closed lower set and a Scott-compact upper set, with the *Egli-Milner order*, where the "sum" of sets is the smallest such set containing their union (cf. [1] for details here). Each of these defines a monad on DCPO (cf. [7]), whose algebras are ordered semilattices; another example is the *probabilistic power domain* $\mathbb{V}$ whose algebras satisfy equations that characterize the probability measures over P (cf. [10]).

One goal of domain theory is to provide a setting in which all of the constructors needed to model a given programming language can be combined. If the aim is to model both nondeterminism and probabilistic choice, then one needs to combine the appropriate nondeterminism monad with the probabilistic power domain monad, so that the laws of each constructor are preserved in the resulting model. This is the function of a *distributive law*, which is a natural transformation $d\colon ST \stackrel{\cdot}{\longrightarrow} TS$ between monads S and T on A satisfying several identities—cf. [2]. The significance of distributive laws is the following theorem of Beck:

Theorem 1.1 (Beck [2]) *Let (T, η^T, μ^T) and (S, η^S, μ^S) be monads on the category A. Then there is a one-to-one correspondence between*

(i) *Distributive laws $d : ST \stackrel{\cdot}{\longrightarrow} TS$;*

(ii) *Multiplications $\mu : TSTS \stackrel{\cdot}{\longrightarrow} TS$, satisfying*
- *$(TS, \eta^T\eta^S, \mu)$ is a monad;*
- *the natural transformations $\eta^T S : S \stackrel{\cdot}{\longrightarrow} TS$ and $T\eta^S : T \stackrel{\cdot}{\longrightarrow} TS$ are monad morphisms;*
- *the following* middle unit law *holds:*



(iii) *Liftings $\tilde{T}$ of the monad T to A^S, the category of S-algebras in A.* □

So, one way to know that the combination of the probabilistic power domain and one of the power domains for nondeterminism provides a model satisfying all the needed laws would be to show there is a distributive law of one of these monads over the probabilistic power domain monad. Unfortunately, it was shown by Plotkin and Varacca [23] that there is no distributive law of $\mathbb{V}$ over $\mathcal{P}_X$, or of $\mathcal{P}_X$ over $\mathbb{V}$ for any of the nondeterminism monads $\mathcal{P}_X$. This led to the work we report on next.

2 Indexed Valuations

We now recall some of the work of Varacca [24] that was motivated by problems associated with trying to support both nondeterminism and probabilistic choice within the same model. Once it was shown that there is no distributive law between $\mathbb{V}$ and any of the nondeterminism monads, Varacca considered the simpler situation of sets, where the model of nondeterminism is the power set monad, and, in the finite case, the probabilistic monad is the family of simple measures on the set. A theorem of Gautam shows why the distributive law doesn't hold in this setting:

Theorem 2.1 (Gautam [3]) *A necessary and sufficient condition for an equational theory to extend from a model X to its power set is that every law of the theory mentions each variable at most once on each side of the equation.* □

Since finite posets are domains, this implies that, even if X is a probabilistic algebra, the operations on X cannot be extended to $\mathcal{P}(X)$ to satisfy the same laws. In fact, both the nondeterminism monad over a set X and the probabilistic monad over X violate the conditions of the theorem. For the nondeterminism monad, it is the law $x \oplus x = x$, while for the probabilistic monad, it is the law $pA \oplus (1-p)A = A$.[4] Then Varacca realized that weakening one of the laws of probabilistic choice could result in a monad that enjoys a distributive law with respect to a monad for nondeterminism. For $0 < p < 1$ and A a domain element,Varacca weakened the law $pA + (1-p)A = A$ in three ways:

$$pA + (1-p)A \sqsupseteq A, \tag{1}$$
$$pA + (1-p)A \sqsubseteq A, \tag{2}$$
$$pA + (1-p)A \text{ and } A \text{ not necessarily related by order.} \tag{3}$$

He called the monad he constructed satisfying (1) the *Hoare indexed valuations*, the one satisfying (2) the *Smyth indexed valuations* and the one satisfying (3), the *Plotkin indexed valuations*. We exploit this last construction—the so-called *Plotkin indexed valuations* over a domain—in the construction of a power domain of discrete random variables.

2.1 Plotkin Indexed Valuations

Definition 2.2 An *indexed valuation* over the poset P is a tuple $(r_i, p_i)_{i\in I}$ where I is an index set,[5] each $r_i \geq 0$ is a non-negative real number and $p_i \in P$ for each $i \in I$.

[4] The operator $pA \oplus (1-p)A$ models probabilistic choice where the left branch is chosen with probability $0 \leq p \leq 1$ and the right branch is chosen with probability $1-p$.

[5] For our discussion, we can assume I is always finite.

Two indexed valuations satisfy $(r_i, p_i)_I \simeq_1 (s_j, q_j)_J$ if $|I| = |J|$ and there is a permutation $\phi \in S(|I|)$ [6] with $r_{\phi(i)} = s_i$ and $p_{\phi(i)} = q_i$ for each i.

If $I' = \{i \in I \mid r_i \neq 0\}$ and similarly for J, then $(r_i, p_i)_I \simeq_2 (s_j, q_j)_J$ if $(r_i, p_i)_{I'} \simeq_1 (s_j, q_j)_{J'}$.

We let $\simeq$ denote the equivalence relation on indexed valuations generated by $\simeq_2$. For an indexed valuation $(r_i, p_i)_I$, we let $\langle r_i, p_i\rangle_I$ denote the equivalence class modulo $\simeq$.

Next, let $\overline{\mathbb{R}_{\geq 0}}$ denote the extended non-negative real numbers, with the usual order. Then given a domain P, Varacca defines a relation $\ll_{IV_P}$ on the family $\{\langle r_i, p_i\rangle_I \mid r_i \in \overline{\mathbb{R}_{\geq 0}} \ \&\ p_i \in P\}$ of indexed valuations over P by

$$\begin{aligned} \langle r_i, p_i\rangle_I \ll_{IV_P} \langle s_j, q_j\rangle_J \quad \text{iff} \quad & (|I| = |J|) \,\&\, (\exists \phi \in S(|I|)) \\ & r_i \ll s_{\phi(i)} \ \&\ p_i \ll_{IV_P} q_{\phi(i)} \ (\forall i \in I).^6 \end{aligned} \tag{4}$$

This forms an abstract basis whose round ideal completion is the family of *Plotkin indexed valuations over* P. We denote this domain by $IV_P(P)$.

We also can "add" indexed valuations $\langle r_i, p_i\rangle_I$ and $\langle s_j, q_j\rangle_J$ by

$$\langle r_i, x_i\rangle_{i\in I} \oplus \langle s_j, y_j\rangle_{j\in J} = \langle t_k, z_k\rangle_{k\in K}$$

where $K = I \dot\cup J$ and

$$(t_k, z_k) = \begin{cases} (r_i, x_i) & \text{if } k = i \in I \\ (s_j, y_j) & \text{if } k = j \in J. \end{cases}$$

This operation of concatenating tuples and taking the equivalence class of the resulting $I \dot\cup J$-tuple forms a continuous operation on indexed valuations that is commutative, by construction. We let $\overline{\mathbb{R}_{\geq 0}}$ act on $\langle r_i, p_i\rangle_I$ by $r \cdot \langle r_i, p_i\rangle_I = \langle r \cdot r_i, p_i\rangle_I$. Varacca's main result for the family of Plotkin indexed valuations can be summarized as follows:

Theorem 2.3 (Varacca [23])

(i) *If P is a continuous poset, then the family of* Plotkin indexed valuations *ordered by $\ll_{IV_P}$ as defined in (4) is an abstract basis.*

(ii) *The round ideal completion of the Plotkin indexed valuations, $IV_P(P)$, admits an addition $\oplus$ and a scalar multiplication by elements of $\overline{\mathbb{R}_{\geq 0}}$ that satisfy the following laws:*

(1)	$A \oplus B = B \oplus A$	*(2)*	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$
(3)	$A \oplus \underline{0} = A$	*(4)*	$0A = \underline{0}$
(5)	$1A = A$	*(6)*	$p(A \oplus B) = pA \oplus pB$
(7)	$p(qA) = (pq)A$	*where*	$p, q \in \mathbb{R}_{\geq 0}$ *and* $A, B, C \in IV_P(P)$.

[6] $S(n)$ denotes the permutation group on an n-element set.

[7] Note that $r \ll s$ iff $r = 0$ or $r < s$ for $r, s \in \overline{\mathbb{R}_{\geq 0}}$.

(iii) *IV_P defines the object level of an endofunctor which is monadic over* DOM*, and each of the power domain monads satisfies a distributive law with respect to the Plotkin indexed valuations monad.* □

A *coherent domain* is one whose *Lawson topology* is compact; it is a standard result of domain theory is that the Plotkin power domain applied to a coherent domain yields another such (cf. [1] for details). A corollary of Theorem 2.3 is that the composition $\mathcal{P}_P \circ IV_P$ defines a monad on CDOM, the category of coherent domains, whose algebras satisfy the laws listed in Theorem 2.3(i) and the laws of the Plotkin power domain:

(i) $X + Y = Y + X$
(ii) $X + X = X$
(iii) $X + (Y + Z) = (X + Y) + Z$

In other words, $\mathcal{P}_P(IV_P(P))$ is the initial domain semilattice algebra over P that also satisfies the laws listed in Theorem 2.3(ii).

3 Bag domains

In this section we develop some results that are fundamental for our main construction. More details about these results are contained in [17]. The earliest comments in the literature about *bag domains*—domains whose elements are *bags* or *multisets* from an underlying domain, are in [20] where Exercise 103 discusses them; Poigné [21] also comments on the existence of free bag domains. But the first paper devoted to bag domains is apparently [6], where a topological approach to their construction is investigated. There is the work of Vickers [25] and of Johnstone [8,9], but these works were inspired by problems in database theory, and the focus is on abstract categorical construction, not on domains as we consider them. To be sure, we also are interested in categorical aspects, but our aim is more in the spirit of [6], which includes analyzing the internal structure of these objects. Our results also allow us to capture the constructions of Varacca [23] more concretely. We begin with a simple result about posets:

Definition 3.1 Let P be a poset and let $n \in \mathbb{N}$. For $\phi \in S(n)$, define a mapping

$$\phi\colon P^n \to P^n \quad \text{by} \quad \phi(d)_i = d_{\phi^{-1}(i)}. \tag{5}$$

Then ϕ permutes the components of d according to ϕ's permutation of the indices $i = 1, \ldots, n$. Next, define a preorder $\preceq_n$ on P^n by

$$d \preceq_n e \quad \text{iff} \quad (\exists \phi \in S(n))\ \phi(d) \leq e \quad \text{iff} \quad d_{\phi^{-1}(i)} \leq e_i\ (\forall i = 1 \ldots, n). \tag{6}$$

Finally, we define the equivalence relation $\equiv_n$ on P^n by

$$\equiv_n\ =\ \preceq_n \cap \succeq_n, \tag{7}$$

and we note that $(P^n/\!\equiv_n, \sqsubseteq_n)$ is a partial order. We denote by $[d]$ the image of $d \in P^n$ in $P^n/\!\equiv_n$.

Lemma 3.2 *Let P be a poset, let $n \in \mathbb{N}$, and let $d, e \in P^n$. Then the following are equivalent:*

(i) $d \sqsubseteq_n e$,

(ii) $\uparrow\{\phi(d) \mid \phi \in S(n)\} \supseteq \uparrow\{\phi(e) \mid \phi \in S(n)\}$.

Proof. For (i) implies (ii), we note that, if $\phi \in S(n)$ satisfies $d_{\phi^{-1}(i)} \leq e_i$, then $d_i \leq e_{\phi(i)}$ for each $i = 1, \ldots, n$, so $\phi^{-1}(e) \in \uparrow d$, and then $\psi(e) \in \uparrow\{(\phi(d) \mid \phi \in S(n)\}$ for each $\psi \in S(n)$ by composing permutations, from which (iii) follows. Conversely, (ii) implies (i) is clear. $\square$

We also need a classic result due to M.-E. Rudin (cf. Lemma III-3.3 of[4].

Lemma 3.3 (Rudin) *Let P be a poset and let $\{\uparrow F_i \mid i \in I\}$ be a filter basis of non-empty, finitely-generated upper sets. Then there is a directed subset $A \subseteq \cup_i F_i$ with $A \cap F_i \neq \emptyset$ for all $i \in I$.* $\square$

Next, let P be a dcpo and let $n > 0$. We can apply the lemma above to derive the following:

Proposition 3.4 *Let P be a dcpo, and let $n > 0$.*

- *If $A \subseteq P^n/\!\equiv_n$ is directed, then there is a directed subset $B \subseteq \bigcup_{[a]\in A}\{\phi(a) \mid \phi \in S(n)\}$ satisfying*

$$\uparrow\{\phi(\sqcup B) \mid \phi \in S(n)\} = \bigcap_{a\in A} \uparrow\{\phi(a) \mid \phi \in S(n)\} \quad \textit{and} \quad [\sqcup B] = \sqcup A. \tag{8}$$

- *In particular, $(P^n/\!\equiv_n, \sqsubseteq)$ is a dcpo, and the mapping $x \mapsto [x] \colon P^n \to P^n/\!\equiv_n$ is Scott continuous.*

Proof. If $A \subseteq P^n/\!\equiv_n$ is directed, then Lemma 3.2 implies that $\{\cup_{\phi\in S(n)} \uparrow \phi(a) \mid [a] \in A\}$ is a filter basis of finitely generated upper sets, and so by Lemma 3.3 there is a directed set $B \subseteq \bigcup_{[a]\in A}\{\phi(a) \mid \phi \in S(n)\}$ with $B \cap \{\phi(a) \mid \phi \in S(n)\} \neq \emptyset$ for each $[a] \in A$. Since $B \subseteq P^n$ is directed, we have $x = \sqcup B$ exists. If $[a] \in A$, then $B \cap \{\phi(a) \mid \phi \in S(n)\} \neq \emptyset$ means there is some $\phi \in S(n)$ with $\phi(a) \in B$, so $\phi(a) \leq x$ by Lemma 3.2. Hence $[a] \sqsubseteq [x]$ for each $[a] \in A$, so $[x]$ is an upper bound for A.

We also note that, since $\sqcup B = x$,

$$\bigcap_{b\in B} \uparrow\{\phi(b) \mid \phi \in S(n)\} = \uparrow\{\phi(x) \mid \phi \in S(n)\}.$$

Indeed, the right hand side is clearly contained in the left hand side since $b \leq x$ for all $b \in B$. On the other hand, if y is in the left hand side, then $b \sqsubseteq y$ for each $b \in B$. Now, since $S(n)$ is finite, there is some $\phi_0 \in S(n)$ and some cofinal subset $B' \subseteq B$ with $\phi_0(b) \leq y$ for each $b \in B'$. But then $\sqcup B' = \sqcup B$, and so $\sqcup\{\phi_0(b) \mid b \in B'\} = \phi_0(x)$, from which we conclude that $\phi_0(x) \leq y$. Thus y is in the right hand side, so the sets are equal.

Now, if $y \in P^n$ satisfies $[a] \sqsubseteq [y]$ for each $[a] \in A$, then since $B \subseteq \bigcup_{[a]\in A}\{\phi(a) \mid \phi \in S(n)\}$, it follows that $[b] \sqsubseteq [y]$ for each $b \in B$. Then $y \in \bigcap_{b\in B} \uparrow\{\phi(b) \mid \phi \in S(n)\} = \uparrow\{\phi(x) \mid \phi \in S(n)\}$, and so $[x] \sqsubseteq [y]$. Thus $[x] = \sqcup A$ in the order $\sqsubseteq_n$. This also shows that $\bigcap_{[a]\in A} \uparrow\{\phi(a) \mid \phi \in S(n)\} = \uparrow\{\phi(x) \mid \phi \in S(n)\}$.

It is clear now that $P^n/{\equiv_n}$ is a dcpo, and the argument we just gave shows that directed sets $B \subseteq P^n$ satisfy $[\sqcup B] = \sqcup_{b\in B}[b]$. This concludes the proof.□

Proposition 3.5 *Let P be a domain and let $n \in \mathbb{N}$. Then*

(i) *$(P^n/{\equiv_n}, \sqsubseteq_n)$ is a domain. In fact, of $d, e \in P^n$, then*

$$[d] \ll [e] \quad \text{iff} \quad (\exists \phi \in S(n))\ \phi((d_i)_n) \ll (e_i)_n.$$

(ii) *If P is* RB *or* FS*, then so is $P^n/{\equiv_n}$.*

(iii) *If P is coherent, then so is $P^n/{\equiv_n}$.*

Proof. $P^n/{\equiv_n}$ is a domain: Proposition 3.4 shows that $(P^n/{\equiv_n}, \sqsubseteq_n)$ is directed complete and that the quotient map is Scott continuous. To characterize the way-below relation on $P^n/{\equiv_n}$, let $x, y \in P^n$ with $x \ll y$. Then $x_i \ll y_i$ for each $i = 1, \ldots, n$, and it follows that $\phi(x) \ll \phi(y)$ for each $\phi \in S(n)$. If $A \subseteq P^n/{\equiv_n}$ is directed and $[y] \sqsubseteq_n \sqcup A$, then there is some $\phi \in S(n)$ with $\phi(y) \leq z$, where $[z] = \sqcup A$. Then Proposition 3.4 shows there is a directed set $B \subseteq \cup_{[a]\in A} \uparrow\{\phi(a) \mid \phi \in S(n)\}$ with $\sqcup B \equiv_n z$. Hence, there is some $\psi \in S(n)$ with $\psi(y) \leq \sqcup B$. Since $\psi(x) \ll \psi(y)$, it follows that there is some $b \in B$ with $\psi(x) \leq b$, so $[x] \sqsubseteq_n [b]$. Hence $[x] \ll [y]$ in $P^n/{\equiv_n}$.

We have just shown that $x \ll y$ in P^n implies that $[x] \ll [y]$ in $P^n/{\equiv_n}$. Since P^n is a domain, $\downarrow\!\!\downarrow y$ is directed, and so the same is true for $\downarrow\!\!\downarrow[y] \in P^n/{\equiv_n}$. Since the quotient map $[\ ]\colon P^n \to P^n/{\equiv_n}$ is continuous, if follows that $[y] = \sqcup[\downarrow\!\!\downarrow y] \sqsubseteq_n \sqcup\downarrow\!\!\downarrow[y] \sqsubseteq_n [y]$, and so $P^n/{\equiv_n}$ is a domain.

$P^n/{\equiv_n}$ is RB if P is: Now suppose the P is in RB. Then, by Theorem 4.1 of [11] there is a directed family $f_k\colon P \to P$ of Scott-continuous maps with $1_P = \sqcup_k f_k$ and $f_k(P)$ finite for each $k \in K$. Then the mappings $(f_k)^n\colon P^n \to P^n$ also form such a family, showing P^n is in RB.

Next, given $k \in K, x \in P^n$ and $\phi \in S(n)$, we have $\phi(f_k^n(x)) = f_k^n(\phi(x))$ since f_k^n is f_k acting on each component of x. It follows that there is an induced map $[f_k^n]\colon P^n/{\equiv_n} \to P^n/{\equiv_n}$ satisfying $[f_k^n]([x]) = [f_k^n(x)]$, and this map is continuous since $[\ ]$ is a quotient map. Finally, $[f_k^n](P^n/{\equiv_n})$ is finite since $f_k^n(P^n)$ is finite, and the fact that $\sqcup_k[f_k^n] = 1_{P^n/{\equiv_n}}$ follows from $\sqcup_k f_k^n = 1_{P^n}$. Thus, $P^n/{\equiv_n}$ is RB if P is.

$P^n/{\equiv_n}$ is FS if P is: The argument is analogous to the one we just gave for the case P is RB.

$P^n/{\equiv_n}$ is coherent if P is: Last, we consider coherent domains. Recall a domain is *coherent* if the Lawson topology is compact, where the Lawson topology has the family of sets $\{U \setminus \uparrow F \mid F \subseteq P \text{ finite } \& \ U \text{ Scott open}\}$ for a basis. Now, if $x \in P^n$, then $\{\phi(x) \mid \phi \in S(n)\}$ is finite, and so if $F \subseteq P^n/{\equiv_n}$ is finite, then $[\uparrow F]^{-1} = \cup_{[x]\in F} \uparrow\{\phi(x) \mid \phi \in S(n)\}$ is finitely generated. It follows that $[\ ]\colon P^n \to P^n/{\equiv_n}$ is Lawson continuous, so if P is coherent, then so are P^n and $P^n/{\equiv_n}$. □

Definition 3.6 Let P and Q be domains, and let $f\colon P \to Q$ be Scott continuous. We let

(i) $\mathcal{B}_n(P) = (P^n/{\equiv_n}, \sqsubseteq_n)$ denote the domain of *n-bags over P*. If $n = 0$, we identify P^0 with $[\langle\rangle]$, the equivalence class of the empty word over P. We also let $\mathcal{B}_n(f)\colon \mathcal{B}_n(P) \to \mathcal{B}_n(Q)$ be the induced map satisfying $\mathcal{B}_n(f)([d]_P) = [f^n(d)]_Q$ for all $d \in P^n$, where $[\ ]_P\colon P^n \to P^n/{\equiv_n}$ is the quotient map, and likewise for $[\ ]_Q$.

(ii) $\mathcal{B}(P) = \dot\cup_n\ \mathcal{B}_n(P)$ denote the disjoint sum [8] of the $\mathcal{B}_n(P)$. We also let $\mathcal{B}(f) = \dot\cup_n\ \mathcal{B}_n(f)$.

(iii) $\mathcal{B}_\perp(P) = \ \oplus_n \mathcal{B}_n(P)$ denote the separated sum [9] of the $\mathcal{B}_n(P)$. We also let $\mathcal{B}_\perp(f)$ be the extension of $\mathcal{B}(f)$ that sends $\perp$ to $\perp$.

Recall that if A is a category of dcpos, then $\mathsf{A}_\perp$ denotes the full subcategory of cpos in A, while $\mathsf{A}_{\perp,!}$ denotes the subcategory of $\mathsf{A}_\perp$ of *strict* Scott-continuous maps, i.e., those that preserve least elements. We call $\mathsf{A}_{\perp,!}$ the *pointed subcategory determined by* A.

Proposition 3.7

(i) *$\mathcal{B}_n$ defines a continuous endofunctor on the categories* DCPO, DOM, CDOM, RB *and* FS *for each $n \in \mathbb{N}$. $\mathcal{B}_n$ also defines a continuous endofunctor of the pointed subcategory determined by each of these categories of dcpos.*

(ii) *$\mathcal{B}$ defines a continuous endofunctor on* DCPO *and* DOM*.*

(iii) *$\mathcal{B}_\perp$ defines a continuous endofunctor on* $\mathsf{CPO}_!$*, the category of cpos and strict Scott-continuous maps, and on* $\mathsf{CDOM}_{\perp,!}$, $\mathsf{RB}_{\perp,!}$ *and* $\mathsf{FS}_{\perp,!}$*.*

Proof. *Ad (i):* Proposition 3.5 shows that $\mathcal{B}_n(P)$ is a domain if P is one. If $f\colon P \to Q$ is Scott continuous, then so is $f^n\colon P^n \to Q^n$, and it is routine to show that the mapping $[d]_P \mapsto [f^n(d)]_Q$, $d \in P^n$ is well-defined and continuous. Thus $\mathcal{B}_n$ is an endofunctor on CDOM, and it is defined as a composition of constructors that define continuous endofunctors on each of the indicated categories, so it is continuous. Clearly $[(\perp, \ldots, \perp)]$ is the least element of $\mathcal{B}_n(P)$ if $\perp$ is the least element of P, from which the claim about pointed subcategories follows.

Ad (ii): For $\mathcal{B}$, we must add the countable separated sum constructor, which it is easy to show leaves DCPO and DOM invariant.

[8] The disjoint sum of dcpos is their disjoint union.

[9] The separated sum of (d)cpos is their disjoint union with a (new) bottom element added.

Ad (iii): It is obvious that $\mathcal{B}_\perp$ leaves CPO invariant. A proof that each of the other pointed categories is closed under coalesced sums can be found in [1]. □

3.1 Commutative domain monoids

By a *domain monoid* we mean a domain S equipped with a monoid operation $\cdot\colon S \times S \to S$ which is Scott continuous. For example, it is well known that the family of finite words over P, $P^* =_{\text{def}} (\dot\cup P^n, \cdot)$ where $\cdot$ is concatenation, is the free monoid over P, and hence the free domain monoid over P, if P is a domain. Of course, our interest is in commutative domain monoids – ones where the monoid operation is commutative.

Notation 3.8 We let CDM denote that category of commutative domain monoids and Scott continuous monoid homomorphisms.

Theorem 3.9 *Then $\mathcal{B}\colon$ DOM $\to$ CDM is left adjoint to the forgetful functor.*

Proof. It is routine to show that $\mathcal{B}(P)$ is a commutative monoid with respect to concatenation, with the empty word as the identity. Is $(S, *)$ is any commutative monoid and $f\colon P \to S$ is Scott continuous, then there is a unique Scott continuous monoid homomorphism $\widehat{f}\colon P^* \to S$ since $(P^*, \cdot)$ is the free domain monoid over P. But since S is commutative, the mapping $\widehat{f}|_{P^n}$ factors through the family $\mathcal{B}_n(P)$ for each $n \geq 0$, and so there is a unique induced monoid homomorphism $\mathcal{B}(f)\colon \mathcal{B}(P) \to S$. □

The interesting question is how to proceed in the case of $\mathcal{B}_\perp(P)$, the separated sum of the family of $\mathcal{B}_n(P)$. There are two "obvious" ways to extend the monoid operation on $\mathcal{B}(P)$ to include the new least element:

(+) Define $\perp \cdot x = x$, so that $\perp$ acts like an additive 0, or

(×) Define $\perp \cdot x = \perp$, so that $\perp$ acts like a multiplicative 0.

In the first case, we would identify $\perp$ with the equivalence class of the empty word, $[\langle\ \rangle]$, and the effect is to make the empty word the least element of $\mathcal{B}_\perp(P)$. We denote this monoid by $\mathcal{B}_+(P)$. As we will see, the order on $\mathcal{B}_+(P)$ is too coarse to support this monoid structure, so it has to be refined. But it plays a crucial role on recapturing Varacca's indexed valuations using our techniques.

Definition 3.10 If P is a domain, we let $\mathcal{B}_+(P) = \oplus_{n>0}\mathcal{B}_n(P)$ denote the separated sum of the family $\{\mathcal{B}_n(P) \mid n > 0\}$, in the order given by

$$[(d_i)_m] \sqsubseteq [(e_j)_n] \text{ iff } (\exists \iota\colon m \hookrightarrow n)^{10}\ d_i \leq e_{\iota(i)} \text{ for } i \in m, \tag{9}$$

with the semigroup operation of $\mathcal{B}(P)$ extended so that $\perp \cdot x = x$ for all $x \in \mathcal{B}_+(P)$.

[10] We identify m with the set $\{0, \ldots, m-1\}$, and similarly for n.

We also let CPM_+ denote the category of commutative monoids whose underlying structure is a continuous poset with least element, and for which the monoid operations are Scott continuous and in which the least element is the identity of the monoid. The morphisms are the Scott-continuous monoid homomorphisms.

Theorem 3.11

(i) *If P is a continuous poset, then $\mathcal{B}_+(P)$ is a continuous poset for which the monoid operation is Scott continuous.*

(ii) *In fact, $\mathcal{B}_+\colon \mathsf{CPOS} \to \mathsf{CPM}_+$ is left adjoint to the forgetful functor.*

Proof. For the first claim, we begin by noting that the order defined in (9) restricts to the usual product order on $\mathcal{B}_n(P)$, so it defines a partial order that refines the order on $\mathcal{B}_\perp(P)$. Next, if $A \subseteq \mathcal{B}_+(P)$ is directed and $[(d_i)_m] \in \mathcal{B}_+(P)$ is an upper bound for A, then each $[(a_i)_n] \in A$ satisfies $n \leq m$. It follows that there is a maximum $m \geq 0$ with $A \cap \mathcal{B}_m(P) \neq \emptyset$, and we let $A_0 = A \cap \mathcal{B}_m(P)$ for this maximum m. Since A is directed, it follows that A_0 is directed, too, and then that $\sqcup A_0 \in \mathcal{B}_m(P)$ also is the least upper bound of A. Thus directed sets in $\mathcal{B}_+(P)$ that have upper bounds are have least upper bounds, and these are computed in $\mathcal{B}_m(P)$, where m is the largest $m \geq 0$ for which the directed set intersects $\mathcal{B}_m(P)$. From this and the fact that $\mathcal{B}_m(P)$ is a continuous poset for each m, it is a straightforward argument to show that $\mathcal{B}_+(P)$ is a continuous poset, and in fact that $\mathord{\downarrow\!\!\!\downarrow}[d] \cap \mathcal{B}_m(P)$ is a cofinal subset of $\mathord{\downarrow\!\!\!\downarrow}[d]$ for each $d \in P^m$. The fact that the monoid operation on $\mathcal{B}(P)$ is continuous implies the same is true of that operation on $\mathcal{B}_+(P)$. Hence, $\mathcal{B}_+(P)$ is a CPM_+-object.

Now, let $(S, *)$ be a continuous poset with a Scott continuous monoid operation for which the identity is the least element, and let $f\colon P \to S$ be a Scott continuous map with P a continuous poset. Then P^n is a continuous poset, and since continuous posets have finite products, the mapping $f^n\colon P^n \to S^n$ by $f^n((d_1, \ldots, d_n)) = (f(d_1), \ldots, f(d_n))$ is continuous. We can follow this by the product mapping $(s_1, \ldots, s_n) \mapsto s_1 * \cdots * s_n\colon S^n \to S$, yielding a continuous mapping. Since S is commutative, this mapping factors through $\mathcal{B}_n(P)$, yielding a Scott-continuous mapping from $f^{[n]}\colon \mathcal{B}_n(P) \to S$ extending f. The problem is to show the family $\{f^{[n]}\}_{n\geq 1}$ of mappings together with $f(\perp) = \epsilon_S$ gives a continuous mapping from $\mathcal{B}_+(P)$ to S.

Since the $\mathcal{B}_+(P) = \dot{\cup}_n \mathcal{B}_n(P)$, and we know $f^{[n]}$ is Scott continuous, the family gives a mapping $f_+\colon \mathcal{B}_+(P) \to S$ by $f_+([(d_i)_m]) = f^{[m]}([(d_i)_m])$ which is well-defined since the $\mathcal{B}_n(P)$s are pairwise disjoint. In fact, f_+ is Scott continuous on each $\mathcal{B}_n(P)$, and we claim that it is Scott continuous on all of $\mathcal{B}_+(P)$ because it is monotone. This is clear on each $\mathcal{B}_n(P)$, so suppose the $[(d_i)_m] \sqsubseteq [(e_j)_n]$. Then $m \leq n$ and there is an injection $\iota\colon m \hookrightarrow n$ with $d_i \sqsubseteq_P e_{\iota(i)}$ for each i. Then $f(d_i) \sqsubseteq_S f(e_{\iota(i)})$ for each i, from which it follows that $f_+([(d_i)_m]) = f^{[m]}([(d_1, \ldots, d_m]) = f(d_1) * \cdots * f(d_m) \sqsubseteq_S f(e_{\iota(1)}) * \cdots f(e_{\iota(m)})$.

Since ϵ_S is the least element of S, if we let s denote the product in S of those $f(e_j)$ with $j \neq \iota(i)$ for any i, then $\epsilon_S \sqsubseteq_S s$, so

$$\begin{aligned} f_+([(d_i)_m]) = f^{[m]}([d_1,\ldots,d_m)] &= f(d_1) * \cdots * f(d_m) * \epsilon_S \\ &\sqsubseteq_S f(e_{\iota(1)}) * \cdots * f(e_{\iota(m)}) * s \\ &= f(e_1) * \cdots * f(e_n) \\ &= f^{[n]}([e_1,\ldots,e_n]) = f_+([(e_j)_n]). \end{aligned}$$

This shows the induced mapping is monotone. Continuity then follows, and the mapping is a homomorphism by design. As before, uniqueness follows from that of $f^n \colon P^n \to S^n$. □

If we let CDM_+ denote the subcategory of CPM_+ whose objects are domains, then each CPM_+-object S has a round ideal completion that is in CDM_+, since the inequations and operations on S extend to its completion. This observation leads to:

Corollary 3.12 *$\mathcal{B}_+$ restricts to a left adjoint to the forgetful functor from CDM_+ to DOM.*

The second possibility for extending the monoid structure on $\mathcal{B}(P)$ to include $\perp$ keeps $\perp$ and $[\langle\ \rangle]$ distinct, with $\perp$ as a multiplicative zero.

Definition 3.13 Let P be a domain, Then we let $\mathcal{B}_\times(P) = \ \oplus_{n\geq 0} \mathcal{B}_n(P)$ denote the separated sum of the $\mathcal{B}_n(P)$, but where we define $\perp \oplus [d] =\perp$ for all $d \in P^n$, for all n. Note that $[\langle\ \rangle]$, the equivalence class of the empty word, is included and is the identity of the monoid.

We also define $\mathsf{CDM}_\times$ to be the category of commutative domain monoids with least element in which the least element acts as a multiplicative 0, and Scott continuous monoid homomorphisms preserving the least element.

Clearly $\mathcal{B}_\times(P)$ is in $\mathsf{CDM}_\times$ for each domain P. In fact, we can say more.

Theorem 3.14 *$\mathcal{B}_\times \colon \mathsf{DOM} \to \mathsf{CDM}_\times$ is left adjoint to the forgetful functor, In fact, $\mathcal{B}_\times$ restricts to a left adjoint to the forgetful functor from the subcategory of $\mathsf{CDM}_\times$ whose objects are in A, for each of the categories A listed in Proposition 3.7(iii).*

Proof. Given the prior results, the argument for $\mathcal{B}_\times$ only requires noting the following. First, for a domain P, defining $\perp *[d] =\perp$ extends concatenation to a continuous operation on $\mathcal{B}_\times(P)$. Further, for a commutative domain monoid S satisfying $s* \perp=\perp$, and a Scott-continuous map $f \colon P \to S$, the mapping $\mathcal{B}_\times(f)(\perp) =\perp_S$ gives the unique strict extension of $\mathcal{B}(f) \colon \mathcal{B}(P) \to S$ to all of $\mathcal{B}_\times(P)$. This establishes the first claim.

For the second, we showed in Proposition 3.7(iii) that $\mathcal{B}_\perp$ restricts to an endofunctor of each of the subcategories listed there, and it then follows from what we just showed that $\mathcal{B}_\times$ restricts to a left adjoint to the forgetful functor from the subcategory of $\mathsf{CDM}_\times$ in each case. □

Example 3.15 We have shown that several Cartesian closed categories of domains are closed under the action of the functor $\mathcal{B}$ and its relatives. Here is an example that demonstrates that the Cartesian closed category of Scott domains does not enjoy this property. Let $P = \{\bot, a, b, \top\}$ be the four-element lattice with a and b incomparable, then $P^2/{\equiv_2}$ is not in BCD: the pair $[a, \bot], [b\ \bot]$ has $[a, b]$ and $[\top, \bot]$ as minimal upper bounds. [11]

4 Reconstructing $IV_P(P)$

We next use our results on bag domains to reconstruct Varacca's Plotkin indexed valuations. We begin by considering how to introduce the non-negative reals into the picture. In fact, what we really want are commutative domain monoids on which the non-negative reals act, so they should be modules over the non-negative reals.

4.1 $\mathbb{R}_+$*-spaces*

The set $\mathbb{R}_+$ of positive reals is a continuous poset in the usual order. We say a continuous poset P is an $\mathbb{R}_+$*-continuous poset* if there is a Scott continuous mapping $\cdot\colon \mathbb{R}_+ \times P \to P$ satisfying *mixed associativity:* $r \cdot (s \cdot p) = (rs) \cdot p$, for $p \in P$ and $r, s \in \mathbb{R}_+$, and *identity:* $1 \cdot p = p$ for all $p \in P$. We let $\mathsf{CPOS}_{\mathbb{R}_+}$ denote the category of $\mathbb{R}_+$-continuous posets and Scott continuous mappings that preserve the action: $f(r \cdot p) = r \cdot f(p)$ for each $r \in \mathbb{R}_+$ and $p \in P$. For any continuous poset P, its easy to create a continuous poset that contains P and on which $\mathbb{R}_+$ acts:

Proposition 4.1 *We define the functor* $\mathcal{B}_{\mathbb{R}_+}\colon \mathsf{CPOS} \to \mathsf{CPOS}_{\mathbb{R}_+}$ *by* $\mathcal{B}_{\mathbb{R}_+}(P) = \mathbb{R}_+ \times P$ *and for* $f\colon P \to Q$, $\mathcal{B}_{\mathbb{R}_+}(f)(r, p) = (r, f(p))$*. Then* $\mathcal{B}_{\mathbb{R}_+}$ *is left adjoint to the forgetful functor.*

Moreover, if we let CPM *denote the category of continuous posets which are also commutative monoids and Scott-continuous monoid maps, and if* $\mathsf{CPM}_{\mathbb{R}_+}$ *denotes the subcategory of* CPM *of commutative monoid continuous posets which admit* $\mathbb{R}_+$ *actions, then* $\mathcal{B}_{\overline{\mathbb{R}_{\geq 0}}}$ *restricts to a left adjoint to the forgetful functor from* $\mathsf{CPM}_{\mathbb{R}_+}$ *to* CPM*.*

Proof. The only interesting point is that the unit is the mapping $p \mapsto (1, p)\colon P \to \mathbb{R}_+ \times P$, which is guaranteed by the identity axiom. $\square$

4.2 Varacca's indexed valuations

As with commutative domain monoids, there are two possibilities we can consider for how commutative domain monoids relate the identity of the monoid and the least element of the domain – one is to identify them, and the other

[11] Thanks to one of the anonymous referees for this example.

is to treat the least element as a multiplicative zero. The first of these leads us to recapturing Varacca's indexed valuations, as we now show.

Definition 4.2 Let P be a continuous poset. We define

$$\mathcal{B}_+^{\mathbb{R}}(P) = \mathcal{B}_+(\mathbb{R}_+ \times P) = \oplus_{n>0}\mathcal{B}_n(\mathbb{R}_+ \times P)$$

to be the separated sum of the family $\{\mathcal{B}_n(\mathbb{R}_+ \times P) \mid n > 0\}$, and we define the monoid operation on $\mathcal{B}_+^{\mathbb{R}}(P)$ to be the semigroup operation of $\mathcal{B}_+(P)$ extended so that the new least element, $\perp$, is the identity as well as the least element. We define an action of $\mathbb{R}_{\geq 0}$ on $\mathcal{B}_+^{\mathbb{R}}(P)$ by

$$r \cdot [(r_i, d_i)_m] = [(rr_i, d_i)_m] \quad \text{and} \quad 0 \cdot [(r_i, d_i)_m] \ = \ 0 \cdot \perp = \ r \cdot \perp = \perp,$$

for $[(r_i, d_i)_m] \in \mathcal{B}_+^{\mathbb{R}}(P)$ and $r \in \mathbb{R}_+$. Then $\mathcal{B}_+^{\mathbb{R}}(P)$ is a commutative monoid whose underlying structure is a continuous poset, on which $\mathbb{R}_{\geq 0}$ acts Scott-continuously.

We also recall the laws enumerated in Theorem 2.3 for a monoid $(P, *)$ admitting an action of $\mathbb{R}_{\geq 0}$:

(1)	$p * q = q * p$	(2)	$p * (q * u) = (p * q) * u$
(3)	$p * \epsilon_P = p$	(4)	$0 \cdot p = \epsilon_P$
(5)	$1 \cdot p = p$	(6)	$r(p * q) = r \cdot p * r \cdot q$
(7)	$r \cdot (s \cdot p) = (rs) \cdot p$	where	$r, s \in \mathbb{R}_{\geq 0}$ and $p, q, u \in P$.

Fig. 1. Varacca's Laws for Actions of $\mathbb{R}_{\geq 0}$ on Monoids

These laws assert that $(P, *)$ is a commutative monoid that admits an action of $\mathbb{R}_{\geq 0}$ so that $1 \in \mathbb{R}_{\geq 0}$ acts like a multiplicative identity and $0 \in \mathbb{R}_{\geq 0}$ acts like a multiplicative zero, leaving the identity ϵ_P fixed. We let $\mathsf{CPOS}_+^{\mathbb{R}}$ denote the category of commutative monoids on continuous posets admitting a Scott-continuous action of $\mathbb{R}_{\geq 0}$ and satisfying the laws in Figure 4.2, and Scott-continuous monoid homomorphisms preserving the $\mathbb{R}_{\geq 0}$ action.

Theorem 4.3 *For a continuous poset P, $\mathcal{B}_+^{\mathbb{R}}(P)$ satisfies the laws of Theorem 2.3. In fact, $\mathcal{B}_+^{\mathbb{R}}$ is the object level of a left adjoint to the forgetful functor from $\mathsf{CPOS}_+^{\mathbb{R}}$ and the category CPOS of continuous posets and Scott-continuous maps.*

Proof. It's clear from the definition that $\mathcal{B}_+^{\mathbb{R}}(P)$ is a commutative monoid whose underlying structure is a continuous poset, and that $\mathbb{R}_{\geq 0}$ acts on this structure so that the laws (4) – (7) are satisfied. The fact that the operations are Scott continuous is also clear from the construction.

For the claim about $\mathcal{B}_+^{\mathbb{R}}$, we note that it is a composition of two left adjoints, $\mathcal{B}_{\mathbb{R}_+}$ followed by $\mathcal{B}_+$, and so their composition is one as well. □

Remark 4.4 Recall that a continuous poset P can be completed into a domain by taking its round-ideal completion, $\mathrm{Id}(P)$. The resulting structure has the same way-below relation as the underlying continuous poset, and in fact it also has the same Scott topology. An equivalent way of realizing this construction is to take the sobrification of P in its Scott topology.

Corollary 4.5 *For a domain P, $IV_P(P) \simeq \mathrm{Id}(\mathcal{B}_+^{\mathbb{R}}(P))$.*

Proof. One proof follows by noting that the mapping $\langle r_i, p_i\rangle_n \mapsto [(r_i, p_i)_n]$ and $\underline{0} \mapsto \underline{0}$ defines a bijection of the indexed valuations over P onto $\mathcal{B}_+^{\mathbb{R}}(P)$ that takes $\ll$ in $IV_P(P)$ to $\ll$ on $\mathcal{B}_+^{\mathbb{R}}(P)$. Since they have isomorphic bases, the domains $IV_P(P)$ and $\mathrm{Id}(\mathcal{B}_+^{\mathbb{R}}(P))$ are also isomorphic.

A perhaps more elegant proof follows by noting that $\mathcal{B}_+^{\mathbb{R}}(P)$ satisfies the same laws as $IV_P(P)$, and both are initial objects over P according to Theorems 2.3 and 4.3. $\Box$

Corollary 4.6 $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}$ *generates a monad on* CPOS*, and each of the power domain monads $\mathcal{P}_X$ lifts to a monad on* $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}$*-algebras.*

Proof. For each of the power domain monads, $\mathcal{P}_X$, Varacca showed that there is a distributive law of IV_P over $\mathcal{P}_X$ in [23], and this implies that $\mathcal{P}_X$ lifts to a monad on the class of $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}$-algebras by Beck's Theorem 1.1 [2] and by Theorem 4.5. In fact, we can easily recover the distributive law that Varacca obtains in [24] as follows: For a domain P, a power domain monad $\mathcal{P}_X$ and an element $[(r_i, X_i)_n] \in \mathcal{B}_+^{\mathbb{R}}\mathcal{P}_X(P)$, the distributive law on the basis elements in $\mathcal{B}_+^{\mathbb{R}}(P)$ is

$$d\colon \mathcal{B}_+^{\mathbb{R}}\mathcal{P}_X \stackrel{\cdot}{\longrightarrow} \mathcal{P}_X\mathcal{B}_+^{\mathbb{R}} \quad \text{by} \quad d_P([(r_i, X_i)_n]) = \langle [(r_i, x_i)_n] \mid x_i \in X_i \in \mathcal{P}_X(P)\rangle,$$

where $\langle - \rangle$ denotes the element of $\mathcal{P}_X\mathcal{B}_+^{\mathbb{R}}(P)$ generated by $\cup_{i\leq n}\{[(r_i, x_i)_n] \mid x_i \in X_i \in \mathcal{P}_X(P)\}$. The result follows from Beck's Theorem 1.1 [2]. $\Box$

Remark 4.7 For all its attractiveness, the shortcoming of Varacca's indexed valuations approach is that it is not clear whether it leaves any ccc's of domains invariant. There is some relevant literature here: Poigné shows that there are no left adjoints for commutative semigroups or commutative idempotent monoids over domains, if one includes a least element in the discussion [21]. Gordon Plotkin has also commented in a personal communication that he once showed that the free commutative semigroup with least element takes one out of the largest ccc of ω-algebraic domains, but that the argument would fail in the continuous case. Finally, Heckmann [6] has shown that there is a lower bag domain construction that does not leave any ccc of algebraic domains invariant, but again his arguments rely on the characterization of bifinite domains, and it is not clear if they can be generalized to the setting where one is dealing with continuous objects.

So, this issue remains unresolved for Varacca's construction. In the next section, we present an alternative that sacrifices one of the laws of Theorem 2.3, but gains the property of staying within ccc's of domains.

4.3 Real domain monoids

We now turn our attention to the action of $\mathbb{R}_+$ on other possible commutative domain monoid with least element, namely one in which the monoid identity is not the least element. Here is a generalization of Varacca's laws to address this situation.

Definition 4.8 We call a commutative domain monoid $(S,*)$ a *real domain monoid* if S has a least element and if $\mathbb{R}_{\geq 0}$ acts on S so that the following laws are satisfied:

(1)	$s*t = t*s$	(2)	$s*(t*u) = (s*t)*u$
(3)	$s*\epsilon = s$	(4′)	$0\cdot s = \perp$
(5)	$1\cdot s = s$	(6)	$r(s*t) = rs*rt$
(7)	$r\cdot(r'\cdot s) = (rr')\cdot s$	where	$r, r' \in \mathbb{R}_+$ and $s,t,u\in S$.

A *morphism* of real domain monoids S and T is a Scott-continuous monoid homomorphism that preserves the action of $\mathbb{R}_{\geq 0}$. We let RDM denote the category of real domain monoids and their morphisms.

Remark 4.9 The laws (1) – (3) just assert that $(S,*)$ is a commutative monoid, while the remaining laws characterize the action of $\mathbb{R}_{\geq 0}$ on S. The difference with Varacca's indexed valuation domains defined in Theorem 2.3 is law (4′), which in his case asserts $0A = \underline{0}$. But for him, $\underline{0} = \perp$, so his objects satisfy these laws. We will find that differentiating ϵ from $\perp$ gives a very different structure which is crucial for our construction of a power domain of discrete random variables in the next section that leaves invariant two Cartesian closed categories of domains.

Our next goal is to characterize the objects we have defined.

Definition 4.10 Let P be a domain.

- If P is a domain, we define $\mathcal{B}^{\mathbb{R}}_\times(P) = \mathcal{B}_\times(\overline{\mathbb{R}_{\geq 0}}\times P)$, the separated sum of the family $\{\mathcal{B}_n(\overline{\mathbb{R}_{\geq 0}}\times P) \mid n\geq 0\}$ with the monoid structure defined in Definition 3.13. We also define the action of $\mathbb{R}_{\geq 0}$ on $\mathcal{B}^{\mathbb{R}}_n(P)$ and on $\mathcal{B}^{\mathbb{R}}_\times(P)$ by
$$r\cdot[(r_i,p_i)_n] = [(rr_i,p_i)_n] \quad\text{and}\quad 0\cdot[(r_i,p_i)_n] = r\cdot\perp = \perp,$$
where $r\in\mathbb{R}_+$. Then $\mathbb{R}_{\geq 0}$ acts Scott-continuously on $\mathcal{B}^{\mathbb{R}}_\times(P)$.
- If $f\colon P\to Q$ is Scott continuous, then we define
$$\mathcal{B}^{\mathbb{R}}_\times(f)\colon \mathcal{B}^{\mathbb{R}}_\times(P)\to\mathcal{B}^{\mathbb{R}}_\times(Q) \text{ by } \mathcal{B}^{\mathbb{R}}_\times(f)([r_i,p_i]_n) = [r_i,f(p_i)]_n \ \wedge \mathcal{B}^{\mathbb{R}}_\times(f)(\perp) = \perp.$$

Theorem 4.11 *$\mathcal{B}^{\mathbb{R}}_\times$ defines a continuous endofunctor on* DOM, on RB *and on* FS. *In fact, $\mathcal{B}^{\mathbb{R}}_\times$ is a left adjoint to the forgetful functor from the subcategory of real domain monoids of each of these categories.*

Proof. The arguments are similar to those given in the proof of Theorem 4.3, with the distinguishing feature of this construction that the least element is not the monoid identity, but instead acts like a multiplicative zero relative to the monoid operation, and that $0 \cdot x = \perp$ for all x. □

Remark 4.12 The difference between $\mathcal{B}_+^{\mathbb{R}}$ and $\mathcal{B}_\times^{\mathbb{R}}$ is that the former creates a structure whose least element is a multiplicative identity, while the latter makes it a multiplicative zero. As we have seen, $\mathcal{B}_+^{\mathbb{R}}$ serves to help us recapture Varacca's indexed valuations, but in so doing, we must form an ideal completion, because the order on $\mathcal{B}_+^{\mathbb{R}}(P)$ is incomplete. This is because elements of the form $[(r_i, d_i)_m]$ and $[(s_j, e_j)_n]$ can compare in $\mathcal{B}_+^{\mathbb{R}}(P)$ even if $m \neq n$. In fact, it is easy to show that $\mathcal{B}_+^{\mathbb{R}}(P)$ is directed, and so there is a largest element in its round ideal completion. It would be interesting to study what role this element plays in the structure on the object.

On the other hand, $\mathcal{B}_\times^{\mathbb{R}}(P)$ is a domain if P is one, so no completion is necessary. Its least element is distinct from the monoid identity, and the order prevents elements of the form $[(r_i, d_i)_m]$ and $[(s_j, e_j)_n]$ from comparing unless $m = n$.

5 Discrete random variables over domains

We now show how to construct two power domains of discrete random variables over a domain, one using each of our constructions, $\mathcal{B}_+^{\mathbb{R}}$ and $\mathcal{B}_\times^{\mathbb{R}}$. We also show that the second of these leaves some Cartesian closed categories of domains invariant.

To begin, recall that a *random variable* is a function $f\colon (X, \mu) \to (Y, \Sigma)$ where (X, μ) is a probability space, (Y, Σ) is a measure space, and f is a measurable function, which means $f^{-1}(A)$ is measurable in X for every $A \in \Sigma$, the specified σ-algebra of subsets of Y. Most often random variables take their values in $\mathbb{R}$, equipped with the usual Borel σ-algebra. For us, X will be a countable, discrete space, and Y will be a domain, where Σ will be the Borel σ-algebra generated by the Scott-open subsets.

Given a random variable $f\colon X \to Y$, the usual approach is to "push the probability measure μ forward" onto Y by defining $f\mu\,(A) = \mu(f^{-1}(A))$ for each measurable subset A of Y. But this defeats one of the attractions of random variables: namely, that there may be several points $x \in X$ which f takes to the same value $y \in Y$. This is 'attractive' because it means that the random variable f makes distinctions that the probability measure $f\mu$ does not, and we would like to exploit this fact. Varacca makes exactly this point in his work [23,24], a point he justifies by showing how to distinguish the random variable f from the probability measure $f\mu$ operationally. We return to this point later. For the moment, we define our power domain of random variables.

Definition 5.1 For a domain P, we define the *power domain of $+$-random* *variables over* P to be the subdomain

$$\mathbb{RV}_+(P) = \bigcup_{n\geq 0}\{[(r_i, d_i)_n] \mid \sum_i r_i \leq 1\} \cup \{\bot\} \subseteq \mathrm{Id}\mathcal{B}^{\mathbb{R}}_+(P).$$

Remark 5.2 We can think of a finite random variable over P as a formal sum $\sum_{i\leq n} r_i\delta_{x_i}$ where some of the x_i can be repeated. But, the order from $\mathcal{B}^{\mathbb{R}}(P)$ distinguishes, for example, $\frac{1}{2}\,\delta_x \oplus \frac{1}{2}\,\delta_x$ from δ_x, while these two would be identified as probability measures.

In order to show that $\mathbb{RV}_+$ generates a monad, we need an enumeration of the laws that a +-random variable algebra should satisfy. These are adapted from the laws for probabilistic algebras first defined by Graham [5]:

Definition 5.3 A *+-random variable algebra* is a domain P with $\bot$, its least element and with a Scott-continuous mapping $+\colon (0,1]\times P\times P \to P$ satisfying:

- $p +_r \bot = p$ for all $0 < r \leq 1$.[12]
- $a +_1 b = a$,
- $a +_r b = b +_{1-r} a$, and
- $(a +_r b) +_s c = a +_{rs} (b +_{\frac{s(1-r)}{1-sr}} c)$,

where $r, s \in (0,1)$ and $a, b, c \in P$.

A *morphism of +-random variable algebras* is a Scott-continuous map $f\colon S \to T$ satisfying $f(s +_r s') = f(s) +_r f(s')$, for all $s, s' \in S$ and all $r \in (0,1]$.

Other than the addition of the first law, the difference between our laws and those from Graham's characterization of probabilistic algebras are that (i) we restrict the value of r in $+_r$ to cases when $0 < r < 1$ in all but the first law (which avoids some annoying side conditions in Graham's listing), and (ii) the law $a +_r a = a$ is missing. This last is exactly the law Varacca weakened to allow a distributive law to hold.

Proposition 5.4 *Let P be a domain, and for $[(r_i, p_i)_m], [(s_j, q_j)_n] \in \mathbb{RV}_+(P)$ and $0 < r \leq 1$, define*

$$[(r_i, p_i)_m] +_r [(s_j, q_j)_n] = [(rr_i, p_i)_m] \oplus [((1-r)s_j, q_j)_n]\,.$$

Then:

(i) $\mathbb{RV}_+(P)$ *is a +-random variable algebra, and*

(ii) $[(r,p)] = [(1,p)] +_r \bot$ *for all $p \in P$ and all $r \in (0,1)$, and*

$$[(r_1,p_1),\ldots,(r_m,p_m)] = [(1,p_1)] +_{r_1} [(\frac{r_2}{(1-r_1)}, p_2), \ldots, (\frac{r_m}{(1-r_1)}, p_m)]$$

for all $[(r_1,p_1),\ldots,(r_m,p_m)] \in \mathbb{RV}_+(P)$.

[12] We use $a +_r b$ as infix notation for $+(r, a, b)$.

Proof. Given a domain P, we can define $+\colon (0,1] \times \mathcal{B}^{\mathbb{R}}(P)^2 \to \mathcal{B}^{\mathbb{R}}(P)$ by

$$+(r, [(r_i, p_i)_m], [(s_j, q_j)_n] = [(rr_i, p_i)_m] \oplus [((1-r)s_j, q_j)_n]\,.$$

Because $\mathbb{R}_+$ acts continuously on $\mathcal{B}_+^{\mathbb{R}}(P)$ and because $\oplus$ is continuous, $+$ is a continuous operation. $\mathbb{RV}_+(P)$ is the subfamily of $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}(P)$ whose real components are bounded by 1, and this family is clearly invariant under the action of $\mathbb{R}_+$, so this defines a continuous mapping $+\colon (0,1] \times \mathbb{RV}_+(P)^2 \to \mathbb{RV}_+(P)$. Using the abbreviation that $[(r_i, p_i)_m] +_r [(s_j, q_j)_n] = [(rr_i, p_i)_m] \oplus [((1-r)s_j, q_j)_n]$, it also is routine to verify that the laws of Definition 5.3 are satisfied.

The results in (ii) are simple calculations. □

We now characterize initial +-random variable algebras over domains.

Theorem 5.5 $\mathbb{RV}_+$ *defines a continuous endofunctor on* DOM*. Moreover, it also defines a left adjoint to the forgetful functor from the category of +-random variable algebra domains and +-random variable maps to* DOM*.*

Proof. $\mathbb{RV}_+$ is obtained by restricting $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}$ in the "real components" to ones whose sum is at most 1. This family is a Scott-closed subset of $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}(P)$. Hence $\mathbb{RV}_+(P)$ is a domain if P is one. Continuous maps $f\colon P \to Q$ extend to $\mathcal{B}_+^{\mathbb{R}}(P)$ by $\mathcal{B}_+^{\mathbb{R}}(f)[(r_i, p_i)_n] = [(r_i, f(p_i))_n]$ and then to its round ideal completion, and the elements in $\mathbb{RV}_+(P)$ are those in $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}(P)$ whose real components sum to at most 1; it follows that $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}(f)(P) \subseteq \mathrm{Id}\mathcal{B}^{\mathbb{R}}(Q)$. Since the endofunctor is composed of components that are locally continuous, it is as well.

For the second part, we first show that $\mathbb{RV}_+$ is left adjoint to the forgetful functor from +-random variable domains into DOM. First, we let $\eta\colon P \to \mathbb{RV}_+(P)$ by $\eta(p) = [(1,p)]$ define the unit of the adjunction.

Next, let S be a+- random variable domain algebra, P a domain, and let $f\colon P \to S$ be a Scott continuous map. We define $\widehat{f}\colon (\mathbb{RV}_+(P) \cap \mathcal{B}_+^{\mathbb{R}}(P)) \to S$ via $\widehat{f}([(r_i, p_i)_m]$ by induction on m, and then extend to it closure, which is $\mathbb{RV}_+(P)$.

If $m = 0$, then $\widehat{f}(\perp) = \perp_S$. In case of $[(r,p)]$, we have $[(r,p)] = [(1,p)] +_r \perp$ by Proposition 5.4, so we define $\widehat{f}([(r,p)]) = f(p) +_r \perp_S$. This mapping is clearly continuous on $P/{\equiv_1} \subseteq \mathbb{RV}_+(P)$, since $P/{\equiv_1}$ inherits its Scott topology from that of $\mathbb{RV}_+(P)$. This is also the unique such function on $P/{\equiv_1}$ satisfying $\widehat{f} \circ \eta = f$.

Continuing the definition of $\widehat{f}$ by induction, assume that we have defined $\widehat{f}$ on $\cup_{k \leq n}(P^k/{\equiv_k})$ uniquely so that it is continuous and satisfies $\widehat{f} \circ \eta = f$. Let $[(r_1, p_1), \ldots, (r_{m+1}, p_{m+1})] \in P^{m+1}/{\equiv_{m+1}}$, and then define

$$\widehat{f}([(r_1, p_1), \ldots, (r_{m+1}, p_{m+1})]) = f(p_1) +_{r_1} \widehat{f}([(\frac{r_2}{1-r_1}, p_2), \ldots, (\frac{r_{m+1}}{1-r_1}, p_{m+1})].$$

This is well-defined by Proposition 5.4(ii), and it is the composition of continuous functions, so it is continuous. It also satisfies $\widehat{f} \circ \eta = f$ because it's restriction to $P/{\equiv_1}$ does by definition. Finally, Proposition 5.4(ii) again shows it is the unique such function.

This shows that $\mathbb{RV}_+$ is left adjoint to the forgetful functor from random variable algebras into DOM, so it generates a monad on DOM. $\square$

Corollary 5.6 *Each of the power domain monads $\mathcal{P}_X$ lifts to a monad on $\mathbb{RV}_+$-algebras.*

Proof. The distributive law given in the proof of Corollary 4.6 clearly restricts to one for $\mathbb{RV}_+$. $\square$

This corollary means we can solve domain equations such as $P \simeq \mathcal{P}_X \circ \mathbb{RV}_+(P)$ for each of the power domain monads $\mathcal{P}_X$. The resulting domain P will be a $\mathcal{P}_X$-algebra and simultaneously a $\mathbb{RV}_+$-algebra.

5.1 Discrete random variables for Cartesian closed categories

We now use our functor $\mathcal{B}^{\mathbb{R}}_{\times}$ to define a second construction of random variables over domains. This one has the advantage of leaving some Cartesian closed categories of domains invariant. Since the results parallel those of the last subsection, with $\mathcal{B}^{\mathbb{R}}_{\times}$ replacing $\mathcal{B}^{\mathbb{R}}_{+}$, we confine the proofs to pointing out those arguments that vary from the ones in the last subsection.

Definition 5.7 For a domain P, we define the *power domain of $\times$-random variables over P* to be the subdomain

$$\mathbb{RV}_\times(P) = \bigcup_{n\geq 0}\{[(r_i,d_i)_n] \mid \sum_i r_i \leq 1\} \cup \{\bot\} \subseteq \mathcal{B}^{\mathbb{R}}_{\times}(P).$$

The laws that a $\times$-random variable algebra should satisfy vary only slightly from those for $+$-random variable algebras from the last subsection:

Definition 5.8 A *$\times$-random variable algebra* is a domain P with $\underline{0}$, its identity element and with a Scott-continuous mapping $+\colon (0,1] \times P \times P \to P$ satisfying:

- $\underline{0} +_r a = a$ and $\bot +_r a = \bot$ for all $0 < r \leq 1$,
- $a +_1 b = a$,
- $a +_r b = b +_{1-r} a$, and
- $(a +_r b) +_s c = a +_{rs} (b +_{\frac{s(1-r)}{1-sr}} c)$,

where $r, s \in (0,1)$ and $a, b, c \in P$.

A *morphism of $\times$-random variable algebras* is a Scott-continuous map $f\colon S \to T$ satisfying $f(\underline{0}_S) = \underline{0}_T$, $f(\bot_S) = \bot_T$ and $f(s +_r s') = f(s) +_r f(s')$, for all $s, s' \in S$ and all $r \in (0,1]$.

The difference between these laws and those characterizing $+$-random variables is the addition of the second part of the first law that asserts $\bot +_r a = \bot$.

This does not hold in the case of +-random variables, since for those, $\underline{0} = \perp$. As before, (i) we restrict the application of the laws involving $+_r$ to cases when $0 < r < 1$ (which avoids some annoying side conditions in Graham's listing), and (ii) the law $a +_r a = a$ is missing.

Proposition 5.9 *Let P be a domain, and for $[(r_i, p_i)_m], [(s_j, q_j)_n] \in \mathbb{RV}_\times(P)$ and $0 < r \leq 1$, define*

$$[(r_i, p_i)_m] +_r [(s_j, q_j)_n] = [(rr_i, p_i)_m] \oplus [((1-r)s_j, q_j)_n] .$$

Then:

(i) $\mathbb{RV}_\times(P)$ *is a $\times$-random variable algebra, and*

(ii) $[(r, p)] = [(1, p)] +_r \underline{0}$ *for all $p \in P$ and all $r \in (0, 1)$, and*

$$[(r_1, p_1), \ldots, (r_m, p_m)] = [(1, p_1)] +_{r_1} [(\frac{r_2}{(1-r_1)}, p_2), \ldots, (\frac{r_m}{(1-r_1)}, p_m)]$$

for all $[(r_1, p_1), \ldots, (r_m, p_m)] \in \mathbb{RV}_\times(P)$.

Proof. Note that the second part of the first law (involving $\perp$) holds in $\mathbb{RV}_\times(P)$ because it is inherited from $\mathcal{B}^{\mathbb{R}}_\times(P)$. The proof of the rest of the first part follows as in the case of +-random variables,

As with +-random variables, the results in (ii) are simple calculations. □

We now characterize initial +-random variable algebras over domains.

Theorem 5.10 $\mathbb{RV}_\times$ *defines a continuous endofunctor on* DOM*, as well as on* RB *and* FS*. Moreover,* $\mathbb{RV}_\times$ *also defines a left adjoint to the forgetful functor from the subcategory of $\times$-random variable algebra domains and $\times$-random variable maps to* DOM*,* RB *and* FS*, respectively.*

Proof. $\mathbb{RV}_\times$ is obtained by restricting $\mathcal{B}^{\mathbb{R}}_\times$ in the "real components" to ones whose sum is at most 1. This family is a Scott-closed subset of $\mathcal{B}^{\mathbb{R}}_\times(P)$. Hence $\mathbb{RV}_\times(P)$ is a domain if P is one. Continuous maps $f \colon P \to Q$ extend to $\mathcal{B}^{\mathbb{R}}_\times(P)$ by $\mathcal{B}^{\mathbb{R}}_\times(f)[(r_i, p_i)_n] = [(r_i, f(p_i))_n]$, and the elements in $\mathbb{RV}_\times(P)$ are those in $\mathcal{B}^{\mathbb{R}}_\times(P)$ whose real components sum to at most 1; it follows that $\mathcal{B}^{\mathbb{R}}_\times(f)(P) \subseteq \mathcal{B}^{\mathbb{R}}(Q)$. Since the endofunctor is composed of components that are locally continuous, it is as well.

For the second part, we first show that $\mathbb{RV}_\times$ is left adjoint to the forgetful functor from $\times$-random variable domains into DOM. First, we let $\eta \colon P \to \mathbb{RV}_\times(P)$ by $\eta(p) = [(1, p)]$ define the unit of the adjunction.

Next, let S be a $\times$-random variable domain algebra, P a domain, and let $f \colon P \to S$ be a Scott continuous map. We define $\widehat{f} \colon (\mathbb{RV}_\times(P) \cap \mathcal{B}^{\mathbb{R}}_\times(P)) \to S$ via $\widehat{f}([(r_i, p_i)_m]$ by induction on m, just as in the case of $\mathbb{RV}_+(P)$, and the argument is virtually the same.

This shows that $\mathbb{RV}_\times$ is left adjoint to the forgetful functor from random variable algebras into DOM, so it generates a monad on DOM. We have already shown that $\mathbb{RV}_\times(P)$ is in RB or FS if P is, so $\mathbb{RV}_\times$ has restrictions to these subcategories that also define monads. □

Corollary 5.11 *Each of the power domain monads $\mathcal{P}_X$ lifts to a monad on $\mathbb{RV}_\times$-algebras.* □

As in the case of $\mathbb{RV}_\times$, we can solve domain equations such as $P \simeq \mathcal{P}_X \circ \mathbb{RV}_\times(P)$ for each of the power domain monads $\mathcal{P}_X$. The resulting domain P will be a $\mathcal{P}_X$-algebra and simultaneously a $\mathbb{RV}_\times$-algebra. What's true now, though, is that these domain equations can be solved within Cartesian closed categories of domains.

5.2 Random variables and probability measures

One might also ask about the relationship between our construction and the traditional probabilistic power domain over a domain. The following provides the answer.

Theorem 5.12 *If P is a domain, then there is an epimorphism $Flat\colon \mathbb{RV}_+(P) \to \mathbb{V}(P)$, the domain of subprobability neasures over P. Similarly, there is an epimorphism $Flat\colon \mathbb{RV}_\times(P) \to \mathbb{V}(P)$.*

Proof. In both cases, the mapping is $Flat([r_i, d_i]_n) = \sum_{i\leq n} r_i\delta_{d_i}$, where in $\mathbb{V}(P)$, summands with the same support are identified. This is easily seen to define a Scott-continuous map in both cases. It is an epimorphism of domains because the simple valuations are dense [10], and clearly they are the range of $Flat$. □

6 Summary and Future Work

We have presented a power domain for finite random variables, and shown that it defines a monad that enjoys a distributive law with respect to each of the power domain monads. Moreover, our construction defines a continuous endofunctor on the Cartesian closed categories RB and FS, as well as on the category DOM. This is where our results on bag domains have their payoff. Indeed, we could have defined our random variables monad as a restriction of $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}$, which is equivalent to Varacca's IV_P, but then we would have had to show $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}$ or at least its restriction leaves ccc's of domains invariant. We fear this is akin to the long standing problem of showing there is a ccc of domains that $\mathbb{V}$ leaves invariant. On the other hand, if we could resolve this question for $\mathrm{Id}\,\mathcal{B}_+^{\mathbb{R}}$ (or even for $\mathcal{B}$ or $\mathcal{B}_+$), the result might shed some light on the situation for $\mathbb{V}$. In any case, we believe trying to attack these issues using abstract bases would be much more difficult. And, since no analogous result is known to hold for the probabilistic power domain, our construction $\mathbb{RV}_\times(P)$ provides an alternative for modeling probabilistic choice on domains that does leave two of the prominent ccc's of domains invariant.

Varacca actually presents three separate indexed valuation constructions, as described in Section 2. Our methods can be adopted to recapture each of them; a discussion of the Hoare indexed valuations from our approach is presented in [17].

One issue we haven't discussed is what sort of operational intuition there is for random variables. Again, we rely on Varacca, who showed in [24] that, for a simple state-based language supporting nondeterminism and probabilistic choice, probabilistic schedulers could distinguish distinct programs in his model. This is similar to refusal testing in CSP: one tests a process at each place where a probabilistic choice is made. In the more traditional approach using probabilistic bisimulation, such as in [19], one tests processes at the end of their computation, not at each choice point. This provides a viable, albeit more complicated method of assigning behaviors to programs.

Our construction only models finite random variables over domains. It is our intention to extend these ideas to encompass discrete random variables, and eventually continuous ones as well. The main issue is how to overcome the reliance on Rudin's Lemma 3.3, which underlies our proof that $P^n/{\equiv_n}$ is a dcpo, and the arguments we need to show that $\mathcal{B}(P)$ is a domain.

Another issue not discussed here is whether one can bring Shannon's information theory into the picture [22]. This is based on bringing entropy into play; there are some very interesting results about domains and entropy in Martin's recent work [14], a line we plan to explore. A particularly appealing issue here is defining an order on random variables over a domain relative to which entropy forms a measurement. If Martin's work is any indication, this will probably be a fairly difficult issue to resolve.

Acknowledgment We wish to thank several people for their comments about this work. The list includes Samson Abramsky, Gavin Lowe, Keye Martin, Dusko Pavlovic, Bill Roscoe and James Worrell. We also would like to thank anonymous referees both of this paper and of its predecessor [18] for many valuable suggestions.

References

[1] Abramsky, S. and A. Jung, "Doman Theory," in: Handbook of Logic in Computer Science, S. Abramsky and D. M. Gabbay and T. S. E. Maibaum, editors, Clarendon Press, 1994, pp. 1—168.

[2] Beck, J., *Distributive laws,* in: *Seminar on Triples and Categorical Homology Theory,* 1969, pp. 119–140.

[3] Gautam, N. D., *The validity of equations of complex algebras,* Archiv für Mathematische Logik und Grundlagenforschung **3** (1957), pp. 117–124.

[4] Gierz, G., K. H. Hofmann, K. Keimel, J. Lawson, M. Mislove and D. Scott, "Continuous Lattices and Domains," Cambridge University Press, 2003.

[5] Graham, S. K., *Closure properties of a probabilistic domain construction,* Lecture Notes in Computer Science **298** (1985), pp. 213–233.

[6] Heckmann, Reinhold, *Lower bag domains.* Fundamenta Informaticae **24** (1995), pp. 259–281.

[7] Hennessy, M. and G. D. Plotkin, *Full abstraction for a simple parallel programming language,* Lecture Notes in Computer Science **74** (1979), pp. 108–120.

[8] Johnstone, P. T., *Partial products, bag domains and hyperlocal toposes,* LMS Lecture Notes Series **77** (1992), pp. 315–339.

[9] Johnstone, P. T., *Variations on a bagdomain theme,* Theoretical Computer Science **136** (1994), pp. 3–20.

[10] Jones, C., "Probabilistic Nondeterminism," PhD Dissertation, University of Edinburgh, Scotland, 1989.

[11] Jung, A., "Cartesian Closed Categories of Domains," CWI Tracts **66** (1989), Centrum voor Wiskunde en Informatica, Amsterdam.

[12] Jung, A. and R. Tix, *The troublesome probabilistic power domain,* Electronic Notes in Theoretical Computer Science **13** (1999), http://www.elsevier.com/locate/entcs/volume13.html

[13] Lowe, G., "Probabilities and Priorities in Timed CSP," DPhil Thesis, Oxford University, 1993.

[14] Martin, K. *Entropy as a fixed point,* Proceedings of ICALP 2004, LNCS **3142**, 2004.

[15] Mislove, M. *Algebraic posets, algebraic cpos and models of concurrency, Proceedings of the Oxford Symposium on Topology*, G. M. Reed, A. W. Roscoe and R. Wachter, editors, Oxford University Press, 75 – 111.

[16] Mislove, M. *Nondeterminism and probabilistic choice: Obeying the laws,* Lecture Notes in Computer Science **1877** (2000), pp. 350–364.

[17] Mislove, M. *Monoids over domains, Mathematical Structures for Computer Science*, to appear.

[18] Mislove, M., *Discrete random variables over domains, Proceedings* of ICALP 2005, Lecture Notes in Computer Science **3580** (2005), pp. 1006–1017.

[19] Mislove, M., J. Ouaknine and J. B. Worrell, *Axioms for probability and nondeterminism,* Proceedings of *EXPRESS 2003*, Electronic Notes in Theoretical Computer Science **91(3)**, Elsevier.

[20] Plotkin, G. D., "Domains," The Pisa Notes, 1983, available at URL www.inf.ed.ac.uk.

[21] Poigné, A., *A remark on variations of power domains,* Bulletin of the EATCS **31** (1987), pp. 38–41.

[22] Shannon, C., *A mathematical theory of information,* Bell Systems Technical Journal **27** (1948), pp. 379–423 & 623–656.

[23] Varacca, D., *The powerdomain of indexed valuations,* Proceedings 17th IEEE Symposium on Logic in Computer Science (LICS 2002), IEEE Press, 2002.

[24] Varacca, D., "Probability, Nondeterminism and Concurrency: Two Denotational Models for Probabilistic Computation," PhD Dissertation, Aarhus University, Aarhus, Denmark, 2003.

[25] Vickers, S. *Geometric theories and databases,* LMS Lecture Notes Series **77** (1992), pp. 288–314.