Approximating Continuous Channels

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How many bars...

- The closer to the tower, the more faithful the reception,
- The further away, the more noise, and the less faithful...
- Cell phone acts like a lossy, noisy channel.

The Full Binary Tree

$$
\begin{array}{c}
\vdots \\
\mathcal{C}_n \simeq 2^n \\
\downarrow \pi_n \\
\mathcal{C}_{n-1} \simeq 2^{n-1} \\
\downarrow \pi_{n-1} \\
\vdots \\
\downarrow \pi_2 \\
\mathcal{C}_1 \simeq 2 \\
\downarrow \pi_1 \\
\mathcal{C}_0 = \{*\}
\end{array}
\qquad
\begin{array}{c}
\mathcal{C}_\infty \simeq \varprojlim (\mathcal{C}_n, \pi_{mn}) \\
\downarrow \pi_n^\infty \\
\mathcal{C}_n \simeq 2^n \\
\downarrow \pi_n \\
\mathcal{C}_{n-1} \simeq 2^{n-1} \\
\downarrow \pi_{n-1} \\
\vdots \\
\downarrow \pi_2 \\
\mathcal{C}_1 \simeq 2 \\
\downarrow \pi_1 \\
\mathcal{C}_0 = \{*\}
\end{array}
$$

Adding probability

$$Prob(\mathcal{C}_\infty) \simeq \varprojlim \, (Prob(\mathcal{C}_n), Prob(\pi_{mn}))$$

$$\Big\downarrow Prob(\pi_n^\infty)$$

$$Prob(\mathcal{C}_n)$$

$$\Big\downarrow Prob(\pi_n)$$

$$Prob(\mathcal{C}_{n-1})$$

$$\Big\downarrow Prob(\pi_{n-1})$$

$$\vdots$$

$$\Big\downarrow Prob(\pi_2)$$

$$Prob(\mathcal{C}_1)$$

$$\Big\downarrow Prob(\pi_1)$$

$$Prob(\mathcal{C}_0)$$

Approximating Measures

Given $\sum_{i\in\mathcal{C}_{n-1}} r_i\delta_i$, can we find $\sum_{j\in\mathcal{C}_n} s_j\delta_j$ with $Prob(\pi_n)(\sum_{j\in\mathcal{C}_n} r_j\delta_j) = \sum_{i\in\mathcal{C}_{n-1}} s_i\delta_i$?

True iff $(\forall i = 1, \cdots, 2^{n-1})\ r_{2i-1} + r_{2i} = s_i$.

So,

$$\sum_{j\leq 2^{n-1}} s_j\delta_{2j-1}$$

would work. Or,

$$\sum_{j\leq 2^n} \frac{1}{2} s_{\frac{j}{2}}\delta_j$$

$=$

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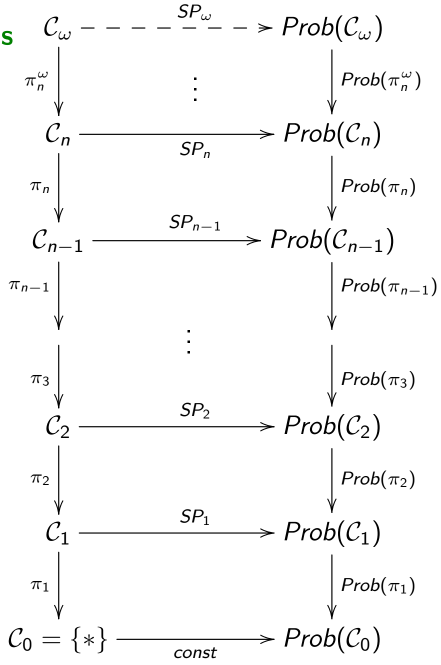
Harmonic Analysis

If we regard $\mathcal{C}_n$ as a group, then $Prob(\mathcal{C}_n)$ is a compact affine monoid, and the partial order of interest is

$$\mu \leq \nu \text{ iff } \mu * Prob(\mathcal{C}_n) \supseteq \nu * Prob(\mathcal{C}_n),$$

where $*$ denotes convolution. Haar measure is the least element.

Tower of Channels



Buildng SP$_n$

$$SP_1 = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$SP_2 = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Then, $SP_1 \circ \pi_2 = Prob(\pi_2) \circ SP_2$ iff

$$\begin{aligned} x_1 &= x_{11} + x_{12} &= x_{21} + x_{22} \\ x_2 &= x_{13} + x_{14} &= x_{23} + x_{24} \\ x_3 &= x_{31} + x_{32} &= x_{41} + x_{42} \\ x_4 &= x_{33} + x_{34} &= x_{43} + x_{44} \end{aligned}$$

Buildng SP$_n$ (con'd)

For example,

$$SP_1 = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$SP_2 = \begin{bmatrix} x_1 & 0 & x_2 & 0 \\ 0 & x_1 & 0 & x_2 \\ 0 & x_3 & x_4 & 0 \\ x_3 & 0 & 0 & x_4 \end{bmatrix}$$

or

$$SP_2 = \frac{1}{2} \begin{bmatrix} x_1 & x_1 & x_2 & x_2 \\ x_1 & x_1 & x_2 & x_2 \\ x_3 & x_3 & x_4 & x_4 \\ x_3 & x_3 & x_4 & x_4 \end{bmatrix}$$

$=$

Buildng SP$_n$ (con'd)

For example,

$$SP_1 = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$SP_2 = \begin{bmatrix} x_1 & 0 & x_2 & 0 \\ 0 & x_1 & 0 & x_2 \\ 0 & x_3 & x_4 & 0 \\ x_3 & 0 & 0 & x_4 \end{bmatrix}$$

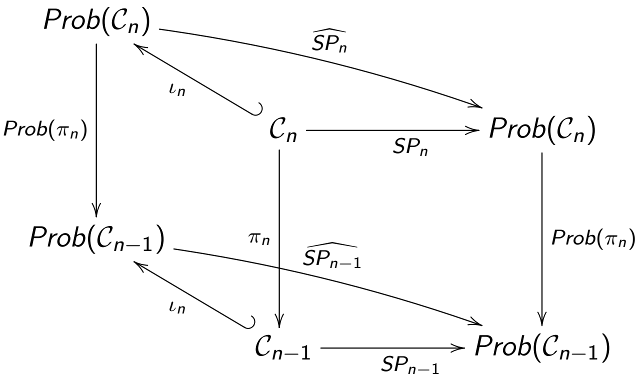
or

$$SP_2 = \frac{1}{2} \begin{bmatrix} x_1 & x_1 & x_2 & x_2 \\ x_1 & x_1 & x_2 & x_2 \\ x_3 & x_3 & x_4 & x_4 \\ x_3 & x_3 & x_4 & x_4 \end{bmatrix}$$

Harmonic Analysis

Again, the partial order on $Prob(\mathcal{C}_n)$ as a compact affine monoid helps here. $Cap(SP_n)$ can be minimized using this analysis.

$\mathbf{SP}_n$ lives in the Kleisli Category for Prob : Comp $\to$ CompAff



The Monad of Subprobability Measures

$$SProb(X) = \{\mu \mid \mu \text{ subprobability measure on } X\}$$

This is a compact affine monoid with 0. In fact, $[0, 1]$ acts on such a monoid, so that $r \cdot \mu$ is defined for each $r \in [0, 1]$. The same analysis we used above allies here as eell

$\mathbf{SP}_n$ also lives in the Kleisli Category for **SProb : Comp $\to$ CompAffMon**