Anatomy of a Domain of Continuous Random Variables II

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Abstract. In this paper we conclude a two-part analysis of recent work of Jean Goubault-Larrecq and Daniele Varacca, who devised a model of continuous random variables over bounded complete domains. Their presentation leaves out many details, and also misses some motivations for their construction. In this and a related paper we attempt to fill in some of these details, and in the process, we discover a flaw in the model they built.

Our earlier paper showed how to construct $\Theta\mathsf{Prob}(A^\infty)$, the bounded complete algebraic domain of *thin probability measures* over A^∞, the monoid of finite and infinite words over a finite alphabet A. In this second paper, we apply our earlier results to construct $\Theta RV_{A^\infty}(D)$, the bounded complete domain of continuous random variables defined on supports of thin probability measures on A^∞ with values in a bounded complete domain D, and we show $D \mapsto \Theta RV_{A^\infty}(D)$ is the object map of a monad. In the case $A = \{0, 1\}$, our construction yields the domain of continuous random variables over bounded complete domains devised by Goubault-Larrecq and Varacca. However, we also show that the Kleisli extension $h^\dagger\colon \Theta RV_{A^\infty}(D) \to \Theta RV(E)$ of a Scott-continuous map $h\colon D \to E$ is not Scott continuous, so the construction does not yield a monad on BCD, the category of bounded complete domains and Scott-continuous maps. We leave the question of whether the construction can be rescued as an open problem.

Key words: Random variable, bounded complete domain, Cartesian closed category

1 Introduction

Domain theory is fundamental for building computational models. Its use dates to Dana Scott's first models of the untyped lambda calculus, and the applications of domains have now spread well beyond the early focus on programming semantics. One of the seminal advances of the work in semantics was Abramsky's use of Stone Duality [1] to tailor a logic to fit precisely any domain constructed using basic components and adhering to Moggi's monadic approach to building models [21].

While there is a broad range of computational effects that fall under this approach, one monad that has caused continuing problems is the *probabilistic power domain.* First explored by Saheb-Djarhomi [23], the Borel probability measures on an underlying domain can be ordered pointwise as valuation maps from the Scott-open sets to the reals. This forms the object level of a free construction over the category DCPO of directed complete partial orders, but it suffers from two flaws: (i) The probabilistic power domain does not satisfy a distributive law with respect to any of the three nondeterminism monads over domains, so Beck's Theorem [3] implies the composition of the probabilistic power domain and any of the nondeterminism domains is not a monad. Second, there there is no Cartesian closed category of domains – dcpos that satisfy the usual approximation assumption – that is known to be invariant under this construct. The best that is known is that the category of *coherent domains* is invariant under the probabilistic choice monad [15], but this category is not Cartesian closed.

To address the first flaw, Varacca and Winskel [25, 26] explored weakening the laws of probabilistic choice, and discovered three monads for probabilistic choice based on weakened laws: (i) $p \leq p +_r p$; (ii) $p \geq p +_r p$; and finally (iii) no relation assumed between p and $p +_r p$ (where $p +_r q$ denotes choosing p with probability r and choosing q with probability $1 - r$, for $0 \leq r \leq 1$). They called these constructions *indexed valuation monads*, and each of them enjoys a distributive law with respect to the monads for nondeterminism.

This author took this work a bit further, showing in [18] that one could use one of the indexed valuation models to define a monad of finite random variables over either the domain RB or the domain FS, the latter of which is a maximal CCC of domains, and both of which are closed under all three nondeterminism monads. More recently, Goubault-Larrecq and Varacca attempted to extend this line of work to show that there is a monad of *continuous* random variables over the CCC of bounded complete domains [10]. The category BCD of *bounded complete domains* is more general than Scott domains, the objects used by Dana Scott in devising the first model of the untyped lambda calculus [24]. While BCD is a CCC, it is not closed under the convex power domain monad, and it is not a maximal CCC. The work of Goubault-Larrecq and Varacca inspired the work in this paper and in the earlier one on this subject [20].

1.1 The model of Goubault-Larrecq and Varacca

The model of continuous random variables devised by Goubault-Larrecq and Varacca [10] restricts probability measures to one particular domain $\mathcal{C}$, which we call the *Cantor fan*, and models probabilistic choice on an arbitrary domain D as the family of (Scott) continuous maps $f\colon \operatorname{supp}\mu \to D$, where $\mu \in \mathsf{Prob}(\mathcal{C})$. To start, the Cantor fan is the ideal completion $\mathcal{C}$ of the rooted full binary tree, where the latter admits the partial order in which the root is the least element, and node m is below node n iff the path from the root to n passes through m. This makes $\mathcal{C}$ a Scott domain whose space of maximal elements is homeomorphic to the middle-third Cantor subset of the unit interval. In addition to its usual convex structure, the domain $\mathsf{Prob}(\mathcal{C})$ of probability measures over $\mathcal{C}$ admits a binary probabilistic choice operator in the spirit of Varacca's *Hoare indexed valuations*, so that $p \leq p +_r p$ holds for each $p \in \mathcal{C}$. The definition of $+_r$ relies on the concatenation operator on $\mathcal{C}$, regarding $\mathcal{C}$ as the family of finite and infinite words over $\{0,1\}$. Since concatenation $\cdot\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is not monotone, let alone Scott continuous, Goubault-Larrecq and Varacca restrict their model to contain only those measures $\mu \in \mathsf{Prob}(\mathcal{C})$ whose support (in the Lawson topology) is an antichain, since such measures have the property that concatenation defines a Scott continuous operation on the sets on which they are concentrated. So the model is the family

$$\Theta RV(D) = \{(\mu, f) \in \mathsf{Prob}(\mathcal{C}) \times [\operatorname{supp}\mu \to D] \mid \operatorname{supp}\mu \text{ is an antichain}\}.$$

(Here Θ stands for "thin", a term adopted by Goubault-Larrecq and Varacca.) They claim that $\Theta RV(D)$ forms a monad over BCD; the monad laws are displayed explicitly in [10], but the definition of the lift of a Scott-continuous map $\phi\colon D \to RV(E)$ to $\phi^{\#}\colon RV(D) \to RV(E)$ leaves a lot to the reader to unravel. In fact, we have identified a flaw in that claim that can be traced to the concatenation operator.

1.2 Our contribution

In this and a preceding paper [20], we clarify the construction devised by Goubault-Larrecq and Varacca. We use an example from probabilistic automata to motivate the order used by Goubault-Larrecq and Varacca. The paper [20] is devoted to understanding the construction of the thin measures over $\mathcal{C}$; this requires a completely different presentation from the one given in [10]. Goubault-Larrecq and Varacca impose the restriction that the only simple measures – affine combinations of finitely many point masses – in $\Theta RV(D)$ are those supported on antichains, and then define $\Theta RV(D)$ to be the least subset of $\mathsf{Prob}(\mathcal{C}) \times [\operatorname{supp} \mu \to D]$ containing these measures in the first component, and closed under directed suprema; in effect, they are giving a basis for the allowable measures, and capturing the rest by taking directed suprema. The alternative approach in [20] shows their definition is the same thing as defining thin measures in the model to be those that are supported on Lawson-closed antichains, using Stone duality to prove this result. Our results allow one to account for all measures in the model as having the form $\pi_A(\mu)$ where $A \subseteq \mathcal{C}$ is a Lawson-closed subset and μ is a probability measure that is supported on a Lawson-closed subset of $\mathrm{Max}(\mathcal{C})$, the Cantor set of maximal elements of $\mathcal{C}$. We completed this analysis by showing the order arises naturally on probabilistic automata. Our results in this paper and in [20] also are broader than those of [10], since they hold for A^∞ for an arbitrary finite alphabet A, whereas Goubault-Larrecq and Varacca restrict themselves to the case $A = \{0, 1\}$.

In this paper we complete our reconstruction and analysis of the continuous random variable model of Goubault-Larrecq and Varacca. Using the results from [20], we define the model of continuous random variables in a fashion similar to that in [10], but we present the complete structure, rather than having to appeal to a completion within a larger domain. This approach also allows us to examine the construction of constituents of the monad, and in particular, the Kleisli extension, in a more accessible way than is given in [10]. We verify that the monad laws hold, but we also show that the Kleisli extension of a Scott-continuous map is not Scott-continuous – in fact, it's not even monotone. This means that the construction yields a monad, but one that does not leave any category of domains and Scott-continuous maps invariant. We leave open the question whether this approach can be rescued to obtain a monad on BCD or any other category of domains.

1.3 The plan of the paper

In the next section, we review some background material from domain theory and the other areas we need. The latter includes a version of Stone duality, a result about the probability measure monad on the category of compact Hausdorff spaces and continuous maps, as well as some results from [20] on Lawson-closed antichains in A^∞ for a finite alphabet A. Section 3 summarizes the main results from [20], starting with the motivating example that informs the order we use to define our model of thin probability measures. The next section constitutes the main part of the paper, where we develop the family of continuous random variables over thin probability measures on A^∞, for any finite alphabet A. We show this family is a bounded complete domain. We also show our construct defines the object map of a monad, but, as commented above, the Kleisli extension of a Scott-continuous map is not Scott continuous, so it is unclear exactly what the right category for this monad is. Finally, we show that our results are the same as those of Goubault-Larrecq and Varacca in case $A = \{0, 1\}$, which implies the flaw we have detected applies equally to their construction. In Section 5 we summarize our results and pose some questions for future research.

2 Background

In this section we present the background material we need for our main results.

2.1 Domains

The basis for our results rely fundamentally on domain theory. Most of the results that we quote below can be found in [2] or [7]; we give specific references for those that are not found there.

To start, a *poset* is a partially ordered set. Antichains play a major role in our development: a subset $A \subseteq P$ of a poset is an *antichain* if any two distinct elements in A are incomparable in the order.

A poset is *directed complete* if each of its directed subsets has a least upper bound; here a subset S is *directed* if each finite subset of S has an upper bound in S. A directed complete partial order is called a *dcpo*. The relevant maps between dcpos are the monotone maps that also preserve suprema of directed sets; these maps are usually called *Scott continuous*.

These notions can be presented from a purely topological perspective: a subset $U \subseteq P$ of a poset is *Scott open* if (i) $U = \uparrow U \equiv \{x \in P \mid (\exists u \in U)\ u \leq x\}$ is an upper set, and (ii) if $\sup S \in U$ implies $S \cap U \neq \emptyset$ for each directed subset $S \subseteq P$. It is routine to show that the family of Scott-open sets forms a topology on any poset; this topology satisfies $\downarrow x \equiv \{y \in P \mid y \leq x\} = \overline{\{x\}}$ is the closure of a point, so the Scott topology is always T_0, but it is T_1 iff P is a flat poset. A mapping between dcpos is Scott continuous in the order-theoretic sense iff it is a monotone map that is continuous with respect to the Scott topologies on its domain and range. We let DCPO denote the category of dcpos and Scott-continuous maps; DCPO is a Cartesian closed category.

If P is a dcpo, and $x, y \in P$, then x *approximates* y iff for every directed set $S \subseteq P$, if $y \leq \sup S$, then there is some $s \in S$ with $x \leq s$. In this case, we write $x \ll y$ and we let $\mathord{\downarrow\!\!\downarrow} y = \{x \in P \mid x \ll y\}$. A *basis* for a poset P is a family $B \subseteq P$ satisfying $\mathord{\downarrow\!\!\downarrow} y \cap B$ is directed and $y = \sup(\mathord{\downarrow\!\!\downarrow} y \cap B)$ for each $y \in P$. A *continuous poset* is one that has a basis, and a dcpo P is a *domain* if P is a continuous dcpo. An element $k \in P$ is *compact* if $x \ll x$, and P is *algebraic* if $KP = \{k \in P \mid k \ll k\}$ forms a basis. Domains are sober spaces in the Scott topology.

We let DOM denote that category of domains and Scott continuous maps; this is a full subcategory of DCPO, but it is not Cartesian closed. Nevertheless, DOM has several Cartesian closed full subcategories. Two of particular interest to us are the full subcategory SDOM of Scott domains, and BCD its continuous analog. Precisely, a domain is *bounded complete* if every non-empty subset has a greatest lower bound. An equivalent statement to the last condition is that every subset with an upper bound has a least upper bound. Bounded complete domains generalize *Scott domains*, which are algebraic domains for which KP is countable and that also satisfy the property that every non-empty subset has a greatest lower bound. We let BCD denote the category of bounded complete domains and Scott-continuous maps.

Example 1. A prototypical example of a Scott domain is the free monoid $A^\infty = A^* \cup A^\omega$ of finite and infinite words over a finite alphabet A, where we use the prefix order on words: $s \leq t \in A^\infty$ iff $(\exists w \in A^\infty)\, sw = t$. Two words compare iff one is a prefix of the other, and the infimum of any set of words is their longest common prefix. As a domain, $KA^\infty = A^*$, which is countable since A is finite.

Note that this same reasoning applies to any *Scott-closed subset* of A^∞ – examples here are the language of a finite state automaton, where the "alphabet" is the product $S \times \mathit{Act}$ of the set of states and the set of actions.

Domains also have a Hausdorff refinement of the Scott topology which will play a role in our work. The *weak lower topology* on P has the sets of the form if $O = P \setminus \uparrow F$ as a basis, where $F \subset P$ is a finite subset. The *Lawson topology* on a domain P is the common refinement of the Scott- and weak lower topologies on P. This topology has the family

$$\{U \setminus \uparrow F \mid U \text{ Scott open } \& \ F \subseteq P \text{ finite}\}$$

as a basis. The Lawson topology on a domain is always Hausdorff.

A domain is *coherent* if its Lawson topology is compact. We denote the closure of a subset $X \subseteq P$ of a coherent domain in the Lawson topology by $\overline{X}^{\Lambda}$.

Example 2. Bounded complete domains are coherent. A basic example of a bounded complete domain is the unit interval; here $x \ll y$ iff $x = 0$ or $x < y$. The Scott topology on the $[0, 1]$ has open sets $[0, 1]$ together with $\twoheaduparrow x = (x, 1]$ for $x \in (0, 1]$. Since BCD has finite products, $[0, 1]^n$ is a domain in the product order, where $x \ll y$ iff $x_i \ll y_i$ for each i; a basis of Scott-open sets is formed by the sets $\twoheaduparrow x$ for $x \in [0, 1]^n$ (this last is true in any domain).

The Lawson topology on [0,1] has basic open sets $(x, 1] \setminus [y, 1]$ for $x < y$ – i.e., sets of the form (x, y) for $x < y$, which is the usual topology. Then, the Lawson topology on $[0, 1]^n$ is the product topology from the usual topology on $[0, 1]$.

Since $[0, 1]$ has a least element, the same results apply for any power of $[0, 1]$, where $x \ll y$ in $[0, 1]^J$ iff $x_j = 0$ for almost all $j \in J$, and $x_j \ll y_j$ for all $j \in J$. Thus, every power of $[0, 1]$ is a bounded complete domain.

A more interesting example of a coherent domain is $\mathsf{Prob}(D)$, the family of probability measures on a coherent domain D, where $\mu \leq \nu$ iff $\mu(U) \leq \nu(U)$ for every Scott-open subset $U \subseteq D$. For example, $\mathsf{Prob}([0, 1])$ is a coherent domain. In fact, the category COH of coherent domains and Scott continuous maps is closed under the application of the functor Prob [15].

While coherent domains having least elements are closed under arbitrary products, the category COH of coherent domains and Scott continuous maps is not Cartesian closed. There is an inclusion of the category of coherent domains and Lawson continuous monotone maps into the category of compact ordered spaces and continuous monotone maps that is obtained by equipping coherent domains with the Lawson topology. This is right adjoint to the functor that associates to a compact ordered space its family of closed order-convex subsets ordered by reverse inclusion, where $C \ll D$ iff $D \subseteq C^{\circ}$. In this case, the Lawson topology is the topology the family inherits from the Vietoris topology on the family of compact subsets of the underlying space.

Finally, we need some results related to power domains, the convex power domain in particular. Details for the following can be found in [19]. For a coherent domain D, the *convex power domain* consists of the family

$$\mathcal{P}_C(D) = \{X \subseteq D \mid X = {\downarrow} X \cap {\uparrow} X \ \text{ is Lawson closed}\}$$

under the *Egli-Milner order:*

$$X \leq Y \quad \text{iff} \quad X \subseteq {\downarrow}Y \ \& \ Y \subseteq {\uparrow}X.$$

$\mathcal{P}_C(D)$ is a coherent domain if D is one, where

$$X \ll Y \quad \text{iff} \quad Y \subseteq ({\uparrow}X)^\circ \ \& \ (\forall x \in X)(\exists y \in Y)\, x \ll y. \tag{1}$$

3 On Lawson-compact antichains and thin probability measures over A^∞

3.1 Lawson-compact antichains in A^∞

The following results are from Section 2.2 of [20]. They present some results about Lawson-closed sets and Lawson-closed antichains in $AC(A^\infty)$ that we need in developing the model of continuous random variables over the next two sections.

Lemma 1. *If $X \subseteq A^\infty$ is a Lawson compact subset of a coherent domain, then ${\downarrow}X$ is a Scott-closed subset of A^∞. Moreover there is a canonical map $\pi_{\downarrow X} \colon A^\infty \to {\downarrow}X$ that is both Scott- and Lawson continuous.*

Corollary 1. *If $X \subseteq A^\infty$ is a Lawson-compact antichain, then there is a Lawson compact subset $Y \subseteq A^\omega$ (which is necessarily an antichain) for which $\pi_{\downarrow X}(Y) = X$.*

Proposition 1. *Let A be a finite alphabet. Then $X \subseteq A^\infty$ is Scott closed iff $Max\, X$ is Lawson closed and $X = {\downarrow}(Max\, X)$.*

Theorem 1. *Let A be a finite alphabet and consider the domain A^∞ in the prefix order. Let*

$$AC(A^\infty) = \{X \subseteq A^\infty \mid X = \overline{X}^\Lambda \text{ is an antichain}\}.$$

Then $AC(A^\infty)$ is a Scott domain that is a subdomain of $\mathcal{P}_C(A^\infty)$. In particular,

1. *If $X, Y \in AC(A^\infty)$, then $X \leq Y$ iff $\pi_{\downarrow X}(Y) = X$.*
2. *the supremum of two antichains $X, Y \in AC(A^\infty)$ with an upper bound is given by $X \vee Y = Max(X \cup Y)$, the set of maximal elements of their union.*

Proof. The proof that $AC(A^\infty)$ is a sub-dcpo of $\mathcal{P}_C(A^\infty)$ and that $X \vee Y = \mathrm{Max}(X \cup Y)$ is contained in [20], so only the proof of 1) above is lacking.

If $X \leq Y \in AC(A^\infty)$, then $X \subseteq {\downarrow}Y$ and $Y \subseteq {\uparrow}X$. This means that every $x \in X$ is below some $y \in Y$, and vice versa, every $y \in Y$ is above some $x \in X$. Since X is an antichain. if $x \leq y \in Y$, then $\pi_{\downarrow X}(y) = x$, which shows $X \subseteq \pi_{\downarrow X}(Y)$. Conversely, if $y \in Y$, then there is some $x \in X$ with $x \leq y$, so again $\pi_{\downarrow X}(y) = x \in X$. Thus $\pi_{\downarrow X}(Y) = X$.

For the converse, if $\pi_{\downarrow X}(Y) = X$, then $x \in X$ implies $x = \pi_{\downarrow X}(y)$ for some $y \in Y$, so $X \subseteq {\downarrow}Y$. On the other hand, if $y \in Y$, then $\pi_{\downarrow X}(y) \in X$ and $y \in {\uparrow}\pi_{\downarrow X}(y)$. Hence $Y \subseteq {\uparrow}X$. □

Proposition 4.47 of [19] implies that the Lawson topology on $\mathcal{P}_C(A^\infty)$ is the same as the topology $\mathcal{P}_C(A^\infty)$ inherits from the Vietoris topology on the family of compact subsets of A^∞, when A^∞ is a coherent domain endowed with the Lawson topology. This implies that the convergence of a directed family of Lawson compact antichains from A^∞ is the same as their convergence in the Vietoris topology. The relevance of this to our work is summarized in the following result.

Theorem 2. *Let A be a finite set, and for each n, let $\pi_n \colon A^\infty \to A^{\leq n} \equiv \{s \in A^* \mid |s| \leq n\}$ be the projection onto the set of words of length at most n. Then π_n is continuous for each n, where we endow A^∞ and $A^{\leq n}$ with either the Scott- or Lawson topologies. Moreover,*

1. *Each Lawson-compact antichain $X \subseteq A^\infty$ satisfies $\{\pi_n(X)\}_n$ is a directed family of finite antichains satisfying $\sup_n \pi_n(X) = X$.*
2. *Conversely, each directed family of finite antichains $F_n \subseteq A^{\leq n}$ satisfies $\sup_n F_n = X$ is a Lawson compact antichain in A^∞ satisfying $\pi_n(X) = F_n$ for each n.*

Some further results We need some additional results about Lawson-compact antichains in A^∞ which are not in [20].

Proposition 2. *Let $X \in AC(A^\infty)$ be a Lawson-compact antichain in A^∞. Then:*

1. *${\downarrow}X$ is a bounded complete domain.*
2. *The relative Lawson- and Scott topologies on X from ${\downarrow}X$ are the same.*

Proof. For 1, Proposition 1 implies ${\downarrow}X$ is a Scott-closed subset of A^∞, and Scott-closed subset of a bounded complete domain is another such:

if $s \in \downarrow X$, then $\Downarrow s \subseteq \downarrow X$, so $\downarrow X$ is continuous, and if $\emptyset \neq S \subseteq \downarrow X$, then $\inf_{A^\infty} S \in \downarrow X$.

For 2, the Lawson topology refines the Scott topology, so we only need to show that each relatively-open subset of X in the Lawson topology is relatively Scott open. A basic open subset of X in the relative Lawson topology has the form $X \cap (U \setminus \uparrow F)$, where $U \subseteq \downarrow X$ is Scott open, and $F \subseteq \downarrow X$ is finite. In fact, we may assume $U = \uparrow s$ for some finite word $s \in A^*$, since A^∞ is algebraic. Then, for each $t \in F$, if t and s have an upper bound, then they must compare, and assuming $\uparrow s \setminus \uparrow F \neq \emptyset$, we conclude that $s < t$. If $x \in X \cap \uparrow s \setminus \uparrow F$, then $s < x$ and $t \not\leq x$ for all $t \in F$. But then we can find $s' \in A^*$ with $s < s' \leq x$ and $s' \not\leq t$ for all $t \in F$ since F is finite. Then $x \in X \cap \uparrow s' \subseteq \uparrow s \setminus \uparrow F$, and $X \cap \uparrow s'$ is relatively Scott open. □

Remark 1. Let $X \in AC(A^\infty)$ and let D be a bounded complete domain. Then

1. Part 1 implies $[\downarrow X \to D]$ is a bounded complete domain, since BCD is Cartesian closed.
2. Part 2 implies $f\colon X \to D$ is continuous from the relative Scott topology on X to the Scott topology on D iff f is continuous from the Lawson topology on X to the Scott topology on D. We denote the family of these maps by $[X \to D]$.
3. If $X \in AC(A^*)$ is a finite antichain and D is bounded complete, then $[X \to D] \simeq D^{|X|}$ is a bounded complete domain, since BCD is closed under products.

Proposition 3. *Let D be a bounded complete domain, and let $X \in AC(A^\infty)$, where A is a finite alphabet. Then $X \leq Y \in AC(A^\infty)$ implies there is an embedding-projection pair*

$$f \mapsto f \circ \pi_X \colon [X \to D] \hookrightarrow [Y \to D]; \quad g \mapsto \widehat{g}\colon [Y \to D] \twoheadrightarrow [X \to D],$$

where $\pi_X\colon Y \to X$ is the projection mapping and $\widehat{g}(x) = \inf g(Y \cap \uparrow x)$.

Proof. Given $f\colon X \to D$, $f \circ \pi_X \colon Y \to D$ is well-defined because $X \leq Y$, and it is continuous because it is a composition of continuous maps.

On the other hand, given $g\colon Y \to D$, we first recall $\mathcal{P}_U(D) = (\{C \subseteq D \mid \emptyset \neq C \text{ Scott compact}\}, \supseteq)$ denotes the upper power domain over D, and that D bounded complete implies $\inf\colon \mathcal{P}_U(D) \to D$ is a Soctt-continuous retraction (cf. [19]). Then we define $\overline{g}\colon \downarrow Y \to \mathcal{P}_U(D)$ by $\overline{g}(s) = \uparrow_D g(\uparrow s \cap Y)$. This is well-defined since $s \in \downarrow Y$ implies $\uparrow s \cap Y \neq \emptyset$ is Lawson, hence Scott compact, and the Scott continuity of g implies $g(\uparrow s \cap Y)$ is Scott compact as well.

If $s \leq t$, then obviously $g(\uparrow s \cap Y) \supseteq g(\uparrow t \cap Y)$, so $\overline{g}$ is monotone. For continuity, suppose that $S \subseteq \downarrow Y$ is directed, and let $t = \sup S$ in $\downarrow Y$. Then $\overline{g}(s) \supseteq \overline{g}(t)$ by monotonicity. Conversely, suppose $x \in \overline{g}(s)$ for each $s \in S$. Then for each $s \in S$, there is some $y_s \in \uparrow s \cap Y$ with $g(y_s) \leq x$. Since Y is compact, $\{y_s\}_{s \in S}$ has a limit point $y \in Y$, and since $\{\uparrow s \cap Y \mid s \in S\}$ is a filter base of compact sets, it follows that $y \in \uparrow s \cap Y$ for each $s \in S$. Thus $y \in \uparrow t \cap Y$, and then $g(t) \leq x$. It follows that $\overline{g}(t) = \sup_{s \in S} \overline{g}(s)$, so $\overline{g} \colon \downarrow Y \to \mathcal{P}_U(D)$ by $\overline{g}(s) = \uparrow_D g(\uparrow s \cap Y)$ is Scott continuous.

Now, $\inf \colon \mathcal{P}_U(D) \to D$ is Scott continuous, and $X \leq Y$ implies $X \subseteq \downarrow Y$, so $\widehat{g} \colon X \to D$ by $\widehat{g}(x) = \inf \overline{g}(x)$ is Scott continuous.

Now, given $f \colon X \to D$,

$$\widehat{f \circ \pi_X}(x) = \inf f(Y \cap \uparrow x) = \inf f(x) = f(x)$$

since $X \leq Y$ and X an antichain imply $\pi_X(Y \cap \uparrow x) = x$.

Conversely, if $g \colon Y \to D$ and $y \in Y$, then

$$\widehat{g} \circ \pi_X(y) = \inf g(Y \cap \uparrow \pi_X(y)) \leq g(y). \qquad \square$$

Notation:

- In the following, we let $\bigoplus_{X \in AC(A^\infty)} [X \to D]$ denote the disjoint sum of the domains $[X \to D]$, as X ranges over $AC(A^\infty)$.
- Given $f \in \bigoplus_{X \in AC(A^\infty)} [X \to D]$, we let $X_f = \mathrm{dom}(f)$. Then $f \in [X \to D]$ iff $X = X_f$.
- We order $\bigoplus_{X \in AC(A^\infty)} [X \to D]$ by

$$f \leq g \quad \text{iff} \quad X_f \leq X_g \text{ and } f \circ \pi_{X_f} \leq g.$$

Theorem 3. *Let A be a finite alphabet and let D be a bounded complete domain.*

1. *If $X \in AC(A^\infty)$, then $[X \to D]$ is a bounded complete domain.*
2. *$\bigoplus_{X \in AC(A^\infty)} [X \to D]$ is a bounded complete domain.*

Proof. We use Theorems 1 and 2 to prove the results. For 1, Theorem 2 implies $\pi_n(X) \in AC(A^{\leq n})$ is a finite antichain for each $n \geq 1$, and so $[\pi_n(X) \to D] \simeq D^{|\pi_n(X)|}$ is bounded complete, since BCD has products. Moreover, the family $\{[\pi_n(X) \to D], f \mapsto \pi_n^m \circ f, g \mapsto \widehat{g}\}_{m \leq n}$ is a family of bounded complete domains and embeddiing-projection pairs, so it has a bilimit, which is also a bounded complete domain. To complete the proof, we show that $[X \to D]$ is that bilimit. This is proved if we show that $\sup_n \widehat{f} \circ \pi_n = f$ for each $f \in [X \to D]$. If $x \in X$, then

$$\sup_n \widehat{f} \circ \pi_n(x) = \sup_n \inf f(X \cap {\uparrow}\pi_n(x)) = \inf f(X \cap {\uparrow}x) = f(x),$$

the second equality following from part 1) of Theorem 2, and the last from the fact that $x \in X \in AC(A^\infty)$.

For part 2), we first show that

$$f \leq g \quad \text{iff} \quad X_f \leq X_g \text{ and } f \circ \pi_{X_f} \leq g$$

is a partial order on $\bigoplus_{X \in AC(A^\infty)}[X \to D]$: indeed, it's clearly reflexive and transitive. If $f \leq g \leq f$, then $X_f \leq X_g \leq X_f$, and so $X_f = X_g$ because $AC(A^\infty)$ is partially ordered. Then $\pi_{X_f}|_{X_g} = \mathrm{id}_{X_f}$. Thus $f = f \circ \pi_{X_f} \leq g = g \circ \pi_{X_g} \leq f$, and they're equal.

Next, let $\mathcal{S} \subseteq \bigoplus_{X \in AC(A^\infty)}[X \to D]$ be a a directed set. Then $\mathcal{S}_0 = \{X_f \mid f \in \mathcal{S}\}$ is a directed family in $AC(A^\infty)$, so it has a least upper bound, $X_0 = \sup_{\mathcal{S}_0} X$. Then $\{[X_f \to D] \mid f \in \mathcal{S}\}$ together with $[X_0 \to D]$ is a cone in $\bigoplus_{X \in AC(A^\infty)}[X \to D]$, using the embedding-projection pairs between $[X_f \to D]$ and $[X_g \to D]$ if $f \leq g \in \mathcal{S}$, and between $[X_f \to D]$ and $[X_0 \to D]$ for each $X_f \in \mathcal{S}_0$. An argument similar to the one in the first part of the proof shows that $\sup_{X \in \mathcal{S}_0} \widehat{f} \circ \pi_X = f$ for each $f \in [X_0 \to D]$, which implies this is a limit cone. This implies that $[X_0 \to D] = \lim_{f \in \mathcal{S}}[X_f \to D]$. Then $(f)_{f \in \mathcal{S}} \in \Pi_{f \in \mathcal{S}}[X_f \to D]$ determines a unique point $h \in \lim_{f \in \mathcal{S}}[X_f \to D] = [X_0 \to D]$. Thus, $\pi_{X_f}(h) = f$ for each $f \in \mathcal{S}$, so $f \leq h$ for each $f \in S$. Likewise, if $f \leq g$ for each $f \in \mathcal{S}$, then $\pi_{X_f}(g) = f$ for each $f \in \mathcal{S}$, and so $\pi_{X_0}(g) = h$ by the definition of the limit. Hence $h = \sup \mathcal{S}$. So, $\bigoplus_{X \in AC(A^\infty)}[X \to D]$ is a dcpo.

Since $[X \to D]$ is a domain for each $X \in AC(A^\infty)$, the same is true of $\bigoplus_{X \in AC(A^\infty)}[X \to D]$ – a basis is the family $\bigoplus_{X \in AC(A^*)}[X \to \mathcal{B}(D)]$, where $\mathcal{B}(D)$ is any basis for D. And since $[X \to D]$ is bounded complete for each $X \in AC(A^\infty)$ and since $AC(A^\infty)$ itself is bounded complete, the same holds for $\bigoplus_{X \in AC(A^\infty)}[X \to D]$. □

Notation. For a bounded complete domain D, we use $\Theta[A^\infty \to D] \equiv \bigoplus_{X \in AC(A^\infty)}[X \to D]$ to denote the family of Lawson continuous maps from some Lawson-compact antichain $X \in AC(A^\infty)$ to D.

Stone duality In modern parlance, Marshall Stone's seminal result states that the category of Stone spaces – compact Hausdorff totally disconnected spaces – and continuous maps is dually equivalent to the category of Boolean algebras and Boolean algebra maps. The dual equivalence sends a Stone space to the Boolean algebra of its compact-open subsets; dually, a Boolean algebra is sent to the set of prime ideals, endowed with the hull-kernel topology. This dual equivalence was used to great effect by Abramsky [1] where he showed how to extract a logic from a domain constructed using Moggi's monadic approach, so that the logic was tailor made for the domain used to build it.

Our approach to Stone duality is somewhat unconventional, but one that also has been utilized in recent work by Gehrke [8, 9]. The idea is to realize a Stone space as a projective limit of finite spaces, a result which follows from Stone duality, as we now demonstrate.

Theorem 4 (Stone Duality). *Each Stone space X can be represented as a projective limit $X \simeq \varprojlim_{\alpha\in A} X_\alpha$, where X_α is a finite space. In fact, each X_α is a partition of X into a finite cover by clopen subsets, and the projection $X \twoheadrightarrow X_\alpha$ maps each point of X to the element of X_α containing it.*

We note that a corollary of this result says that it is enough to have a basis for the family of finite Boolean subalgebras of $\mathcal{B}(X)$ in order to realize X as a projective limit of finite spaces, where by a *basis*, we mean a directed family whose union generates all of $\mathcal{B}(X)$. The following example illustrates this point.

Example 3. Let C denote the middle third Cantor set from the unit interval. This is Stone space, and so it can be realized as a projective limit of finite spaces $C \simeq \varprojlim_{\alpha\in A} C_\alpha$. But since C is second countable, we can define a countable family of finite spaces C_n for which $C \simeq \varprojlim_n C_n$. Indeed, we can use the construction of C from $[0, 1]$ to define these finite spaces:

- $C_0 = [0, 1]$ is the entire space.
- $C_1 = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$ is the result of deleting the middle third from $[0, 1]$.

  $\vdots$
- $C_n = \{[0, \frac{1}{3^n}], \ldots, [\frac{3^n-1}{3^n}, 1]\}$.

  $\vdots$

Note that C_n has 2^n elements – this is the "top down" approach to building C, as opposed the "bottom up" approach obtained by viewing C as the set of maximal elements of the Cantor fan. In categorical parlance, the approach via Stone duality realizes C as an F-algebra, whereas the

Cantor fan realizes C as a (final) F-coalgebra, where F is the functor that sends a space X to $X \dot{\cup} X$, the disjoint sum of two copies of X.

The Prob monad on Comp It is well known that the family of probability measures on a compact Hausdorff space is the object level of a functor which defines a monad on Comp, the category of compact Hausdorff spaces and continuous maps. As outlined in [11], this monad gives rise to several related monads:

- On Comp, it associates to a compact Hausdorff space X the free *barycentric algebra* over X, the name deriving from the counit $\epsilon\colon \mathsf{Prob}(S) \to S$ which assigns to each measure μ on a probabilistic algebra S its barycenter $\epsilon(\mu)$.
- On the category $\mathsf{CompMon}$ of compact monoids and continuous monoid homomorphisms, Prob gives rise to a monad that assigns to a compact monoid S the free compact affine monoid over S.
- On the category $\mathsf{CompGrp}$ of compact groups and continuous homomorphisms, Prob assigns to a compact group G the free compact affine monoid over G; in this case the right adjoint sends a compact affine monoid to its group of units, as opposed to the inclusion functor, which is the right adjoint in the first two cases.

If we let $\mathsf{SProb}(X)$ denote the family of subprobability measures on a compact Hausdorff space X, then it's routine to show that SProb defines monads in each of the cases just described, where the only change is that the objects now have a 0 (i.e., they are affine structures with 0-element, allowing one to define scalar multiples $r \cdot x$ for $r \in [0,1]$ and $x \in \mathsf{SProb}(X)$, as well as affine combinations).

There is a further result we need about Prob which relates to its role as an endofunctor on Comp and its subcategories. The following result is due to Fedorchuk:

Theorem 5 (Fedorchuk [5]). *The functor $\mathsf{Prob}\colon \mathsf{Comp} \to \mathsf{Comp}$ is normal; in particular, Prob preserves inverse limits.*

Remark 2. If we combine this result with the results at the end of Subsection 2.1, then we see that the family of probability measures supported on a Lawson-compact antichain X in A^∞ can be written as the inverse limit of the measures supported on finite subsets $\pi_n(X)$; this follows from our having shown that $X = \sup_n \pi_n(X)$ and the fact (quoted from [19]) that the Lawson topology on the family of antichains is the same as the Vietoris topology, which coincides with the topology used to form the inverse limit.

3.2 A Motivating Example

The following example is from Section 3 of [20].

Definition 1. *A* probabilistic automaton *is a tuple* (S, A, q_0, D) *where* S *is a finite set of states,* A *a finite set of actions,* $q_0 \in S$ *a start state, and* $D \subseteq S \times \mathsf{Prob}(A \times S)$ *a transition relation that assigns to each state* s_0 *a probability distribution* $\sum_{A\times S} r_{(s_0,(a,s))}\delta_{(a,s)}$ *on* $A \times S$.

If we start such an automaton in its start state – which amounts to assigning it the starting distribution δ_{q_0}, and then follow the automaton as it evolves, then we see a sequence of *global trace distributions* that describe the step-by-step evolution of the automaton:

1. δ_{q_0},
2. $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}\delta_{q_0 a_1 s_1}$,
3. $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}(\sum_{(a_2,s_2)\in A\times S} r_{(s_1,(a_2,s_2))}\delta_{q_0 a_1 s_1 a_2 s_2})$,

   $\vdots$

If we strip away the probabilities, we have a nondeterministic finite state automaton (albeit one without final states), and the resulting automaton generates a language that is a subset of $(S \times A)^\infty$. This automaton generates the sequence

$$\{q_0\}, \{(q_0 s_1 a_1 \mid r_{(q_0,(s_1,a_1))} \neq 0\}, \{q_0 s_1 a_1 s_2 a_2 \mid r_{(q_0,(s_1,a_1))}, r_{(s_1,(a_2,s_2))} \neq 0\}, \ldots.$$

Note that the sequence of sets of states this automaton generates is a family of finite antichains, which we showed in Section 2 is a Scott subdomain of $\mathcal{P}_C((S \times A)^\infty)$ under the Egli-Milner order. Moreover, the projections $\pi_{mn}\colon (S \times A)^{\leq n} \to (S \times A)^{\leq m}$ for $m \leq n$ map the antichain of possible states at the n^{th} stage to those at the m^{th} stage, by truncation.

Since Prob is a monad on Comp, the mappings π_{mn} lift to mappings $\mathsf{Prob}(\pi_{mn})\colon \mathsf{Prob}((S \times A)^{\leq n}) \to \mathsf{Prob}((S \times A)^{\leq m})$. Using the mappings π_{mm+1}, we see that each succeeding distribution is projected onto the previous distribution. For example, the second distribution $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}\delta_{q_0 a_1 s_1}$ collapses to δ_{q_0}, and the third distribution $\sum_{(a_1,s_1)\in A\times S} r_{(q_0,(a_1,s_1))}(\sum_{(a_2,s_2)\in A\times S} r_{(s_1,(a_2,s_2))}\delta_{q_0 a_1 s_1 a_2 s_2})$ collapses to the second. Thus, Prob lifts the order on $AC((S \times A)^\infty)$ to $\mathsf{Prob}(AC((S\times A)^\infty))$, and it is this order we will use in defining the order on the family of thin probability measures, and eventually on the domain of continuous random variables over a bounded complete domain. We now make this observation precise.

3.3 A Bounded Complete Domain of Thin Measures

The following form the main results from [20]; they appear in Sections 4 and 5.

Definition 2. *If Y is a compact Hausdorff space and $X \subseteq Y$ is a compact subspace of Y, then for $\mu \in \mathsf{Prob}(Y)$, then we say μ has* full support on X *if* $\operatorname{supp}\mu = X$. *We denote by $\mathsf{Prob}^\dagger(X)$ the family of $\mu \in \mathsf{Prob}(Y)$ having full support on X.*

Definition 3. *For a finite alphabet A, we define $\Theta\mathsf{Prob}(A^\infty) \equiv \bigoplus_{X\in AC(A^\infty)} \mathsf{Prob}^\dagger(X)$ to be the direct sum of the family of probability measures in $\mathsf{Prob}^\dagger(X)$ as X ranges over $AC(A^\infty)$. These are the* thin probability measures *on A^∞, those that are fully supported on Lawson-compact antichains in A^∞. We order $\Theta\mathsf{Prob}(A^\infty)$ by $\mu \leq \nu$ iff $\pi_{\downarrow(\operatorname{supp}\mu)}(\nu) = \mu$.*

The result summarizes a series of results from [20] about the structure of $\Theta\mathsf{Prob}(A^*)$.

Proposition 4. *Let A be a finite alphabet and let $AC(A^\infty)$ be the family of Lawson-compact antichains in A^∞. Then:*

1. *If $f\colon X \to Y$ is a continuous map between compacta, then $f(\mu) = \nu$ implies $f(\operatorname{supp}\mu) = \operatorname{supp}\nu$.*
2. *The family $(\Theta\mathsf{Prob}(A^\infty), \leq)$ is a dcpo.*
3. *The mapping* $\operatorname{supp}\colon \Theta\mathsf{Prob}(A^\infty) \to AC(A^\infty)$ *sending each measure μ to its support in the Lawson topology is Scott continuous.*
4. *If $\mu \in \Theta\mathsf{Prob}(A^\infty)$ and $F \subseteq A^*$ is finite with $\pi_F(\operatorname{supp}\mu) = F$, then $\pi_F(\mu) \ll \mu$ in $\mathsf{Prob}^\dagger(X)$.*

Theorem 6. *If A is a finite alphabet, then $\Theta\mathsf{Prob}(A^\infty)$ is a bounded complete algebraic domain.*

n-ary Probabilistic Choice Algebras In [10], the authors define *coin algebras* as domains P that have a continuous operation $+\colon [0,1] \times P \times P \to P$ satisfying $x \leq x +_p x$ and $x +_1 y$ and $x +_0 y$ are independent of their second and first arguments, respectively. They also show that their family of continuous random variables over a domain X are free coin algebras. We now define a similar class of algebras and prove a similar freeness result.

Definition 4. *For $n > 0$, let $\Delta_n = \{(r_1, \ldots, r_n) \in [0,1]^n \mid \sum_i r_i = 1\}$. An n-*ary probabilistic algebra *is a domain P that supports an operation $+_n \colon \Delta_n \times P^n \to P$ satisfying the properties:*

1. *$+_n((r_1, \ldots, r_n), (p_1, \ldots, p_n)) \equiv \sum_{i \leq n} r_i p_i \colon \Delta_n \times P^n \to P$ is Scott continuous, and*
2. *For each $i \leq n$, if $(r_1, \ldots, r_n) \in \Delta_n$ and $r_i = 0$, then $(p_1, \ldots, p_n) \mapsto \sum_{j \leq n} r_j p_j$ is independent of its i^{th} input.*

For $A = \{a_1, \ldots, a_n\}$, define $+_n$ on $\Delta_n \times \bigoplus_{X \in AC(A^*)} \mathsf{Prob}^\dagger(X)$ as follows:

- Given $\mu_1, \ldots, \mu_n \in \bigoplus_{X \in AC(A^*)} \mathsf{Prob}^\dagger(X)$, let $S = \mathrm{Max}(\bigcup_{i \leq n} \mathrm{supp}\, \mu_i)$, and for $x \in \mathrm{supp}\, \mu_i$, let $S(x) = \uparrow x \cap S$.
- If $\mu_i = \sum_{x \in \mathrm{supp}\, \mu_i} r_x \delta_x$, then define $\phi^S_{a_i}(\mu_i) = \sum_{x \in \mathrm{supp}\, \mu_i} \frac{r_x}{|S(x)|} \sum_{y \in S(x)} \delta_{y a_i}$.
- Then define

$$+_n \,:\, \Delta_n \times \bigoplus_{X \in AC(A^*)} \mathsf{Prob}^\dagger(X) \to \bigoplus_{X \in AC(A^*)} \mathsf{Prob}^\dagger(X) \quad \text{by}$$
$$+_n((r_1, \ldots, r_n), (\mu_1, \ldots, \mu_n)) = \sum_{i \leq n} r_i \phi^S_{a_i}(\mu_i).$$

Proposition 5. *If $A = \{a, \ldots, a_n\}$ is a finite alphabet, then $\Theta\mathsf{Prob}(A^\infty)$ is an n-ary probabilistic algebra under the continuous extension of the operation given above to all of $\Theta\mathsf{Prob}(A^\infty)$.*

Theorem 7. *If P is an n-ary probabilistic algebra and A is a finite alphabet with $|A| = n$, then given any monotone map $f \colon A^\infty \to P$, there is a unique continuous map $F \colon \Theta\mathsf{Prob}(A^\infty) \to P$ satisfying $F(\sum_{i \leq n} r_i \mu_i) = \sum_{i \leq n} r_i f(\mu_i)$.*

4 Continuous random variables

Recall that a *random variable* is a measurable function $f \colon (X, \Sigma_X) \to (Y, \Sigma_Y)$, where Σ_X and Σ_Y are σ-algebras on X and Y, respectively, where f is measurable iff $f^{-1}(A) \in \Sigma_X$ for each $A \in \Sigma_Y$. If X and Y have topologies that are used to generate Σ_X and Σ_Y, then these algebras are called *Borel σ-algebras*. We are interested in the case that X and Y arise from coherent domains, and the Σ_X and Σ_Y are the Borel algebras generated by the Scott topologies. We note that these are the same as the Borel algebras generated by the Lawson topologies.

If $f\colon X \to Y$ is continuous with respect to topologies on X and Y, respectively, and if the σ-algebras Σ_X and Σ_Y are the Borel algebras for these topologies, then f is measurable. In our setting, the topologies will either be the Scott- or Lawson topologies X and Y inherit from their ambient domains, but the σ-algebras they generate are the same.

Definition 5. *Let A be a finite alphabet, and let $X \in AC(A^\infty)$ be a Lawson-compact antichain. If D is a bounded complete domain, we let $[X \to D] = \{f\colon X \to D \mid f$ Lawson continuous$\}$, where we endow X with the Lawson topology inherited from A^∞ and D with its Scott topology. We let*

$$\Theta RV_{A^\infty}(D) = \bigoplus_{X\in AC(A^\infty)} \mathsf{Prob}^\dagger(X) \times [X \to D]$$

endowed with the partial order

$$(\mu, f) \leq (\nu, g) \quad \textit{iff} \quad \pi_X(\nu) = \mu \;\&\; f \circ \pi_X|_{\operatorname{supp}\nu} \leq g.$$

Theorem 8. *If A is a finite alphabet and D is a bounded complete domain, then $\Theta RV_{A^\infty}(D)$ is a bounded complete domain where*

$$\begin{aligned}(\mu, f) \ll (\nu, g) \quad \textit{iff} \quad & \mu \leq \nu, \operatorname{supp}\mu \subseteq A^* \textit{ finite, and} \\ & f \circ \pi_{\operatorname{supp}\mu}(x) \ll g(x) \ (\forall x \in \operatorname{supp}\nu).\end{aligned}$$

Proof. (Sketch) We know from Theorem 6 that $\Theta\mathsf{Prob}(A^*)$ is a bounded complete algebraic domain in the indicated order, and Proposition 4 shows that $\mu \ll \nu$ iff $\pi_F(\mu) \ll \mu$ for each $\mu \in \mathsf{Prob}^\dagger(X)$, for each $X \in AC(A^\infty)$. Further, Theorerm 3 implies $\Theta[A^\infty \to D]$ is bounded complete with $[X \to D] \leq [Y \to D]$ iff $X \leq Y \in AC(A^\infty)$. Then the product $\Theta\mathsf{Prob}(A^*) \times \Theta[A^\infty \to D]$ is bounded complete. Thus, a directed set $\mathcal{S} \subseteq \Theta RV_{A^\infty}(D)$ has a supremum in $\Theta\mathsf{Prob}(A^*) \times \Theta[A^\infty \to D]$ of the form (μ, f) where $\operatorname{supp}\mu = X$ and $f \in [X \to D]$ by the proof of Theorem 3, so $(\mu, f) \in \Theta RV_{A^\infty}(D)$, showing $\Theta RV_{A^\infty}(D)$ is directed complete. The facts that $\Theta\mathsf{Prob}(A^*) \times \Theta[A^\infty \to D]$, as well as each of its factors are bounded complete domains imply the same is true of the family $\Theta RV_{A^\infty}(D)$. □

4.1 Adding structure to $\Theta RV_{A^\infty}(D)$

We want to show that $\Theta RV_{A^\infty}(D)$ is the object level of a monad, but to do that, we need some algebraic structure on this family. We start by noting that, for a finite alphabet A, the concatenation operation $\cdot\colon A^\infty \times A^\infty \to$

A^∞ is continuous with respect to the Lawson topology; in fact, $(A^\infty, \cdot)$ is the free compact monoid over A with this topology (this is an easy exercise, beginning with the observation that $\{s\}$ is open in the Lawson topology for any finite word s, since A is finite, and using the fact that concatenation is monotone in the second argument). But concatenation is not monotone: $s \leq t$ does not imply $s \cdot w \leq t \cdot w$. A way around this is to avoid words that compare – this is the reason we have been focusing on measures supported on Lawson-compact antichains, since concatenation is monotone on such subsets.

Next, we can apply the probability monad $\mathsf{Prob}\colon \mathsf{CompMon} \to \mathsf{CompMon}$ on compact Hausdorff monoids, and concatenation lifts to *convolution of measures:* $(\mu, \nu) \mapsto \mu * \nu\colon \Theta\mathsf{Prob}(A^\infty) \times \Theta\mathsf{Prob}(A^\infty) \to \Theta\mathsf{Prob}(A^\infty)$ which makes $(\Theta\mathsf{Prob}(A^\infty), *)$ a compact monoid (the identity is $\delta_{\langle\rangle}$, point mass over the empty word):

Proposition 6. *Let A be a finite alphabet, then convolution $*\colon \Theta\mathsf{Prob}(A^\infty) \times \Theta\mathsf{Prob}(A^\infty) \to \Theta\mathsf{Prob}(A^\infty)$ is Lawson continuous.*

Proof. Since the support of each measure is an antichain, and since convolution is Lawson continuous, it also is monotone. Thus, the only issue is whether $\mu * \nu$ is supported on a Lawson-compact antichain if μ and ν are. But from [11], we know that $\operatorname{supp} \mu * \nu = \operatorname{supp} \mu \cdot \operatorname{supp} \nu$, where we are extending the concatenation operation to subsets of A^∞. On any compact monoid, this is a well-defined, continuous operation, and if $\operatorname{supp} \mu$ and $\operatorname{supp} \nu$ are antichains, then so is $\operatorname{supp} \mu \cdot \operatorname{supp} \nu$: if $x, x' \in \operatorname{supp} \mu$ and $y, y' \in \operatorname{supp} \nu$, then x and x' are incomparable, and so are y and y'. But then $x \cdot y$ is incomparable with $x' \cdot y'$: if $x \cdot y \leq x' \cdot y'$, then $x \leq x' \cdot y'$. Since $x \not\leq x'$, this means there is some w with $x' \cdot w = x$, which implies $x' \leq x$, a contradiction. □

Example 4. Since convolution is Lawson continuous, it might be tempting to assume that it is also monotone, and hence Scott continuous when restricted to antichains. This is not the case. For example, if $s, t \in A^*$ satisfy $s < t$, and if we choose $u \in A^*$ with $su \not\leq tu$, then we have an example where concatenation $\cdot\colon A^\infty \times A^\infty \to A^\infty$ is not monotone – namely, at $(s, u) \leq (t, u) \in A^\infty \times A^\infty$. This example lifts to $*\colon \Theta\mathsf{Prob}(A^\infty) \times \Theta\mathsf{Prob}(A^\infty) \to \Theta\mathsf{Prob}(A^\infty)$ via $*(\delta_s, \delta_u) = \delta_{su}$ and $*(\delta_t, \delta_u) = \delta_{tu}$.

We will revisit this example when examine the nature of the monad structure on $\Theta RV_{A^\infty}(D)$ for a bounded complete domain D in Example 5.

For the next result, we recall the notation used in Proposition 5. If $\mu_1, \ldots, \mu_n \in \Theta\mathsf{Prob}(A^*)$, then

- $S = \mathrm{Max}(\bigcup_{i\leq n} \operatorname{supp}\mu_i)$, let $S_i = \uparrow\operatorname{supp}\mu_i \cap S$, and for $x \in \operatorname{supp}\mu_i$, let $S(x) = \uparrow x \cap S$.
- $\phi^S_{a_i}(\mu_i) = \sum_{x\in\operatorname{supp}\mu_i} \frac{\mu_i(x)}{|S(x)|} \sum_{y\in S(x)} \delta_{ya_i}$.

Theorem 9. *Let $A = \{a_1, \cdots, a_n\}$ be a finite alphabet, and let D be a bounded complete domain. Then $\Theta RV_{A^\infty}(D)$ is an n-ary probabilistic algebra where*

$$\sum_{i\leq n} r_i(\mu_i, f_i) = \left(\sum_i r_i\phi^S_{a_i}(\mu_i), \dot{\bigcup}_{i\leq n} f_i \circ \pi_{\operatorname{supp}\mu_i}|_{S_i}\right).$$

Proof. From Proposition 5 we know $\Theta\mathsf{Prob}(A^*)$ is an n-ary probabilistic algebra using the definition above for the first component. The proof of Proposition 6 shows that the concatenation of antichains is an antichain. In particular, if $X_1, \ldots, X_n \in AC(A^\infty)$, then $\phi^S_{a_1}(X_1), \ldots, \phi^S_{a_n}(X_n)$ is a family of pairwise disjoint antichains by construction. This implies the function $\dot{\bigcup}_i f_i \circ \pi_{\operatorname{supp}\mu_i}|_{S_i} \colon \bigcup_i \phi_i(X_i) \to D$ is well-defined and it's continuous because the f_i's and the $\pi_{\operatorname{supp}\mu_i}|_{S_i}$'s are. The proof of the rest is routine. $\square$

4.2 Towards a monad

Following the development in [10], the results we have established allow us to show that $\Theta RV_{A^\infty}(D)$ is the object map of a monad.

Theorem 10. *If A is a finite alphabet, the $D \mapsto \Theta RV_{A^\infty}(D)$ is the object map of a monad.*

Proof. We define the unit of the monad by $\eta_D \colon D \to \Theta RV_{A^\infty}(D)$ by $\eta_D(x) = (\delta_{\langle\rangle}, \chi_x)$, where $\chi_x(\langle\rangle) = x$, and $\langle\rangle$ denotes the empty word.

For $h \colon D \to \Theta RV_{A^\infty}(E)$ with E a bounded complete domain, the definition of $h^\dagger \colon \Theta RV_{A^\infty}(D) \to \Theta RV_{A^\infty}(E)$ is more complicated. We define $h^\dagger$ on the basis $(\sum_{i\leq n} r_i\delta_{s_i}, f)$, where $\{s_i \mid i \leq n\} \subseteq A^*$ is a finite antichain and $f \colon \{s_i \mid i \leq n\} \to \mathcal{B}(D)$, a basis for D, and then extend by continuity.

We begin by noting that $h \colon D \to \Theta RV_{A^\infty}(E)$ means $h(x) = (\mu_x, f_x)$, so using π_1 and π_2 to denote the obvious projections to $\bigoplus_{X\in AC(A^\infty}(X)$ and to $[X \to E]$, respectively, we can write $h(x) = (\pi_1 \circ h(x), \pi_2 \circ h(x))$. Then we can define the mapping $h^\dagger \colon \Theta RV_{A^*}(D) \to \Theta RV_{A^\infty}(E)$ by

$$h^\dagger(\mu, f) = h^\dagger(\sum_{x\in\operatorname{supp}\mu} r_x\delta_x, f) = \left(\sum_{x\in\operatorname{supp}\mu} r_x \left(\delta_x * (\pi_1 \circ h \circ f)(x)\right), g\right),$$

where $*$ denotes convolution and $g\colon \bigcup_{x\in \operatorname{supp}\mu} x\cdot \operatorname{supp}(\pi_1\circ h\circ f)(x)\to E$ is $g(x\cdot y)=(\pi_2\circ h\circ f)(x)(y)$; this makes sense because $x\in\operatorname{supp}\mu$ implies $f(x)\in D$, which in turn implies $(\pi_2\circ h\circ f)(x)\in[\operatorname{supp}(\pi_1\circ h\circ f)(x)\to E]$, and $y\in\operatorname{supp}(\pi_1\circ h\circ f)(x)$.

Note that $(\pi_1\circ h\circ f)(x)$ is a thin probability measure on A^∞, so its support is an antichain. It follows from Proposition 6 that $\operatorname{supp}\delta_x * (\pi_1\circ h\circ f)(x)$ is an antichain for each $x\in\operatorname{supp}\mu$, and since $\operatorname{supp}\mu$ is an antichain, it follows that $\sum_{x\in\operatorname{supp}\mu} r_x\,(\delta_x*(\pi_1\circ h\circ f)(x))$ is one as well. Hence $\pi_1(h^\dagger(\mu,f))$ is a thin probability measure on A^∞.

By definition $(\pi_2\circ h\circ f)(x)\in[\operatorname{supp}(\pi_1\circ h\circ f)(x)\to E]$ is continuous, and since $\bigcup_{x\in\operatorname{supp}\mu} x\cdot\operatorname{supp}(\pi_1\circ h\circ f)(x)$ is a union of pairwise disjont compact antichains in A^∞, it follows that $\pi_2(h^\dagger(\mu,f))=g\colon \bigcup_{x\in\operatorname{supp}\mu} x\cdot \operatorname{supp}(\pi_1\circ h\circ f)(x)\to E$ is continuous.

We now prove $h\mapsto h^\dagger$ satisfies the monad laws:

$\eta_D^\dagger=\mathrm{id}_{\Theta RV_{A^\infty}(D)}$:

$$\begin{aligned}\eta_D^\dagger(\mu,f)&=(\sum_{x\in\operatorname{supp}\mu} r_x(\delta_x*(\pi_1\circ\eta_D\circ f)),(\pi_2\circ\eta_D\circ f))\\&=(\sum_{x\in\operatorname{supp}\mu} r_x(\delta_x*(\delta\langle\,\rangle)),\chi_{f(x)})=(\mu,f).\end{aligned}$$

$h^\dagger\circ\eta_D=h$:

$$\begin{aligned}h^\dagger\circ\eta_D(x)&=h^\dagger(\delta_{\langle\rangle},\chi_x)=(\delta_{\langle\rangle}*(\pi_1\circ h\circ\chi_x),\pi_2\circ\chi_x)\\&=((\pi_1\circ h)(x),(\pi_2\circ h)(x))=h(x)\end{aligned}$$

$k^\dagger\circ h^\dagger=(k^\dagger\circ h)^\dagger$: We assume $k\colon E\to\Theta RV_{A^\infty}(F)$. Then

$$\begin{aligned}k^\dagger\circ h^\dagger(\mu,f)&=k^\dagger(\sum_{x\in\operatorname{supp}\mu} r_x\left(\delta_x*(\pi_1\circ h\circ f)(x)\right),(\pi_2\circ h\circ f)(x)))\\&=k^\dagger(\sum_{x\in\operatorname{supp}\mu} r_x(\delta_x*(\mu_{h\circ f)(x)}),g_{(h\circ f)(x)})\end{aligned}$$

Assuming $\mu_{(h\circ f)(x)}=\sum_{y\in\operatorname{supp}\mu_{(h\circ f)(x)}} s_y\delta_y$, we can rewrite this as

$$\begin{aligned}k^\dagger\circ h^\dagger(\mu,f)&=k^\dagger(\sum_{x\in\operatorname{supp}\mu} r_x(\delta_x*(\sum_{y\in\operatorname{supp}\mu_{h\circ f)(x)}} s_y\delta_y)),g_{(h\circ f)(x)})\\=(\sum_{x\in\operatorname{supp}\mu}\sum_{y\in\operatorname{supp}\mu_{(h\circ f)(x)}} &r_xs_y\delta_x*\delta_y*(\pi_1\circ k\circ g_{(h\circ f)(x)}(y)),\\&\pi_2\circ k\circ g_{(h\circ f)(x)}(y))\end{aligned}\tag{2}$$

where throughout we rewrite $\pi_1 \circ k \circ g_{(h\circ f)(x)} = \mu_{k\circ g_{(h\circ f)(x)}}$ and $\pi_2 \circ k \circ g_{(h\circ f)(x)} = g_{k\circ g_{(h\circ f)(x)}}$.

Starting on the other end, we find

$$
\begin{aligned}
(k^\dagger \circ h)^\dagger(\mu, f) &= (\sum_{x\in \operatorname{supp}\mu} r_x(\delta_x * (\pi_1 \circ k^\dagger \circ h \circ f)(x)), (\pi_2 \circ k^\dagger \circ h \circ f)(x)) \\
&= (\sum_{x\in \operatorname{supp}\mu} r_x(\delta_x * \mu_{k^\dagger((h\circ f)(x))}), g_{k^\dagger((h\circ f)(x))}). \qquad (3)
\end{aligned}
$$

Now,

$$
\begin{aligned}
k^\dagger((h\circ f)(x)) &= k^\dagger(\mu_{(h\circ f)(x)}, g_{(h\circ f)(x)}) = k^\dagger(\sum_{y\in \operatorname{supp}\mu_{(h\circ f)(x)}} s_y\delta_y, g_{(h\circ f)(x)}) \\
&= (\sum_{y\in \operatorname{supp}\mu_{(h\circ f)(x)}} s_y\delta_y * (\pi_1 \circ k \circ g_{(h\circ f)(x)}(y)), \pi_2 \circ k \circ g_{(h\circ f)(x)}(y))
\end{aligned}
$$

Substituting this last in Equation 2 then yields Equation 3, which proves the result. □

Example 5. The observant reader will have noticed two things: first, we haven't said on what category the construction $D \mapsto \Theta RV_{A^\infty}(D)$ forms a monad, and second, we haven't shown that the Kleisli extension $h^\dagger$ is Scott continuous. The fact is that the second is not true, as we now demonstrate, and this implies that the construction is not a monad on any category of domains and Scott continuous maps.

Consider two elements $s, t \in A^*$ from Example 4 with $s < t$ and the element $u \in A^*$ with $su \not\leq tu$. Then $\delta_s < \delta_t \in \Theta\mathsf{Prob}(A^\infty)$. We take $D = A^\infty$, and let $h\colon A^\infty \to \Theta RV_{A^\infty}(A^\infty)$ by $h(w) = (\delta_w, \iota_w)$, where $\iota_x\colon \{x\} \to A^\infty$ is the inclusion of $x \in A^\infty$ into A^∞. Finally, let $f\colon \{s\} \to A^\infty$ and $g\colon \{t\} \to A^\infty$ satisfy $f(s) = g(t) = u$. Then, $(\delta_s, f) \leq (\delta_t, g) \in \Theta RV_{A^\infty}(A^\infty)$ and our definition of $h^\dagger$ implies

$$\pi_1 \circ h^\dagger(\delta_s, f) = \delta_{su} \quad \text{and} \quad \pi_1 \circ h^\dagger(\delta_t, g) = \delta_{tu}.$$

But $s < t$ and our choice of u imply $su \not\leq tu$, which in turn implies

$$\pi_1 \circ h^\dagger(\delta_s, f) = \delta_{su} \not\leq \delta_{tu} = \pi_1 \circ h^\dagger(\delta_t, g),$$

from which it follows that $h^\dagger(\delta_s, f) \not\leq h^\dagger(\delta_t, g)$, so $h^\dagger\colon \Theta RV_{A^\infty}(A^\infty) \to \Theta RV_{A^\infty}(A^\infty)$ is not monotone, hence not Scott continuous.

4.3 Relation to the results of Goubault-Larrecq and Varacca

This paper and [20] were inspired by the work of Goubault-Larrecq and Varacca in [10]. Our goal has been to understand their approach in terms of domain-theoretic constructions, and to reveal in more detail what is taking place. While their presentation is necessarily sparse (given the limitations of a conference submission), we have taken more time to develop the approach in detail. We also have chosen a more general setting — instead of focusing on the case of the Cantor fan, we have developed our results assuming we are working over an arbitrary finite alphabet. Nevertheless, our results subsume theirs for $A = \{0,1\}$, which is to say our construction yields their construction in the case $A = \{0,1\}$. The proof of this relies on checking that our constructions agree with theirs in the case of the bases for $\Theta\mathsf{Prob}(\{0,1\}^*)$ and of $\bigoplus_{X \in AC(\{0,1\}^\infty)}[X \to D]$, for D a bounded complete domain. This is a routine check to carry out. Of course, the main consequence is that there is a flaw in their work. Our example above applies equally in their setting, so their construction is not a monad over BCD.

5 Summary and Future Work

In this paper and in [20] we have presented a reconstruction of the model of continuous random variables over bounded complete domains first devised by Goubault-Larrecq and Varraca in [10]. We also have extended the results to apply to an arbitrary finite alphabet, instead of limiting the focus to the case $A = \{0,1\}$. Our motivation is a more general development that would be directly applicable to settings such as process calculi over finite alphabets, where one wants to add probabilistic choice to an existing model. Our main contributions are the clarification that the structure of the model relies fundamentally on the family of Lawson-compact antichains in the domain A^∞. We have also shown that the monad construction does not lie within BCD – or any category of domains and Scott-continuous maps. We leave as an open problem how to repair this problem – we believe a new idea is needed, since the internal monoid structure on A^∞ using concatenation is not a monotone operation, and so the convolution operation it induces on $\mathsf{Prob}(A^\infty)$ is not monotone either.

Nevertheless, the proof that the monad laws hold – a proof essentially taken from [10] – is valid, so there is a monad. The question is what category it is on. We believe the right category here is one involving monoids and their probability measures, and continuous maps into (bounded complete) domains. But how to make sense of this for computational applications is not clear to us. We also remain intrigued by the construction of the monad, which uses convolution in a way we have not seen before – the second component in the convolved product is parameterized by the first; we'd like to understand this better. This is one reason we believe the probability monad Prob on monoids is at play here, but we do not understand exactly how.

Another problem we are interested in exploring is the relation between automata with discrete state spaces and those with continuous state spaces, e.g., the unit interval. We believe there is a role for the models described here in understanding such systems. As pointed out by one of the anonymous referees, this idea is potentially related to the approximation of labelled Markov processes over continuous state spaces by ones with finite state space, as explored in [6].

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