Corrigendum to Topology, Domain Theory and Theoretical Computer Science

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Abstract

Jean Goubault-Larrecq pointed out an error in the proof of Proposition 4.45 of the published version of this paper [2]. In this corrigendum, I give a corrected proof, and I also give a simpler proof of Corollary 4.48.

We begin this corrigendum by recalling some definitions. First, a *domain* is a continuous cpo, i.e., a dcpo with least element in which $\Downarrow y = \{x \in P \mid x \ll y\}$ is directed and $y = \sqcup \Downarrow y$ for each $y \in P$. P is *coherent* if the Lawson topology on P is compact. For subsets $X, Y \subseteq P$, we write

- $X \sqsubseteq_L Y$ iff $X \subseteq \downarrow Y = \{x \in P \mid (\exists y \in Y)\ x \sqsubseteq y\}$.
- $X \sqsubseteq_U Y$ iff $Y \subseteq \uparrow X = \{y \in P \mid (\exists x \in X)\ x \sqsubseteq y\}$.
- $X \sqsubseteq_D Y$ iff $X \subseteq_L Y$ & $X \sqsubseteq_U Y$.

We define

- $\mathcal{P}_L(P) = (\Gamma_0(P), \sqsubseteq_L)$, the family of non-empty Scott-closed subsets of P in the lower order,
- $\mathcal{P}_U(P) = (\mathcal{C}(P), \sqsubseteq_U)$, where $\mathcal{C}(P)$ is the family of non-empty Scott-compact upper sets of P.
- $\mathcal{P}_D(P) = (\mathcal{D}(P), \sqsubseteq_D)$, where

$$\mathcal{D}(P) = \{X \subseteq P \mid X = \langle X \rangle \equiv \downarrow X \cap \uparrow X \ \&\ \downarrow X \in \Gamma_0(P) \ \&\ \uparrow X \in \mathcal{C}(P)\}$$

  consists of order-convex subsets of P; if P is coherent, then these sets are Lawson compact.

Proposition 4.45 of [2] is incorrect both in its statement and in its proof. Here is the correct version.

Proposition 4.45. *If P is a coherent continuous dcpo, then $(\mathcal{D}(P), \sqsubseteq_D)$ is a continuous dcpo in which*

$$C \ll D \quad \text{iff} \quad (\exists F \subseteq P \text{ finite})\ C \sqsubseteq_D \langle F\rangle \sqsubseteq_D D, D \subseteq \Uparrow F \ \&\ F \subseteq \Downarrow D,$$

and for which the operation

$$(C, D) \mapsto \langle C, D\rangle \equiv \uparrow(C \cup D) \cap \downarrow(C \cup D) \colon \mathcal{D}(P) \times \mathcal{D}(P) \to \mathcal{D}(P)$$

is continuous.

Proof. First assume that $\mathcal{F} \subseteq \mathcal{D}(P)$ is $\sqsubseteq_D$-directed and let $A = \overline{\bigcup\{\downarrow C \mid C \in \mathcal{F}\}}^{\sigma}$ be the Scott closure of the union of the lower sets of the members of $\mathcal{F}$. Then A is Scott closed, and hence also Lawson closed. Now, the sets $\{A \cap \uparrow C \mid C \in \mathcal{F}\}$ form a filtered intersection, and each is non-empty and Lawson compact since P is coherent, so their intersection also is non-empty and Lawson compact (e.g., by Theorem 4.35). Let B be that intersection. We claim that $B = \bigsqcup_{\mathcal{D}(P)} \mathcal{F}$. Indeed, it is obvious that $B \subseteq \uparrow C$ for all $C \in \mathcal{F}$. For the other direction, that $C \subseteq \downarrow B$ for each $C \in \mathcal{F}$, given $x \in C \in \mathcal{F}$, the fact that $\mathcal{F}$ is $\sqsubseteq_D$-directed implies that $\uparrow x \cap (A \cap \uparrow C')$ is non-empty and compact for each $C' \in \mathcal{F}$, so Theorem 4.35 shows the same is true of the $\uparrow x \cap B$. Thus B is an upper bound for $\mathcal{F}$, and a similar argument shows that B is the least upper bound of $\mathcal{F}$ in the order $\sqsubseteq_D$, and so $(\mathcal{D}(P), \sqsubseteq_D)$ is a dcpo.

Next, if $C \in \mathcal{D}(P)$, then C is a convex Lawson-closed subset of P, and $\uparrow C$ is Scott compact. So, we can write $\uparrow C$ as the filtered intersection of sets $\uparrow F$ where $C \subseteq \Uparrow F$ and F is finite. Clearly we can arrange it so that $F \subseteq \Downarrow C$ for each such F, by restricting to those $x \in F$ for which $\Uparrow x \cap C \neq \emptyset$. Then $\langle F\rangle = \downarrow F \cap \uparrow F \in \mathcal{D}(P)$, and $\langle F\rangle \sqsubseteq_D C$. Moreover, since $\uparrow C \subseteq \Uparrow F$, if $\mathcal{F} \subseteq \mathcal{D}(P)$ is directed and $C \sqsubseteq_D \bigsqcup \mathcal{F}$, then $\bigsqcup \mathcal{F} \subseteq \uparrow C \subseteq \Uparrow F$. The first part of the proof implies $\bigsqcup \mathcal{F} = \bigcap\{A \cap \uparrow C' \mid C' \in \mathcal{F}\}$, where $A = \overline{\bigcup\{\downarrow C' \mid C' \in \mathcal{F}\}}$. Since this expresses $\bigsqcup \mathcal{F}$ as a filtered intersection, Theorem 4.35 implies there is some $D_0 \in \mathcal{F}$ with $\uparrow D_0 \cap \downarrow A \subseteq \Uparrow F$. Thus, $D_0 \subseteq \uparrow D_0 \cap \downarrow A \subseteq \Uparrow F \subseteq \uparrow\langle F\rangle$.

On the other hand, $C \sqsubseteq_D \bigsqcup \mathcal{F}$ also implies that $C \sqsubseteq \downarrow \bigsqcup \mathcal{F}$, and so $F \subseteq \Downarrow C \subseteq \Downarrow \bigsqcup \mathcal{F}$. The first part of the proof shows that $\bigsqcup \mathcal{F} = \bigcap_{C' \in \mathcal{F}} (\overline{\downarrow \bigcup \mathcal{F}} \cap \uparrow C')$. Now $F \subseteq \Downarrow \bigsqcup \mathcal{F}$, and since F is finite, there is some $D_1 \in \mathcal{F}$ with $F \subseteq \Downarrow D_1$. Since $\mathcal{F}$ is directed, we can choose a $D_2 \in \mathcal{F}$ such that $D_0, D_1 \sqsubseteq_D D_2$, and it then follows that $\langle F\rangle \sqsubseteq_D D_2$, $D_2 \subseteq \Uparrow F$ and $F \subseteq \Downarrow D_2$. This all goes to show that $\langle F\rangle \ll C$ in $\mathcal{D}(P)$.

We next show that the family

$$\mathcal{F}_C = \{\langle F\rangle \mid \langle F\rangle \sqsubseteq_D C, C \subseteq \Uparrow F \ \&\ F \subseteq \Downarrow C\}$$

is $\sqsubseteq_D$-directed for each $C \in \mathcal{D}(P)$. First, note that if $F \in \mathcal{F}_C$, then $F \subseteq \Downarrow C \ \&\ C \subseteq \Uparrow F$, and this implies $F \subseteq \downarrow C \ \&\ C \subseteq \uparrow F$, which in turn implies $\langle F\rangle \subseteq \downarrow F \subseteq \downarrow C \ \&\ C \subseteq \uparrow F = \uparrow\langle F\rangle$. Hence $\langle F\rangle \sqsubseteq_D C$ if $C \subseteq \Uparrow F \ \&\ F \subseteq \Downarrow C$. Next, suppose that $F_1, F_2 \in \mathcal{F}_C$. Then $C \subseteq \Downarrow F_1 \cap \Downarrow F_2$ is Scott open, and since C is compact, there is a finite set G_0 with $C \subseteq \uparrow G_0 \subseteq \Downarrow F_1 \cap \Downarrow F_2$. Conversely, if $x_1 \in F_1$, then $F_1 \subseteq \Downarrow C$ implies there is some $c \in C$ with $c \ll c_x$. Since $C \subseteq \Downarrow F_2$,, there is some $x_2 \in F_2$

with $x_2 \ll c$. Then there is some x with $x_1, x_2 \ll x \ll c$. Since F_1 is finite, we can choose finitely many such x, and call the resulting set G_1. Then $F_1 \subseteq \Downarrow G_1 \subseteq \Downarrow C$ and $G_1 \subseteq \Downarrow F_1 \cap \Downarrow F_2$. Dually, there is a finite set G_2 with $F_2 \subseteq \Downarrow G_2 \subseteq \Downarrow C$. If $G = G_0 \cup G_1 \cup G_2$, it follows that $F_1, F_2 \subseteq \Downarrow G \subseteq \Downarrow C$, and $C \subseteq \Uparrow G \subseteq \Uparrow F_1 \cap \Uparrow F_2$. This shows $\mathcal{F}_C$ is directed.

It now follows that $\mathcal{D}(P)$ is continuous and that $\mathcal{F}_C$ is a basis for the way-below set of each C in $\mathcal{D}(P)$. Hence, if $C' \ll C$ in $\mathcal{D}(P)$, then there is some $F \subseteq P$ finite with $C' \sqsubseteq_D \langle F \rangle \sqsubseteq_D C$, $C \subseteq \Uparrow F$ and $F \subseteq \Downarrow C$.

The proof that $(C, D) \mapsto \langle C, D \rangle \colon \mathcal{D}(P) \times \mathcal{D}(P) \to \mathcal{D}(P)$ is continuous is straightforward. $\square$

Corollary 4.48 of [2] shows that $\mathcal{P}_D(P)$ is coherent if P is. The proof presented there had two purposes: first, show that $\mathcal{P}_D(P)$ is coherent, but second, that the Lawson topology on $\mathcal{P}_D(P)$ with respect $\sqsubseteq_D$ is the same as the Lawson topology $\mathcal{P}_D(P)$ inherits from the family of non-empty Lawson compact subsets of P in the order of reverse containment. While this latter is an interesting (remarkable?) result, it makes for a rather difficult proof. For those who just want to see a proof that $\mathcal{P}_D(P)$ is coherent, we present an alternate proof. This proof can also be found in [1]

Corollary 4.48. *If P is a coherent dcpo, then so is $\mathcal{P}_D(P)$.*

Proof. Recall that $\mathcal{P}_L(P)$ and $\mathcal{P}_U(P)$ are Scott domains, where

$X \ll_L Y \in \mathcal{P}_L(P)$ iff there is a finite set F with $X \subseteq {\downarrow} F \subseteq \Downarrow Y$, and

$X \ll_U Y \in \mathcal{P}_U(P)$ iff there is a finite set F with $Y \subseteq \Uparrow F \subseteq {\uparrow} F \subseteq X$.

Then $\mathcal{P}_L(P) \times \mathcal{P}_U(P)$ is a Scott domain, where $(X, Y) \sqsubseteq (X', Y')$ iff $X \subseteq X'$ and $Y' \subseteq Y$. Since this is a Scott domain, it is compact in its Lawson topology. Let

$$\mathcal{C} = \{(X, Y) \in \mathcal{P}_L(P) \times \mathcal{P}_U(P) \mid X \cap Y \neq \emptyset\}.$$

Then $\mathcal{C}$ is closed in the Lawson topology, hence it is a compact, Hausdorff space in the inherited topology. We define a retraction $r \colon \mathcal{C} \to \mathcal{C}$ by $r(X, Y) = ({\downarrow}(X \cap Y), {\uparrow}(X \cap Y))$. By assumption, $(X, Y) \in \mathcal{C}$ implies $X \cap Y \neq \emptyset$, so $r(X, Y)$ is well-defined. It is routine to verify this is a retraction, so $r(\mathcal{C})$ is a compact Hausdorff space in the inherited topology. On the other hand, for $(X, Y) \in r(\mathcal{C})$, $X = {\downarrow}(X \cap Y)$ and $Y = {\uparrow}(X \cap Y)$. So, if $(X, Y) \in U \subseteq \mathcal{P}_L(P) \times \mathcal{P}_U(P)$ and U is Lawson open, then there are finite subsets $F_1, G_1 \subseteq P$ and elements $X_1, \ldots, X_m \in \mathcal{P}_L(P), Y_1, \ldots, Y_n \in \mathcal{P}_U(P)$ satisfying $(X, Y) \in (\Uparrow F_1 \setminus (\cup_{i \leq m} {\uparrow} X_i) \times (\Uparrow G_1 \setminus (\cup_{j \leq n} {\uparrow} Y_j)$. This means

- $F_1 \subseteq \Downarrow X = \Downarrow(X \cap Y)$, and $X_i \setminus (X \cap Y) \neq \emptyset$ for $1 \leq i \leq m$,
- $X \cap Y \subseteq Y \subseteq \Uparrow G_1$, and $(X \cap Y) \setminus Y_j \neq \emptyset$ for $i \leq j \leq n$.

We define a map $\phi \colon r(\mathcal{C}) \to \mathcal{P}_D(P)$ by $\phi(X, Y) = X \cap Y$. ϕ also has an inverse, namely $X \mapsto ({\downarrow} X, {\uparrow} X) \colon \mathcal{P}_D(P) \to \mathcal{P}_L(P) \times \mathcal{P}_U(P)$ actually has its image in $\mathcal{C}$. It is routine to show that these mappings both preserve the order (the order on $r(\mathcal{C})$ being the one it inherits from $\mathcal{P}_L(P) \times \mathcal{P}_U(P)$), so ϕ and its inverse are order isomorphisms, and hence they are Lawson continuous.

$\square$

References

[1] J. D. Lawson, The versatile continuous order, LNCS **298** (1987), pp. 134–160.

[2] M. Mislove. Topology, domain theory and theoretical computer science. *Topology and Its Applications*, 89:3–59, 1998.