Domain Theory and Task-structured Probabilistic Input/Output Automata

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Joint Work With
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Outline

- General Setting

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- Task-structured Probabilistic Input/Output Automata

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- Summary and Future work

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$\mathcal{P} \leq_S \mathcal{F}$ iff $(\forall \mathcal{A}dv)(\exists \mathcal{I})(\forall \mathcal{E}) \quad \mathcal{P} \| \mathcal{A}dv \| \mathcal{E} \simeq \mathcal{F} \| \mathcal{I} \| \mathcal{E}$

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- $\leq_S$: Trace distributions of observable events from $(\mathcal{P} \| \mathcal{A}dv \| \mathcal{E})$ all appear in the trace distributions of $(\mathcal{F} \| \mathcal{I} \| \mathcal{E})$.

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- *Semantic model:* Devise model that accommodates concerns arising in protocol analysis
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- *Focus of talk:* **Task-structured Probabilistic Input/Output Automata**

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- S - countable set of states, s_0 - start state
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Transition determinism: $(s, a, \mu), (s, a, \nu) \in D \;\Rightarrow\; \mu = \nu$

Input enabling: $(\forall s \in S)(\forall a \in I)(\exists \mu)\ (s, a, \mu) \in D$

How to compose PIOAs

$\mathcal{A}_i = (S_i, s_{\mathcal{A}_i}, I_i, O_i, H_i, D_i),\ \ i = 1, 2$ are *compatible* if

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- $S_{\mathcal{A}_1,\mathcal{A}_2} = S_1 \times S_2$
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$\mathcal{A}$ is *closed* if $I_{\mathcal{A}} = \emptyset$

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- **Tasks:** Equivalence relation $\mathcal{R} \subseteq (O \cup H) \times (O \cup H)$.

  Task: Any equivalence class of $\mathcal{R}$.

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- **Application:** Via *task schedules* - $\rho = T_1 T_2 \cdots$

Example

- *Player:* Flips coin and announces result
- *Opponent:* Announces *Heads* or *Tails*
- If they match, *Player* wins, otherwise *Opponent* wins.
  - *Player:*
    States: ⊥, flipped, announced, initially ⊥
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  - Task Schedules:
    Flip,P-Announce,O-Announce-Heads
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PIOA Semantics

$\mathcal{A} = (S, s_0, I, O, H, D)$

$\mathsf{Frags}^*(A) = \bigcup_n \{\alpha \in (S \times \mathsf{Act})^n \times S \mid \alpha \text{ finite execution fragment}\}$

Execution fragment:

$\alpha = s_1 a_1 s_2 a_2 \cdots$ with $s_{i+1} \in \mathrm{supp}(\mu_i)$ & $(s_i, a_i, \mu_i) \in D$

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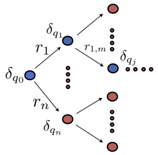

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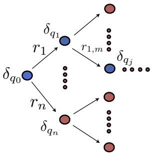

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Apply task schedule $\rho = T_1 T_2 \cdots$

$$\delta_{q_0} \xrightarrow{T_1} \sum_q \mu_{q_0, a_{T_1}}(q) \delta_{q_0 a_{T_1} q} \xrightarrow{T_2} \cdots$$

Defining Apply

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For $\rho = T_1 \cdots T_n$

$$\mathsf{Apply}(\mu, \rho) = \mathsf{Apply}(\mathsf{Apply}(\mu, T_1), T_2 \cdots T_n))$$

Defining Apply (cont'd)

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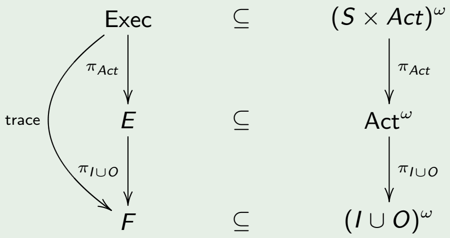
Hence $\mathsf{Apply}(\mu, \rho) = \sup_n \mathsf{Apply}(\mu, T_1 \cdots T_n)$ is well-defined.

Semantics of Observable Events



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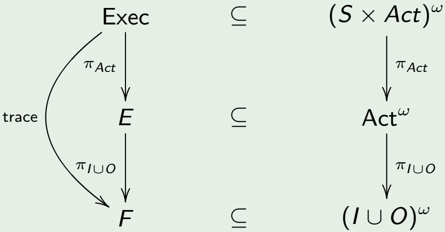

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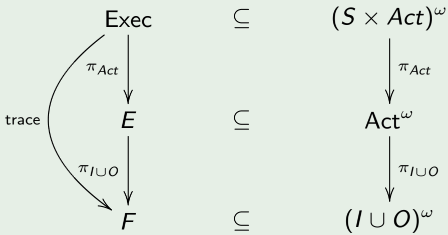

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Given $\mathcal{A}$ and $\rho = T_1 \cdots$, we'd like to have a *scheduler* – determined in advance – that would represent applying ρ to any $\mu \in \mathsf{Prob}(\mathsf{Frags}^*(A))$.

Task Schedulers

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Measures from Schedulers - Original Definition

If $\sigma\colon \mathsf{Frags}^*(A) \to \mathbb{V}(\mathsf{Act})$ is a scheduler and $\alpha \in \mathsf{Frags}^*(A)$ then define $\epsilon_\sigma\colon \mathsf{Prob}(\mathsf{Frags}^*(A)) \to \mathsf{Prob}(\mathsf{Frags}^*(A))$ by

$$\epsilon_{\sigma,\alpha}(\uparrow \alpha') = \begin{cases} 0 & \text{if } \alpha' \not\leq \alpha \not\leq \alpha' \\ 1 & \text{if } \alpha' \leq \alpha \\ \epsilon_{\sigma,\alpha}(\uparrow \alpha'')\sigma(\alpha'')(a)\mu_{\alpha'',a}(s) & \text{if } \alpha \leq \alpha' = \alpha'' as, \end{cases}$$

where $\mu_{\alpha'',a}(s)$ is the probability of landing in state s starting from $\mathsf{ls}(\alpha)$ after executing action a.

Measures from Schedulers - Our Definition

Let $\mathcal{A}$ be a task PIOA and let $\sigma\colon \mathsf{Frags}^*(\mathcal{A}) \to \mathbb{V}(\mathsf{Act})$ be a task scheduler. If $\alpha \in \mathsf{Frags}^*(\mathcal{A})$, we define

$$\epsilon'_{\sigma,\alpha} = (1 - ||\sigma(\alpha)||)\delta_\alpha + \sum_{a \in \mathsf{Act}} \sigma(\alpha)(a) \left(\sum_s \mu_{\alpha,a}(s) \epsilon'_{\sigma,\alpha a s} \right)$$

Then, $\epsilon'_{\sigma,\alpha} = \epsilon_{\sigma,\alpha}$ for all schedulers σ and $\alpha \in \mathsf{Frags}^*(\mathcal{A})$.

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Neat fact:

$$\begin{aligned} \epsilon'_{\sigma,\alpha,0} &= \delta_\alpha \\ \epsilon'_{\sigma,\alpha,n+1} &= (1 - ||\sigma(\alpha)||)\delta_\alpha + \sum_{a\in \mathrm{Act}} \sigma(\alpha)(a) \left(\sum_s \mu_{\alpha,a}(s)\epsilon'_{\sigma,\alpha a s,n} \right) \end{aligned}$$

implies $\epsilon'_{\sigma,\alpha} = \sup_n \epsilon'_{\sigma,\alpha,n}$

Schedulers vs. Task Schedules

Theorem

Let $\mu \in \mathsf{Prob}(\mathsf{Frags}^*(A))$ have support consisting of incomparable fragments, and let ρ be a task schedule. Then there is a scheduler $\sigma_\rho \colon \mathsf{Frags}^*(\mathcal{A}) \to \mathbb{V}(\mathsf{Act})$ such that $\mathsf{Apply}(\mu, \rho) = \epsilon_{\sigma_\rho, \mu}$.

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In fact, for $\mu = \sum_\alpha \mu(\alpha)\delta_\alpha$ and $\rho = \rho' T$, the scheduler σ_ρ is deterministic:

$$\sigma_\rho(\alpha) = \begin{cases} \delta_{a_{\mathrm{ls}(\alpha)},T} & \text{if } \alpha \in A_T \cap \mathrm{supp}\,\mu, \\ \sigma_{\rho'}(\alpha) & \text{if } \sigma_{\rho'}(\alpha) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho = T_1 \cdots \text{ infinite implies } \sigma_\rho = \cup_n \sigma_{T_1 \cdots T_n}.$$

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Corollary Let $\mathcal{A}$ be a Task PIOA.

- For each task schedule ρ, there is a deterministic task scheduler σ_ρ satisfying

$$\mathsf{Apply}(\delta_{s_0}, \rho) = \epsilon_{\sigma_\rho,\delta_{s_0}}.$$

Schedulers vs. Task Schedules

Theorem

Let $\mu \in \mathsf{Prob}(\mathsf{Frags}^*(A))$ have support consisting of incomparable fragments, and let ρ be a task schedule. Then there is a scheduler $\sigma_\rho \colon \mathsf{Frags}^*(\mathcal{A}) \to \mathbb{V}(\mathsf{Act})$ such that $\mathsf{Apply}(\mu, \rho) = \epsilon_{\sigma_\rho, \mu}$.

Corollary Let $\mathcal{A}$ be a Task PIOA.

- For each task schedule ρ, there is a deterministic task scheduler σ_ρ satisfying
$$\mathsf{Apply}(\delta_{s_0}, \rho) = \epsilon_{\sigma_\rho, \delta_{s_0}}.$$
- $\mathsf{tdist}(\mathcal{A}) \subseteq \{\mathsf{trace}(\epsilon_{\sigma, \delta_{s_0}}) \mid \sigma \text{ deterministic scheduler}\}$.

Simulations

Recall

$$(\mathcal{P} \| \mathcal{A}dv \| \mathcal{E}) \simeq (\mathcal{F} \| \mathcal{I} \| \mathcal{E})$$

For us,

$$\mathrm{tdist}[\![\mathcal{P} \| \mathcal{A}dv \| \mathcal{E}]\!] \subseteq \mathrm{tdist}[\![\mathcal{F} \| \mathcal{I} \| \mathcal{E}]\!]$$

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Let $(\mathcal{A}_i, \mathcal{R}_i)$ be two task PIOAs and let $f\colon \mathcal{R}_1^* \times \mathcal{R}_1 \to \mathcal{R}_2^*$ be a function. We define $\text{full}(f)\colon \mathcal{R}_1^* \to \mathcal{R}_2^*$ by

$$\begin{aligned} \text{full}(f)(\langle\ \rangle) &= \langle\ \rangle \\ \text{full}(f)(\rho T) &= \text{full}(f)(\rho)\hat{\ } f(\rho, T) \end{aligned}$$

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$R \subseteq \text{Prob}(\text{Exec}(\mathcal{A}_1)) \times \text{Prob}(\text{Exec}(\mathcal{A}_2))$ is a *simulation* if

- $(\mu_1, \mu_2) \in R \;\Rightarrow\; \text{tdist}(\mu_1) \subseteq \text{tdist}(\mu_2)$
- $(\delta_{s_{0,1}}, \delta_{s_{0,2}}) \in R$
- $(\exists f : \mathcal{R}_1^* \times \mathcal{R}_1 \to \mathcal{R}_2^*)(\forall \rho \in \mathcal{R}_1^*)(\forall T \in \mathcal{R}_1)$

$$(\mu_1, \mu_2) \in R \;\wedge\; \text{supp}(\mu_1) \subseteq \rho \;\wedge\; \text{supp}(\mu_2) \subseteq \text{full}(f)(\rho)$$
$$\Rightarrow\; (\text{Apply}(\mu_1, T), \text{Apply}(\mu_2, \text{full}(f)(\rho, T)) \in \mathcal{E}(R)$$

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Theorem (Canetti, et al)

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be comparable task-PIOAs that are closed and action-deterministic. If there exists a simulation relation from $\mathcal{A}_1$ to $\mathcal{A}_2$, then $\text{tdist}(\mathcal{A}_1) \subseteq \text{tdist}(\mathcal{A}_2)$.

Expansions and Monads

$(X,\Sigma_X),(Y,\Sigma_Y)$ measure spaces, $R \subseteq X \times Y$. The *lift of R* is $\widehat{R} \subseteq \mathbb{V}X \times \mathbb{V}Y$ defined by

$$(\sum_x r_x\delta_x, \sum_y s_y\delta_y) \in \widehat{R} \;\Leftrightarrow\; \exists t\colon X \times Y \to [0,1] \text{ with}$$

- $r_x = \sum_y t(x,y)\ (\forall x)$
- $\sum_x t(x,y) \leq s_y\ (\forall y)$
- $t(x,y) > 0 \;\Rightarrow\; (x,y) \in R$

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But, (X, Σ_X) a measure space implies $\mathbb{V}X$ is a measure space
And $R \subseteq \mathbb{V}X \times \mathbb{V}Y$ implies $\widehat{R} \subseteq \mathbb{V}(\mathbb{V}X) \times \mathbb{V}(\mathbb{V}Y)$

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$\mathbb{V}$: Meas $\to$ Meas is a *monad*, and $R \mapsto \mathcal{E}(R)$ utilizes the *lifting* and *multiplication* of the monad.

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Future Work

- More about the use of the monad $\mathbb{V}$
- Application to Dining Cryptograhers
- Application to other protocols, combining the UC of oblivious transfer due to Canetti, et al.