From Haar to Lebesgue via Domain Theory

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Lebesgue Measure and Unit Interval

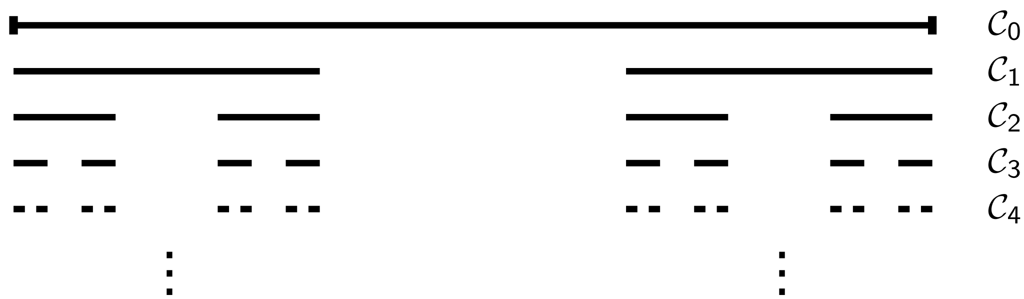
- $[0, 1] \subseteq \mathbb{R}$ inherits *Lebesgue measure*: $\lambda([a, b]) = b - a$.
- *Translation invariance:* $\lambda(x + A) = \lambda(A)$ for all (Borel) measurable $A \subseteq \mathbb{R}$ and all $x \in \mathbb{R}$.
- **Theorem** (Haar, 1933) Every locally compact group G has a unique (up to scalar constant) left-translation invariant regular Borel measure μ_G called *Haar measure*.

  If G is compact, then $\mu_G(G) = 1$.

  Example: $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ with quotient measure from λ.

  If G is finite, then μ_G is normalized counting measure.

The Cantor Set



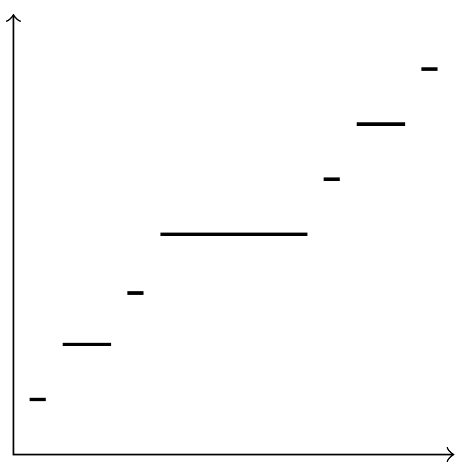
$$\mathcal{C} = \bigcap_n \mathcal{C}_n \subseteq [0,1] \text{ compact 0-dimensional, } \lambda(\mathcal{C}) = 0.$$

Theorem: $\mathcal{C}$ is the unique compact Hausdorff 0-dimensional second countable perfect space.

Cantor Groups

- *Canonical Cantor group:* $\mathcal{C} \simeq {\mathbb{Z}_2}^{\mathbb{N}}$ is a compact group in the product topology.

  $\mu_{\mathcal{C}}$ is the product measure $(\mu_{\mathbb{Z}_2}(\mathbb{Z}_2) = 1)$

  Theorem: (Schmidt) The Cantor map $\mathcal{C} \to [0, 1]$ sends Haar measure on $\mathcal{C} = {\mathbb{Z}_2}^{\mathbb{N}}$ to Lebesgue measure.

  Goal: Generalize this to *all* group structures on $\mathcal{C}$.

Cantor Groups

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- $G = \prod_{n>1} \mathbb{Z}_n$ is also a Cantor group.

  μ_G is the product measure $(\mu_{\mathbb{Z}_n}(\mathbb{Z}_n) = 1)$

- $\mathbb{Z}_{p^\infty} = \varprojlim_n \mathbb{Z}_{p^n}$ – p-adic integers.

  $x \mapsto x \bmod p \colon \mathbb{Z}_{p^{n+1}} \to \mathbb{Z}_{p^n}$.

- $H = \prod_n S(n)$ – $S(n)$ symmetric group on n letters.

  Definition: A *Cantor group* is a compact, 0-dimensional second countable perfect space endowed with a topological group structure.

Two Theorems and a Corollary

- **Theorem:** If G is a compact 0-dimensional group, then G has a neighborhood basis at the identity of clopen normal subgroups.
- *Proof:*
  1. G is a Stone space, so there is a basis $\mathcal{O}$ of clopen neighborhoods of e.
     If $O \in \mathcal{O}$, then $e \cdot O = O \;\Rightarrow\; (\exists U \in \mathcal{O})\, U \cdot O \subseteq O$
     $U \subseteq O \;\Rightarrow\; U^2 \subseteq U \cdot O \subseteq O$. So $U^n \subseteq O$.
     Assuming $U = U^{-1}$, the subgroup $H = \bigcup_n U^n \subseteq O$.
  2. Given $H < G$ clopen, $\mathcal{H} = \{xHx^{-1} \mid x \in G\}$ is compact.
     $G \times \mathcal{H} \to \mathcal{H}$ by $(x, K) \mapsto xKx^{-1}$ is continuous.
     $K = \{x \mid xHx^{-1} = H\}$ is clopen since H is, so G/K is finite.
     Then $|G/K| = |\mathcal{H}|$ is finite, so $L = \bigcap_{x \in G} xHx^{-1} \subseteq H$ is clopen and normal.

Two Theorems and a Corollary

- **Theorem:** If G is a compact 0-dimensional group, then G has a neighborhood basis at the identity of clopen normal subgroups.
- **Corollary:** If G is a Cantor group, then $G \simeq \varprojlim_n G_n$ with G_n finite for each n.
- **Theorem:** (Fedorchuk, 1991) If $X \simeq \varprojlim_{i \in I} X_i$ is a strict projective limit of compact spaces, then $Prob(X) \simeq \varprojlim_{i \in I} Prob(X_i)$.
- **Lemma:** If $\varphi \colon G \twoheadrightarrow H$ is a surmorphism of compact groups, then $Prob(\varphi)(\mu_G) = \mu_H$.

  Proof: $A \subseteq H$ measurable $\Rightarrow$

$$
\begin{aligned}
Prob(\varphi)\, \mu_G(hA) &= \mu_G(\varphi^{-1}(hA)) \\
&= \mu_G(\varphi^{-1}(h)\varphi^{-1}(A)) \\
&= \mu_G((g \ker \varphi) \cdot \varphi^{-1}(A)) \qquad \text{(where } \varphi(g) = h\text{)} \\
&= \mu_G(g \cdot (\ker \varphi \cdot \varphi^{-1}(A)) \\
&= \mu_G(\ker \varphi \cdot \varphi^{-1}(A)) \\
&= \mu_G(\varphi^{-1}(A)) = Prob(\varphi)\, \mu_G(A).
\end{aligned}
$$

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- **Corollary:** If G is a Cantor group, then $G \simeq \varprojlim_n G_n$ with G_n finite for each n.
- **Theorem:** (Fedorchuk, 1991) If $X \simeq \varprojlim_{i \in I} X_i$ is a strict projective limit of compact spaces, then $Prob(X) \simeq \varprojlim_{i \in I} Prob(X_i)$.

  In particular, if $X = G$ and $X_i = G_i$ are compact groups,

  then $\mu_G = \lim_{i \in I} \mu_{G_i}$ in $Prob(\prod_i G_i)$.

Two Theorems and a Corollary

- **Theorem:** If G is a compact 0-dimensional group, then G has a neighborhood basis at the identity of clopen normal subgroups.
- **Corollary:** If G is a Cantor group, then $G \simeq \varprojlim_n G_n$ with G_n finite for each n.
  Moreover, $\mu_G = \lim_n \mu_n$, where μ_n is normalized counting measure on G_n.

It's all about Abelian Groups

- **Theorem:** If $G = \varprojlim_n G_n$ is a Cantor group, there is a sequence $(\mathbb{Z}_{k_i})_{i>0}$ of cyclic groups so that $H = \varprojlim_n (\oplus_{i \le n} \mathbb{Z}_{k_i})$ has the same Haar measure as G.

  Proof: Let $G \simeq \varprojlim_n G_n$, $|G_n| < \infty$.

  Assume $|H_n| = |G_n|$ with H_n abelian.

  Define $H_{n+1} = H_n \times \mathbb{Z}_{|G_{n+1}|/|G_n|}$. Then $|H_{n+1}| = |G_{n+1}|$,

  so $\mu_{H_n} = \mu_n = \mu_{G_n}$ for each n, and $H = \varprojlim_n H_n$ is abelian.

  Hence $\mu_H = \lim_n \mu_n = \mu_G$.

Combining Domain Theory and Group Theory

$\mathcal{C} = \varprojlim_n H_n$, $H_n = \oplus_{i\leq n} \mathbb{Z}_{k_i}$

Endow H_n with *lexicographic order* for each n; then

$\pi_n \colon H_{n+1} \to H_n$ by $\pi_n(x_1, \ldots, x_{n+1}) = (x_i, \ldots, x_n)$ &

$\iota_n \colon H_n \hookrightarrow H_{n+1}$ by $\iota_n(x_1, \ldots, x_n) = (x_i, \ldots, x_n, 0)$ form

embedding-projection pair: $\pi_n \circ \iota_n = 1_{H_n}$ and $\iota_n \circ \pi_n \leq 1_{H_{n+1}}$.

$\mathcal{C} \simeq \text{bilim}\,(H_n, \pi_n, \iota_n)$ is bialgebraic total order:

$\varphi \colon K(\mathcal{C}) \to [0,1]$ by $\varphi(x_1, \ldots, x_n) = \sum_{i\leq n} \frac{x_i}{k_1\cdots k_i}$ strictly monotone

induces $\widehat{\varphi} \colon \mathcal{C} \to [0,1]$ monotone, Lawson continuous.

$\mu_{\mathcal{C}} = \lim_n \mu_n$ implies for $0 \leq m \leq p \leq k_1 \cdots k_n$:

$\mu_{\mathcal{C}}(\widehat{\varphi}^{-1}[\frac{m}{k_1\cdots k_n}, \frac{p}{k_1\cdots k_n}]) = \frac{p-m}{k_1\cdots k_n} = \lambda([\frac{m}{k_1\cdots k_n}, \frac{p}{k_1\cdots k_n}])$

Then inner regularity implies $Prob(\widehat{\varphi})(\mu_{\mathcal{C}}) = \lambda$.

If $\mathcal{C}' = \varprojlim_n G'_n$ with G'_n finite, then

$\widehat{\varphi}^{-1} \circ \widehat{\varphi}' \colon \mathcal{C}' \setminus K(\mathcal{C}') \to \mathcal{C} \setminus K(\mathcal{C})$ is a Borel isomorphism.

Lagniappe: Non-measurable Subgroups

In 1985 S. Saeki and K. Stromberg published the following question: *Does every infinite compact group have a subgroup which is not Haar measurable?*

Some known results: • Every infinite compact abelian group has a non-measurable subgroup (Comfort, Raczkowski, and Trigos-Arrieta 2006)

• With the possible exception of metric profinite groups, every infinite compact group has a non-measurable subgroup (Hernández, Hofmann and Morris 2014)

Proposition (Brian & M. 2014) Let G be an infinite compact group.

1. It is consistent with ZFC that G has a non-measurable subgroup.
2. If G is an abelian Cantor group, then G has a nonmeasurable subgroup.

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Ad 1: By Hernández, *et al.*, we can assume G is metric and profinite, so G is a Cantor group. Our results show Haar measure on $G \simeq \mathcal{C}$ is the same as for an abelian group structure, for which $\widehat{\phi}\colon \mathcal{C} \to [0,1]$ takes Haar measure to Lebesgue measure.

Fact: There is a model of ZFC that admits a countable subset $X \subseteq [0,1]$ that is not Lebesgue measurable (cf. Kechris).

Then $Y = \widehat{\phi}^{-1}(X) \subseteq \mathcal{C}$ is not Haar-measurable.

$H = \langle Y \rangle$ is a countable subgroup of G. Then H is not measure 0 since then Y would be measurable, while $\mu_G(H) > 0$ implies H is open, which implies $|H| = 2^{\aleph_0}$. Thus H is not Haar measurable.

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Ad 2: We first prove something stronger:

1.) If G is an infinite abelian group and $p \in G \setminus \{e\}$, then there is a maximal subgroup $M < G \setminus \{p\}$ satisfying $p \in \langle x, M \rangle$ for all $x \in G \setminus M$.

2.) G/M abelian $\Longrightarrow \exists \phi \colon G/M \to \mathbb{R}/\mathbb{Z}$ with $\phi(p) \neq e$.

$\ker \phi < G/M, M$ maximal wrt not containing $p + M \implies \ker \phi = M$.

Thus $G/M \simeq K < \mathbb{R}/\mathbb{Z}$.

$p \in \langle x, M \rangle \implies pM \in \langle xM \rangle \ (\forall x \in G) \implies pM = (xM)^{n_x} \ (\exists n_x \in \mathbb{Z})$.

$g \in \mathbb{R}/\mathbb{Z} \implies g$ has countably many roots, so G/M is countable.

Choosing $Q < \mathcal{C}$ dense and proper and then $Q < M$ implies M is proper, dense and has countable index. $\square$

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A result of Hofmann and Morris implies the remaining case is $\mathcal{C} = \varprojlim_n G_n$, G_n nonabelian simple groups for each $n > 0$.