Adjunctions Between Categories of Domains

Michael W. Mislove*
Department of Mathematics
Tulane University
New Orleans, LA 70118
and
Frank J. Oles
Mathematical Sciences Department
IBM Thomas J. Watson Research Center
Yorktown Heights, NY 10598

Fundamenta Informaticae, to appear

Abstract

In this paper we show that there is no left adjoint to the inclusion functor from the full subcategory $\mathcal{C}_0$ of Scott domains (i.e., consistently complete $\omega-$algebraic cpo's) to $\mathcal{SFP}$, the category of $\mathcal{SFP}$-objects and Scott-continuous maps. We also show there is no left adjoint to the inclusion functor from $\mathcal{C}_0$ to any larger category of cpo's which contains a simple five-element domain. As a corollary, there is no left adjoint to the inclusion functor from $\mathcal{C}_0$ to the category of L-domains.

We also investigate adjunctions between categories which contain $\mathcal{C}_0$, such as $\mathcal{SFP}$, and subcategories of $\mathcal{C}_0$. Of course, it is well-known that each of the three standard power domain constructs gives rise to a left adjoint. Since the Hoare and Smyth power domains are Scott domains, we can regard each of these two adjunctions as left adjoints to inclusion functors from appropriate subcategories of $\mathcal{C}_0$. But, our interest here is in adjunctions for which the target of the left adjoint is a *lluf subcategory* of $\mathcal{C}$; such a subcategory has all Scott domains as objects, but the morphisms are more restrictive than being Scott continuous. We show that three such adjunctions exist.

The first two of these are based on the Smyth power domain construction. One is a left adjoint to the inclusion functor from the category $\mathcal{C}$ of consistently complete algebraic cpo's and Scott-continuous maps preserving finite, non-empty infima to the category of coherent algebraic cpo's and Scott-continuous maps. The same functor has a restriction to the subcategory of coherent algebraic cpo's whose morphisms also are Lawson continuous to the lluf subcategory of $\mathcal{C}$ whose morphisms are those Scott-continuous maps which preserve all non-empty infima.

The last adjunction we derive is a generalization of the Hoare power domain which satisfies the property that, if D is a nondeterministic algebra, then the image of D under the left adjoint enjoys an additional semigroup structure under which the original algebra D is among the set of idempotents. In this way, we expand the Plotkin power domain $\mathcal{P}(D)$ over the Scott domain D into a Scott domain.

1 Introduction

The category $\mathcal{C}_0$ of consistently complete ω-algebraic cpo's, or *Scott domains,* as they are also called, and Scott-continuous maps is a cartesian closed category which is especially well-suited for use in the denotational semantics of sequential, imperative languages. Scott [12, 82] has shown how this category can be used to provide models for computation. Larsen and Winskel [7] explain how this category is equivalent to the category of information systems, where domain equations can be solved exactly (not just up to isomorphism). But the study of

*Work partially supported by the Office of Naval Research

languages supporting nondeterminism and concurrency led to new constructs not all of which could be realized within the category of Scott domains. In particular, the Plotkin power domain of a Scott domain is not necessarily consistently complete. Plotkin [11] proposed the category $\mathcal{SFP}$ of $\mathcal{SFP}$-objects and Scott-continuous maps as being better suited for investigations of nondeterministic languages. He characterized $\mathcal{SFP}$-objects as those algebraic cpo's which can be expressed as the inverse limit of a sequence of finite posets under embedding-projection pairs, and he showed that the category $\mathcal{SFP}$ is closed under the formation of all the usual domain-theoretic constructs, including the Plotkin power domain. Plotkin also conjectured that $\mathcal{SFP}$ was the largest cartesian closed category of $\omega-$algebraic cpo's, a conjecture which was proved correct by Smyth [14]. Later, Jung [5] showed that $\mathcal{SFP}$ is one of but two maximal cartesian closed categories of algebraic cpo's, the other being the category $\mathcal{LD}$ of L-domains. But $\mathcal{SFP}$ has remained the category of most interest for those working on concurrency because it is the smallest cartesian closed category which accommodates all the necessary constructs. Consequently, much work has been done in trying to explicate clearly the structure of $\mathcal{SFP}$-objects (see, e.g. [6]).

In this paper we investigate the relationship between the full subcategory $\mathcal{C}$ of consistently complete algebraic cpo's and the larger category $\mathcal{A}$ of algebraic cpo's and Scott-continuous maps. Our first result is a proof that there is no left adjoint to the inclusion functor from $\mathcal{C}$ to $\mathcal{A}$. This example also shows that there is no left adjoint to the inclusion functor from $\mathcal{C}_0$ to any containing category which also contains a simple five-element domain, and this applies to the "interesting" subcategories of $\mathcal{A}$ such as $\mathcal{SFP}$ and $\mathcal{LD}$.

On the positive side, we investigate adjunctions between lluf subcategories of $\mathcal{C}$ and $\mathcal{A}$. Recall that a subcategory $\mathcal{M}$ of a category $\mathcal{N}$ is called a *lluf subcategory* if $\mathcal{M}$ contains all the objects of $\mathcal{N}$. After establishing the result that there is no left adjoint to the inclusion functor from $\mathcal{C}$ to $\mathcal{A}$, we investigate the existence of a left adjoint to the inclusion functor from various lluf subcategories of $\mathcal{C}$ to appropriate subcategories of $\mathcal{A}$. We show that the Smyth power domain functor gives rise to two such left adjoints. The first is adjoint to the inclusion functor from the lluf subcategory of $\mathcal{C}$ whose morphisms are Scott-continuous maps preserving finite, non-empty infima, to the category $\mathcal{A}$; this is the usual Smyth power domain construct which was shown by Hennessy and Plotkin ([3]) to be universal for a certain class of nondeterministic algebras (this simply means that D has a commutative, idempotent semilattice operation which is Scott continuous).

The second adjunction we present is a left adjoint to the inclusion functor from the lluf subcategory of $\mathcal{C}$ whose morphisms are the Scott-continuous maps preserving all non-empty infima, and whose target is the full subcategory of $\mathcal{A}$ of *coherent domains*. Recall that an algebraic cpo is *coherent* if, for each pair of compact elements k and k', there is a finite set F of compact elements $\uparrow k \cap \uparrow k' = \uparrow F$; equivalently, D is coherent if and only if D is compact in the Lawson topology.

Most of the paper is devoted to the presentation of a third adjunction between a lluf subcategory of $\mathcal{C}$ and $\mathcal{A}$. This adjunction generalizes the Hoare power domain construction, and it is left adjoint to the inclusion functor from the lluf subcategory of $\mathcal{C}$ whose morphisms preserve all existing non-empty suprema. The left adjoint uses a new construction which associates to a domain D a certain family $\mathbf{C}(D)$ of Scott-closed subsets of D.

Each algebraic cpo D is completely determined by its set of compact elements, $K(D)$. Indeed, if we define an *order ideal* of $K(D)$ to be a directed lower set, then D is isomorphic to $(Id(K(D)), \subseteq)$, the space of order ideals of $K(D)$ endowed with the inclusion order, under the mapping:

$$x \mapsto K(x) = \{k \in K(D) \mid k \leq x\} \colon D \to Id(K(D)).$$

Using the fact that the family of Scott-closed subsets of an algebraic cpo D is isomorphic to the family of lower sets of the compact elements $K(D)$, we can describe $\mathbf{C}(D)$ as

$$\mathbf{C}(D) = \{J \subseteq K(D) \mid \emptyset \neq J = \downarrow J \ \&\ (\exists I \in Id(K(D)))\ J \subseteq I\},$$

the family of non-empty lower sets of $K(D)$ which are contained in some order ideal, again with the inclusion order.

In case that D is *conditionally bounded* (a term we define below), then $\mathbf{C}(D)$ is the family of bounded Scott-closed subsets of D. We show that the conditionally bounded domains are exactly those which are compact in the lower topology; as a result, each coherent domain is conditionally bounded, so $\mathcal{SFP}$-objects are in this class.

In addition to describing $\mathbf{C}(D)$ in terms of the Scott-closed subsets of D, we also describe $\mathbf{C}(D)$ as an information system [7], for the case that D is conditionally bounded. This description is quite pleasing, since it is given in terms of the family $\mathcal{P}_{<\omega}(K(D))$ of non-empty, finite subsets of the set $K(D)$ of compact elements of D. Thus, $\mathbf{C}(D)$ can be viewed in much the same way that each of the standard power domains, the Hoare, Smyth and Plotkin power domains over D is usually described (cf. [3]). Moreover, we describe the action of $\mathbf{C}$ on morphisms in the category $\mathcal{A}$ in terms of approximable mappings, the information system analogue of Scott-continuous maps between Scott domains. However, $\mathbf{C}(D)$ is not a nondeterministic algebra, and so this is not a new power domain construction.

Power domains are the domain-theoretic analogue of the power set; they are needed to model nondeterministic languages. In [3], Hennessy and Plotkin characterize each of the standard power domains as a left adjoint; in each case, they construct an appropriate category of *nondeterministic algebras*, and show that the power domain construct in question is a left adjoint to an obvious inclusion functor. We show that, if $(D, +)$ is a nondeterministic algebra in the sense of [3], then $\mathbf{C}(D)$ carries an additional structure which is analogous to the nondeterministic $+$ operator on D. This operation on $\mathbf{C}(D)$ is commutative and associative, but it is not idempotent. We characterize those elements of $\mathbf{C}(D)$ which are idempotent under the induced operator; they include the image of D under the embedding of D into $\mathbf{C}(D)$. We define the notion of a *nondeterministic semigroup*, a generalization of a nondeterministic algebra, and of a *Scott semigroup*, a Scott domain which is equipped with a Scott-continuous, commutative semigroup operation. Then we show that $\mathbf{C}$ induces a left adjoint to the inclusion functor from the category of Scott semigroups and Scott semigroup maps to the category of nondeterministic semigroups and nondeterministic semigroup maps. In this way, we construct an analogue for the Plotkin power domain in the category of Scott domains, namely the Scott semigroup $\mathbf{C}(\mathcal{P}_P(D))$ over the Plotkin power domain $\mathcal{P}_P(D)$ over D.

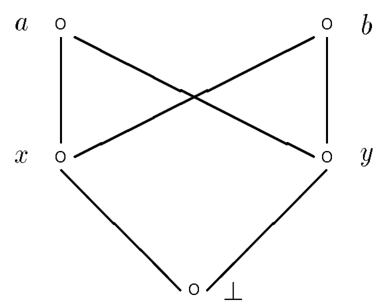
2 A Simple Counterexample

We begin our presentation with a review of the relevant facts concerning algebraic cpo's, and then we present a simple example which shows that there is no adjunction to the inclusion functor from the full subcategory $\mathcal{C}$ of consistently complete algebraic cpo's to the category $\mathcal{A}$ of algebraic cpo's and Scott-continuous maps. In fact, this example shows that no such adjunction can exist between any category of consistently complete algebraic cpo's and Scott-continuous maps and any containing category of algebraic cpo's which also contains a simple five-element domain.

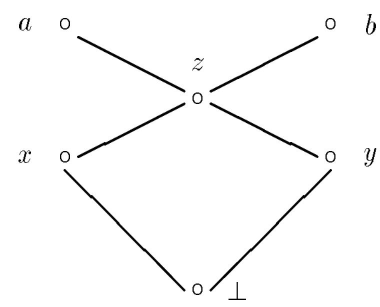
To begin, a subset $X \subseteq D$ of a partially ordered set D is *directed* if each finite subset of X has an upper bound in X. Note that directed sets are always non-empty, since the empty set is finite. A *complete partial order (cpo)* is a partially ordered set D in which each directed subset has a least upper bound and which has a least element, which we denote by $\perp$. An element $k \in D$ is *compact* if, for every directed subset $X \subseteq D$ with $k \leq \bigsqcup X$, there is some $x \in X$ with $k \leq x$. The set of compact elements of D is denoted $K(D)$, and clearly $\perp \in K(D)$. If $x \in D$, then we use the notation $\downarrow x = \{y \in D \mid y \leq x\}$ and $\uparrow x = \{y \in D \mid x \leq y\}$. Also, $K(x) = \downarrow x \cap K(D)$ for each $x \in D$. The cpo D is *algebraic* if $K(x)$ is directed and $x = \bigsqcup K(x)$ for all $x \in D$. A *domain* is an algebraic cpo whose set of compact elements is countable. The algebraic cpo D is *consistently complete* if each subset of D which has an upper bound has a least upper bound. A *Scott domain* is a consistently complete domain. Lastly, an L-

domain is a domain D in which $\downarrow x$ is a complete lattice for each $x \in D$. These domains were introduced by Jung [5], where he showed that L-domains and Scott-continuous maps form a maximal cartesian closed category of domains. We denote by $\mathcal{A}$ the category of algebraic cpo's and Scott-continuous maps, and by $\mathcal{C}$ (respectively, $\mathcal{C}_0$, $\mathcal{SFP}$, $\mathcal{LD}$), the full subcategory of consistently complete algebraic cpo's (respectively, Scott domains, $\mathcal{SFP}$-objects, L-domains). Lastly, by an *algebraic lattice* we mean a complete lattice which also is an algebraic cpo.

Example Let $D_0 = \{\perp, x, y, a, b\}$ denote the five-element poset which satisfies $\perp < x, y < a, b$, and no other pairs are related. Also, let $E_0 = \{\perp, x, y, z, a, b\}$ denote the six-element poset with $\perp < x, y < z < a, b$, and no other pairs are related. The Hasse diagrams of these posets are given below.



D_0



E_0 □

Definition Let D be an algebraic cpo and let E be a consistently complete algebraic cpo. Then E is *weakly universal* for D if there is a Scott-continuous map $e: D \to E$ such that, given any consistently complete algebraic cpo D' and any Scott-continuous map $f: D \to D'$, there is a Scott-continuous map $f': E \to D'$ with $f' \circ e = f$. E is *universal* for D if E is weakly universal and the fill-in map f' is unique.

Lemma 2.1 *The six-element poset E_0 is weakly universal for D_0.*

Proof Clearly E_0 is consistently complete, and the natural embedding $e: D_0 \to E_0$ is Scott continuous. If D is a consistently complete algebraic cpo, and $f: D_0 \to D$ a Scott-continuous map, then $f(x), f(y) \leq f(a)$ implies that $\{f(x), f(y)\}$ has a least upper bound $c \in D$. We can then define $f': E_0 \to D$ by $f'(e(d)) = f(d)$, for $d \in D_0$, and $f'(z) = c$. Clearly f' is monotone (and hence Scott continuous), and $f' \circ e = f$ also is clear. □

Lemma 2.2 *Let $D_1 = \{\perp, x, y, r, s, a, b\}$ with the order $\perp < x, y < r < s < a, b$. Then there are two Scott-continuous maps $f_i: E_0 \to D_1$, $i = 1, 2$ with $f_i \circ e = j$, where $j: D_0 \to D_1$ is the natural embedding.*

Proof There are two Scott-continuous maps $f_i: E_0 \to D_1$ which satisfy $f_i(e(d)) = d$ for each $d \in D_0$; namely, $f_1(z) = r$ and $f_2(z) = s$. □

Theorem 2.3 *There is no universal consistently complete algebraic cpo for D_0.*

Proof Suppose D_0 has a universal consistently complete algebraic cpo D, and let $\iota: D_0 \to D$ be the Scott-continuous map from D_0 to D which extends uniquely. Since $e: D_0 \to E_0$ is Scott continuous and E_0 is consistently complete, there is a Scott-continuous map $f': D \to E_0$ such that $f'(\iota(d)) = e(d)$ for each $d \in D_0$. Then $\iota(x), \iota(y) \leq \iota(a)$ implies $t = \iota(x) \vee \iota(y)$ exists in D, and $\iota(x), \iota(y) \leq \iota(a), \iota(b)$. Hence $x = f'(\iota(x)), y = f'(\iota(y)) \leq f'(t) \leq a = f'(\iota(a)), b = f'(\iota(b))$, and so $f'(t) = z$. But, then $f_i \circ f': D \to D_1$ are two distinct Scott-continuous maps. This contradicts the universality of D. □

Corollary 2.4 *Let $\mathcal{B}$ be a category of algebraic cpo's and Scott-continuous maps which contains the consistently complete algebraic cpo's and the domain D_0. Then there is no left adjoint to the inclusion functor from $\mathcal{C}$ into $\mathcal{B}$. In particular,*

i) *There is no left adjoint to the inclusion functor from $\mathcal{C}$ into $\mathcal{A}$.*

ii) *There is no left adjoint to the inclusion functor from $\mathcal{C}_0$ to $\mathcal{SFP}$.*

iii) *There is no left adjoint to the inclusion functor from $\mathcal{C}_0$ to $\mathcal{LD}$.* □

Despite this negative result, it is appealing to believe that the Scott domain E_0 is somehow universal for D_0; indeed, Lemma 2.1 implies as much. And, it is clear that E_0 is the smallest Scott domain into which D_0 can be embedded. The remaining results we present are an outgrowth of our attempt to make this intuitive result mathematically precise. As we show, there is a left adjoint to the inclusion functor from a lluf subcategory of $\mathcal{C}_0$ if we restrict the morphisms between Scott domains in any one of three ways:

1. Scott domains and Scott-continuous maps preserving finite, non-empty infima,

2. Scott domains and Scott-continuous maps preserving all non-empty infima, and

3. Scott domains and Scott-continuous maps preserving existing non-empty suprema.

The first two of these results are achieved using the Smyth power domain, while the third uses the family of bounded, Scott-closed subsets. And, on the object level, E_0 is the free Scott domain associated to D_0 in each of these cases.

3 A Basic Adjunction

In this section we review the basic results about the Scott topology on an algebraic cpo. We also take the opportunity to present the Hoare power domain as an adjoint using the topological tools we construct. This adjunction is between the category $\mathcal{A}$ of algebraic cpo's and Scott-continuous maps and the category $\mathcal{AL}_0$ of (complete) algebraic lattices and maps preserving all non-empty suprema. But this adjunction is not one of the types we are most interested in, since the domain for the inclusion functor is a category of algebraic lattices, rather than a category whose objects consist of all Scott domains (or, more generally, all consistently complete algebraic cpo's). However, this result is still important for the ones to come later, since it provides a motivation for the construction we give using the bounded, Scott-closed subsets of an algebraic cpo D. The spectral theory of sober spaces and complete Heyting algebras plays a central role in our development, so relevant portions of that theory are also included.

Definition Let D be an algebraic cpo. A subset $U \subseteq D$ is *Scott open* if:

1. $U = \uparrow U$, and

2. $(\forall X \subseteq D \text{ directed}) \bigsqcup X \in U \;\Rightarrow\; X \cap U \neq \emptyset$.

The family of Scott-open subsets of D is denoted $\sigma(D)$; dually, the family of Scott-closed subsets of D is denoted $\Gamma_0(D)$.

The following result summarizes the basic facts about the Scott topology on a algebraic cpo D; all the proofs are straightforward.

Theorem 3.1 *Let D be an algebraic cpo. Then:*

i) *The family $\{\uparrow k \mid k \in K(D)\}$ is a base for the Scott topology on D. For all $x \in D$, $\overline{\{x\}} = \downarrow x$, and $\uparrow x \in \sigma(D)$ if and only if $x \in K(D)$.*

ii) *The Scott topology on D is T_0, compact, and locally compact (i.e., each point has a neighborhood basis of compact sets).*

iii) *If E is an algebraic cpo, then the function $f: D \to E$ is Scott continuous if and only if $f(\bigsqcup X) = \bigsqcup f(X)$ for all $X \subseteq D$ directed.*

iv) *If $X = \downarrow X \subseteq D$ is a lower set in D, then*

$$\overline{X} = \{x \in D \mid K(x) \subseteq X\}.$$

□

Now, consider the lattice $\Gamma(D)$ of *non-empty* Scott-closed subsets of the algebraic cpo D. Since D has a least element, $\bot$, $\Gamma(D)$ has as least element $\{\bot\}$; consequently, $\Gamma(D)$ is a complete lattice. And, $\Gamma(D)$ is a *coHeyting algebra* or *Brouwerian lattice*; i.e., it satisfies the infinite distributivity law:

$$C \cup (\bigcap_{i \in I} C_i) = \bigcap_{i \in I}(C \cup C_i),$$

for any subset $\{C_i \mid i \in I\} \subseteq \Gamma(D)$. Moreover, the map $\eta_D: D \to \Gamma(D)$ defined by $\eta_D(x) = \overline{\{x\}} = \downarrow x$ is a one-to-one map of D into $\Gamma(D)$. In fact, each closed set from D of the form $\overline{\{x\}}$ is a $\cup$−prime in $\Gamma(D)$: if A and B are Scott-closed sets and $\overline{\{x\}} \subseteq A \cup B$, then $\overline{\{x\}} \subseteq A$ or $\overline{\{x\}} \subseteq B$ (such sets are also called *irreducible*). If we denote by $\mathrm{Pr}_{\cup}\Gamma(D)$ the family of $\cup$−primes of $\Gamma(D)$ (including $\{\bot\}$!), then $\eta_D(D) \subseteq \mathrm{Pr}_{\cup}\Gamma(D)$.

Definition The topological space X is *sober* if, for each $\cup$−prime A in the lattice $\mathcal{C}(X)$ of closed subsets of X, there is a unique $x \in X$ with $A = \overline{\{x\}}$. In other words, the space X is sober if and only if the map $x \mapsto \overline{\{x\}}: X \to \mathcal{C}(X)$ is a bijection from X onto the set of $\cup$−primes of $\mathcal{C}(X)$.

Since each $\cup$−prime of $\mathcal{C}(X)$ is the closure of a unique point, it is clear that a sober space is T_0, but the converse may fail.

Proposition 3.2 *If D is an algebraic cpo, then D is a sober space in the Scott topology.*

Proof Let $C \subseteq D$ be an irreducible closed set, and let $K(C) = K(D) \cap C$ denote the family of compact elements in C. Then part iv) of Theorem 3.1 implies that $C = \overline{K(C)}$. We claim that $K(C)$ is directed. Indeed, if $k, k' \in K(C)$ have no upper bound in C, then $\uparrow k \cap \uparrow k' \cap C = \emptyset$. But, then $C = (C \setminus \uparrow k) \cup (C \setminus \uparrow k')$ is the union of proper closed subsets, since $\uparrow k$ and $\uparrow k'$ are Scott open. Thus $K(C)$ is directed. But, then $x = \bigsqcup K(C)$ exists in D since D is a cpo, and $x \in C$ since C is Scott closed. But, $K(C) \subseteq \downarrow x$ implies $C \subseteq \downarrow x$, so $C = \downarrow x = \overline{\{x\}}$. Hence D is sober. □

The subset $\mathrm{Pr}_{\cup}\Gamma(D)$ of $\Gamma(D)$ can be given a topology directly from $\Gamma(D)$, called the *hull-kernel topology.* Define the closed subsets of $\mathrm{Pr}_{\cup}\Gamma(D)$ to be the family of sets of the form

$$\downarrow C \cap \mathrm{Pr}_{\cup}\Gamma(D) = \{\overline{\{x\}} \mid \overline{\{x\}} \subseteq C\},$$

for $C \in \Gamma(D)$. For $C \in \Gamma(D)$, we calculate what this closed set is:

$$\begin{aligned} \downarrow C \cap \mathrm{Pr}_{\cup}\Gamma(D) &= \{\eta_D(x) \mid \eta_D(x) \subseteq C\} \\ &= \{\eta_D(x) \mid x \in C\} \\ &= \eta_D(C). \end{aligned}$$

That is, the hull-kernel topology on $\mathrm{Pr}_{\cup}\Gamma(D)$ from $\Gamma(D)$ is the Scott topology on D, and so the map $\eta_D: D \to \mathrm{Pr}_{\cup}\Gamma(D)$ is a homeomorphism. (This actually holds for all sober spaces, but we restrict our attention to those of interest, algebraic cpo's endowed with the Scott topology.)

A lattice L is *bialgebraic* if L is algebraic in both its given order, $\leq$, and in the dual order, $\leq^{\mathrm{op}}$. The following result shows that $\Gamma(D)$ is rather special in that it is bialgebraic. The compact elements for both orders also are singled out.

Proposition 3.3 *If D an algebraic cpo, then $\Gamma(D)$ is a bialgebraic lattice:*

i) $K(\Gamma(D),\subseteq) = \{\downarrow F \mid F \subseteq K(D) \text{ is finite}\}$.

ii) $K(\Gamma(D),\supseteq) = \{D\backslash\uparrow F \mid F \subseteq K(D) \text{ is finite}\}$.

Moreover, the hull-kernel topology on $\mathrm{Pr}_\cup\Gamma(D)$ is the inherited Scott topology from $\Gamma(D)$.

Proof It is routine to verify that $\downarrow F \in K(\Gamma(D),\subseteq)$ for any finite subset $F \subseteq K(D)$. Furthermore, given any $X \in \Gamma(D)$, it is easy to see that

$$X = \overline{\bigcup\{\downarrow F \mid F \subseteq X \cap K(D) \text{ finite}\}} = \bigsqcup\{\downarrow F \mid F \subseteq X \cap K(D) \text{ finite}\}.$$

This set also is directed for each $X \in \Gamma(D)$, and it follows that

$$K(\Gamma(D),\subseteq) = \{\downarrow F \mid F \subseteq K(D) \text{ is finite}\}.$$

Thus, $(\Gamma(D),\subseteq)$ is an algebraic lattice.

Dually, if $F \subseteq K(D)$ is finite, then the subset $D\backslash\uparrow F$ is a Scott-closed set with the property that:

$$\bigcap_{i\in I} X_i \subseteq D\backslash\uparrow F \ \&\ \{X_i\}_{i\in I} \text{ directed } \Rightarrow (\exists i \in I)\ X_i \subseteq D\backslash\uparrow F.$$

That is, $D\backslash\uparrow F$ is a compact element of $(\Gamma(D),\supseteq)$. Again, it is easy to show that each $X \in \Gamma(D)$ is the filtered intersection of the subsets $D\backslash\uparrow F$ which contain X, and so $(\Gamma(D),\supseteq)$ also is algebraic. Thus, $\Gamma(D)$ is a bialgebraic lattice.

We have already noted that the hull-kernel topology on $\mathrm{Pr}_\cup\Gamma(D)$ is the image of the Scott topology on D via the mapping η_D. And, clearly this topology is contained in the topology $\mathrm{Pr}_\cup\Gamma(D)$ inherits from the Scott topology on $\Gamma(D)$, since $\{B \in \Gamma(D) \mid B \subseteq C\}$ is Scott closed for each $C \in \Gamma(D)$. Conversely, if $\mathcal{U} \subseteq \Gamma(D)$ is a basic Scott-open set, then $\mathcal{U} = \Uparrow(\downarrow F)$, where

$$\Uparrow(\downarrow F) = \{X \in \Gamma(D) \mid \downarrow F \subseteq X\}$$

denotes the upper set of $\downarrow F \in \Gamma(D)$, and $F \subseteq K(D)$ is finite. Then

$$\mathcal{U} = \Uparrow(\downarrow F) = \{X \in \Gamma(D) \mid \downarrow F \subseteq X\} = \{X \in \Gamma(D) \mid F \subseteq X\}.$$

Hence,

$$\begin{aligned}\mathrm{Pr}_\cup\Gamma(D)\setminus\mathcal{U} &= \{\downarrow x \mid \downarrow F \not\subseteq \downarrow x\} = \{\downarrow x \mid F \not\subseteq \downarrow x\} = \bigcup_{i=1}^{n}\{\downarrow x \mid k_i \not\leq x\}\\ &= \bigcup_{i=1}^{n}(\mathrm{Pr}_\cup\Gamma(D) \cap \downarrow(D\backslash\uparrow k_i)),\end{aligned}$$

where $F = \{k_1,\ldots,k_n\}$. Since $k_i \in K(D)$, $D\backslash\uparrow k_i$ is closed, and so $\mathrm{Pr}_\cup\Gamma(D)\setminus\mathcal{U}$ is a finite union of hull-kernel closed sets, hence itself is hull-kernel closed. □

Since $\Gamma(D)$ also is distributive, it follows that $\Gamma(D)$ is completely distributive; in fact, $\Gamma(D)$ is a complete ring of sets (see, e.g., [9]).

Given a Scott-continuous map $f\colon D \to E$ between algebraic cpo's, it is easy to see that the map $\Gamma(f)\colon\Gamma(D) \to \Gamma(E)$ by $\Gamma(f)(C) = \overline{f(C)}$ is Scott continuous, and it also is easy to show that this map preserves union-closures, which are non-empty suprema in $\Gamma(D)$. This mapping also preserves $\cup$-primes, since $\Gamma(f)(\overline{\{x\}}) = \overline{\{f(x)\}}$. Since the Scott topology on $\mathrm{Pr}_\cup\Gamma(D)$ is the inherited topology from the Scott topology on $\Gamma(D)$, it follows that the restriction of $\Gamma(f)$ to $\mathrm{Pr}_\cup\Gamma(D)$ is Scott continuous. In fact, these results can be formulated as an equivalence; to state the result precisely, we first define $\mathcal{CDA}'$ to be the category of completely distributive algebraic lattices and maps preserving all non-empty suprema and all sup-primes.

Theorem 3.4 ([10]) *The functor* $\Gamma: \mathcal{A} \to \mathcal{CDA}'$ *is equivalent to the functor* $\mathrm{Pr}_{\cup}: \mathcal{CDA}' \to \mathcal{A}$ *which assigns to a completely distributive algebraic lattice* L *its family of* $\vee$*-primes and to the morphism* $\phi: L \to M$ *the restriction of* ϕ *to* $\mathrm{Pr}_{\vee} L$ *and* $\mathrm{Pr}_{\vee} M$. □

As we explain more fully in Section 7, Hennessy and Plotkin [3] define a *nondeterministic algebra* to be a domain D which has a Scott-continuous semilattice operation $\oplus: D \times D \to D$. A morphism of nondeterministic algebras is a Scott-continuous map which preserves binary sums. They single out two subcategories of such algebras:

$\mathcal{ND}_H$ – the full subcategory of nondeterministic algebras D satisfying $x \le x \oplus y$ for all $x, y \in D$.

$\mathcal{ND}_S$ – the full subcategory of nondeterministic algebras D satisfying $x \ge x \oplus y$ for all $x, y \in D$.

The first of these, $\mathcal{ND}_H$ arises via the Hoare power domain, which can be defined via the functor Γ (cf. [15]). Thus, if D is a domain, then the Hoare power domain $\Gamma(D)$ for D is an $\mathcal{ND}_H$-object, and as Hennessy and Plotkin noted, it is the free $\mathcal{ND}_H$-object over D. It has been observed in [PL81] that the category $\mathcal{ND}_H$ is simply the category $\mathcal{AL}$ of algebraic lattices and maps preserving all non-empty suprema. By using the inclusion functor $J: \mathcal{CDA}' \to \mathcal{AL}$, we can relate the Hoare power domain to the equivalence we gave above; indeed, the first adjunction in the following result is the universal property of the Hoare power domain.

Theorem 3.5

i) *The functor* $J \circ \Gamma: \mathcal{A} \to \mathcal{AL}$ *is left adjoint to the inclusion functor from* $\mathcal{AL}$ *to* $\mathcal{A}$.

ii) *If* $\mathcal{A}_\perp$ *denotes the subcategory of* $\mathcal{A}$ *of algebraic cpo's and strict Scott-continuous maps, then the restriction functor* $J \circ \Gamma_\perp: \mathcal{A}_\perp \to \mathcal{AL}$ *is left adjoint to the inclusion functor.*

Proof Proposition 3.3 shows that $\Gamma(D)$ is an algebraic lattice, and our comments preceding Theorem 3.4 show that, for each continuous map $f: D \to E$ between algebraic cpo's, the induced mapping $\Gamma(f): \Gamma(D) \to \Gamma(E)$ by $G(f)(X) = \overline{f(X)}$ preserves all non-empty suprema. Moreover, our remarks following Theorem 3.1 show that the mapping $\eta_D: D \to \Gamma(D)$ is continuous since it is continuous into $\mathrm{Pr}_{\cup}\Gamma(D)$ in the inherited topology. Now, it is an easy exercise to show that the sup map $X \mapsto \bigvee X: \Gamma(L) \to L$ preserves all suprema for any algebraic lattice L. So, given a Scott-continuous map $f: D \to L$ from the algebraic cpo D to the algebraic lattice L, we can define the map $\hat{f}: \Gamma(D) \to L$ by $\hat{f}(X) = \bigvee X$, and this is the unique map from $\Gamma(D)$ to L preserving all suprema and satisfying $\hat{f} \circ \eta_D = f$. This shows the first adjunction.

The second statement follows from the observation that strict maps $f: D \to E$ between algebraic cpo's are characterized by the fact that the associated map $\hat{f}: \Gamma_0(D) \to \Gamma_0(E)$ takes the supremum of the empty set in $\Gamma_0(D)$, namely $\{\perp_D\}$, to $\{\perp_E\}$, which is the supremum of the empty set in $\Gamma_0(E)$. □

4 The Smyth Power Domain as an Adjoint

We now show how to use the Smyth power domain to construct a left adjoint to the inclusion functor from the lluf subcategory $\mathcal{C}_\wedge$ of Scott domains and Scott-continuous maps preserving finite, non-empty infima to $\mathcal{A}$, and from $\mathcal{C}_\sqcap$, the category of Scott domains and Scott-continuous maps preserving all non-empty infima, to the full subcategory of $\mathcal{A}$ consisting of coherent algebraic cpo's.

Definition Let X be a topological space and let $A \subseteq X$. The *saturation* of A is the set $sat(A) = \bigcap\{U \subseteq X \mid A \subseteq U \text{ is open}\}$. The subset A of X is *saturated* if $A = sat(A)$. We denote by $\mathcal{Q}(X)$ the family of non-empty saturated compact subsets of X, i.e.

$$\mathcal{Q}(X) = \{C \subseteq X \mid \emptyset \neq C \text{ is saturated and compact}\}.$$

Since $sat(A) \subseteq U$ if and only if $A \subseteq U$, for every open set U, the saturation of a set is compact if and only if the set is compact. If D is an algebraic cpo, then $C \subseteq D$ is saturated if and only if $C = \uparrow C$ is an upper set; this follows from the fact that every Scott-open subset is an upper set and the intersection of upper sets is an upper set. The following result is crucial to our development.

Theorem 4.1 ([4] Proposition 2.09) *Let X be a sober space. Then the family $\mathcal{Q}(X)$ is a complete partial order under reverse containment. Moreover, if $\mathcal{F}$ is a filter basis in $\mathcal{Q}(X)$ and $\bigcap \mathcal{F} \subseteq U$ and U is open, then $F \subseteq U$ for some $F \in \mathcal{F}$.* □

Since the empty set is open, it follows that if X is sober and $\mathcal{F}$ is a filter basis of compact saturated sets whose intersection is empty, then one of the sets in $\mathcal{F}$ must be empty.

Corollary 4.2 ([15]) *If D is an algebraic cpo, then $\mathcal{Q}(D)$ is an algebraic cpo whose compact elements are the sets*

$$\uparrow F = \{d \in D \mid (\exists x \in F)\ x \leq d\},$$

where $F \subseteq K(D)$ is finite and non-empty.

Proof According to Theorem 4.1, $(\mathcal{Q}(D), \supseteq)$ is closed under the formation of directed suprema, and D is clearly the smallest element of $\mathcal{Q}(D)$ under reverse containment, so $\mathcal{Q}(D)$ is a cpo.

We now show that $\uparrow F \in K(\mathcal{Q}(D))$ for each $F \subseteq K(D)$ finite and non-empty. Indeed, since $F \subseteq K(D)$ is finite and non-empty, it follows that $\uparrow F \in \mathcal{Q}(D)$. Suppose that $\mathcal{X}$ is a directed subfamily of $\mathcal{Q}(D)$ with $\uparrow F \leq \bigsqcup \mathcal{X}$. Since the order of $\mathcal{Q}(D)$ is reverse containment and $\mathcal{X}$ is directed, $\mathcal{X}$ is a filter-basis, and $\bigsqcup \mathcal{X} = \bigcap \mathcal{X}$. Moreover, since $\uparrow F$ is open in D, it follows from Theorem 4.1 that $X \subseteq \uparrow F$ for some $X \in \mathcal{X}$. Hence $\uparrow F \in K(\mathcal{Q}(D))$ for each $F \subseteq K(D)$ finite and non-empty.

Let $C \subseteq D$ be saturated and compact, and let $x \in D \setminus C$. Since C is saturated, C is the intersection of open sets, so $C = \uparrow C$. Since $x \notin C = \uparrow C$, it follows that for each $c \in C$, there is some $k \in K(D)$ with $k \leq c$ and $k \not\leq x$, and the compactness of C implies we can find finitely many such k_i, $i = 1, .., n$, with $C \subseteq \bigcup_i \uparrow k_i$ and $k_i \not\leq x$ for each i. Then $F = \{k_1, ..., k_n\}$ is a finite non-empty subset of $K(D)$ with $C \subseteq \uparrow F$ and $x \notin \uparrow F$. It follows that

$$C = \bigcap\{\uparrow F \mid C \subseteq \uparrow F \ \&\ F \subseteq K(D) \text{ is finite and non-empty}\},$$

and it is routine to show this family is directed. Thus $K(C)$ is a directed subset of $\mathcal{Q}(D)$ whose supremum in $\mathcal{Q}(D)$ is C. Since C is an arbitrary element of $\mathcal{Q}(D)$, it follows that $\mathcal{Q}(D)$ is an algebraic cpo and that $K(\mathcal{Q}(D)) = \{\uparrow F \mid \emptyset \neq F \subseteq K(D) \text{ is finite}\}$. □

Definition Let P be a poset. If $x, y \in P$, then a *complete set of upper bounds* for $\{x, y\}$ is a family $M \subseteq P$ such that $\uparrow x \cap \uparrow y = \uparrow M$. An algebraic cpo D is *coherent* if each pair of compact elements of D has a finite, complete set of minimal upper bounds.

It is easy to show that, in an algebraic cpo D, a minimal upper bound of a pair of compact elements is again compact. Thus, for a coherent domain D, the set of minimal upper bounds of a pair of compact elements consists of compact elements. It's an easy exercise to show that the class of coherent domains contains all consistently complete domains. Somewhat harder to show is the fact that $\mathcal{SFP}$-objects also fall within this class (cf. [5]). However, not all L-domains are coherent, as simple examples demonstrate (cf., e.g., the example at the beginning of Section 5).

Lemma 4.3 *Let D be an algebraic cpo, and let $\uparrow F, \uparrow G \in K(\mathcal{Q}(D))$ be compact elements of $\mathcal{Q}(D)$. Then $\uparrow F \vee \uparrow G$ exists in $\mathcal{Q}(D)$ if and only if $\uparrow F \vee \uparrow G = \uparrow F \cap \uparrow G$.*

Proof Suppose that $\uparrow F, \uparrow G \in K(\mathcal{Q}(D))$ have an upper bound $C \in \mathcal{Q}(D)$. Then $C \subseteq \uparrow F \cap \uparrow G$. Conversely, let $x \in \uparrow F \cap \uparrow G$. Since D is algebraic and $\uparrow F \cap \uparrow G$ is Scott open, there is some compact element $k \in K(x)$ with $x \in \uparrow k \subseteq \uparrow F \cap \uparrow G$. And, since $\uparrow k \in \mathcal{Q}(D)$, it follows that $\uparrow F, \uparrow G \sqsubseteq \uparrow k$ in $\mathcal{Q}(D)$. Thus, $x \in \uparrow k \subseteq \uparrow F \vee \uparrow G$. Since $x \in \uparrow F \cap \uparrow G$ is arbitrary, we have $\uparrow F \vee \uparrow G = \uparrow F \cap \uparrow G$, as claimed.

Conversely, if $\uparrow F \cap \uparrow G \in K(\mathcal{Q}(D))$, then clearly $\uparrow F \vee \uparrow G = \uparrow F \cap \uparrow G$, so $\uparrow F \vee \uparrow G$ exists in $\mathcal{Q}(D)$. □

Proposition 4.4 *For an algebraic cpo D, the following are equivalent:*

1. *D is coherent.*
2. *$\mathcal{Q}(D)$ is consistently complete.*
3. *The intersection of compact, saturated subsets of D is compact.*

Proof Suppose that D is coherent. To show 2) holds, we first consider the case of two compact elements of $\mathcal{Q}(D)$. Corollary 4.2 shows that $K(\mathcal{Q}(D)) = \{\uparrow F \mid \emptyset \neq F \subseteq K(D) \text{ is finite}\}$. If $\uparrow F, \uparrow G \in K(\mathcal{Q}(D))$ have an upper bound in $\mathcal{Q}(D)$, then $\uparrow F \cap \uparrow G \neq \emptyset$. Since D is coherent, the set $F \cup G$ has a finite, complete set H of minimal upper bounds, as a simple induction on the size of $F \cup G$ shows. It is clear that $\uparrow F \cap \uparrow G = \uparrow H$, and since minimal upper bounds of compact elements are again compact, it follows that $\uparrow H \in \mathcal{Q}(D)$. Thus, $\uparrow F \vee \uparrow G = \uparrow F \cap \uparrow G = \uparrow H$, so bounded pairs of compact elements in $\mathcal{Q}(D)$ have sups. But, in an algebraic cpo, if every pair of compact elements having an upper bound has a supremum, then every pair of elements having some upper bound also has a supremum. Furthermore, any cpo in which every pair of elements having an upper bound has a supremum is consistently complete. Thus, $\mathcal{Q}(D)$ is consistently complete, so 2) holds.

Suppose now that 2) holds. To show 3), we first consider the case of a pair of compact, open subsets of D; i.e., of compact elements of $\mathcal{Q}(D)$. If $\uparrow F, \uparrow G \in K(\mathcal{Q}(D))$, then Lemma 4.3 implies that $\uparrow F \vee \uparrow G = \uparrow F \cap \uparrow G$ if $\uparrow F \vee \uparrow G$ exists. Now, every element of $\mathcal{Q}(D)$ is the filtered intersection of elements of the form $\uparrow F$, for $\uparrow F \in K(\mathcal{Q}(D))$. Since the supremum of any pair of elements of $K(\mathcal{Q}(D))$ is their intersection, when the supremum exists, it follows from Theorem 4.1 that the supremum of any pair of elements of $\mathcal{Q}(D)$ is their intersection, when the supremum exists. Thus 3) holds for pairs of compact, saturated sets whose intersection is non-empty. But, for a pair of compact, saturated sets with empty intersection, their intersection – the empty set – is clearly compact and saturated. Thus 3) holds.

Finally, if 3) holds, then Lemma 4.3 implies that the intersection of the sets $\uparrow k$ and $\uparrow k'$, for $k, k' \in K(D)$, has the form $\uparrow H$ for some finite, non-empty subset $H \subseteq K(D)$, if $\uparrow k \cap \uparrow k' \neq \emptyset$. But, then H contains a set of minimal upper bounds for k and k', and so D is coherent. □

Proposition 4.5 *If D be a consistently complete algebraic cpo, then $(x, y) \mapsto x \wedge y \colon D \times D \to D$ is Scott continuous. Moreover, the mapping $C \mapsto \bigwedge C \colon \mathcal{Q}(D) \to D$ is a Scott-continuous mapping which preserves finite, non-empty infima.*

Proof Since D is consistently complete, given $x, y \in D$, the set $K(x) \cap K(y)$ is a directed subset of D, and so it has a least upper bound, z. Clearly $z \leq x, y$, and if $w \leq x, y$, then $K(w) \subseteq K(x) \cap K(y)$ implies that $w = \bigvee K(w) \leq \bigvee(K(x) \cap K(y)) = z$. Thus $z = x \wedge y$ exists for each $x, y \in D$.

Since $k = k \wedge k$ for each $k \in K(D)$, it follows that

$$\{(x, y) \in D \times D \mid k \leq x \wedge y\} = \{(x, y) \mid k \leq x, y\} = \uparrow k \times \uparrow k,$$

and this set is clearly Scott open in $D \times D$ if k is compact. Thus $(x, y) \mapsto x \wedge y$ is Scott continuous.

Next, suppose that $C \in \mathcal{Q}(D)$. Then Corollary 4.2 implies that

$$C = \bigcap\{\uparrow F \mid C \subseteq \uparrow F \ \&\ F \subseteq K(D) \text{ finite and non-empty}\},$$

and this family is directed. Since $\bigwedge F = \bigwedge \uparrow F$ for any finite subset of D,

$$\bigwedge C = \bigvee\{\bigwedge F \mid C \subseteq \uparrow F \ \&\ F \subseteq K(D) \text{ is finite and non-empty}\},$$

so $\bigwedge C$ is well-defined. Moreover, this same argument shows that the mapping $C \mapsto \bigwedge C \colon \mathcal{Q}(D) \to D$ is Scott continuous.

If $C, C' \in \mathcal{Q}(D)$, then $C \wedge_{\mathcal{Q}(D)} C' = C \cup C'$, so

$$\wedge C \ \wedge \ \wedge C' \geq \wedge(C \cup C')$$

is clear. But, if $C, C' \subseteq \uparrow(\wedge(C \cup C'))$, so $\wedge C, \wedge C' \geq \wedge(C \cup C')$, from which it follows that $\wedge(C \cup C') \leq \wedge C \ \wedge \ \wedge C'$. Hence the two are equal, and $\wedge \colon \mathcal{Q}(D) \to D$ preserves finite, non-empty infima. $\square$

Theorem 4.6 *Let $\mathcal{C}_\wedge$ denote the category of consistently complete algebraic cpo's and Scott-continuous maps preserving finite, non-empty infima. The inclusion functor from $\mathcal{C}_\wedge$ to $\mathcal{A}$ has a left adjoint $\mathcal{Q} \colon \mathcal{C}_\wedge \to \mathcal{A}$ which associates to an algebraic cpo D the family $\mathcal{Q}(D)$ of Scott-compact saturated subsets of D, and to a Scott continuous map $f \colon D \to E$ the mapping $\mathcal{Q}(f) \colon \mathcal{Q}(D) \to \mathcal{Q}(E)$ given by $\mathcal{Q}(f)(X) = \uparrow f(X)$.*

Proof If D is an algebraic cpo, then the map $\eta_D \colon D \to \mathcal{Q}(D)$ by $\eta_D(x) = \uparrow x$ is Scott continuous: indeed, if $k \in K(x)$, then $\eta_D(k) = \uparrow k \supseteq \uparrow x = \eta_D(x)$, and $\eta_D(k) \in K(\mathcal{Q}(D))$ by Corollary 4.2.

If E is an algebraic cpo and $f \colon D \to E$ is Scott continuous, then $f(C)$ is compact for each $C \in \mathcal{Q}(D)$ since f is Scott continuous, and so $\mathcal{Q}(f)(C) = \uparrow f(C) \in \mathcal{Q}(E)$ follows. The characterization of the compact elements of $\mathcal{Q}(D)$ in Corollary 4.2 also implies that the map $\mathcal{Q}(f)$ is Scott continuous. Indeed, if $F \subseteq K(E)$ is finite and non-empty, then

$$\begin{aligned} \mathcal{Q}(f)^{-1}(\uparrow F) &= \{C \in \mathcal{Q}(D) \mid f(C) \subseteq \uparrow F\} \\ &= \{C \in \mathcal{Q}(D) \mid C \subseteq f^{-1}(\uparrow F)\}, \end{aligned}$$

which is Scott open in $\mathcal{Q}(D)$, by Corollary 4.2 and the fact that f is Scott continuous.

Next we show that $\mathcal{Q}(f)$ preserves finite, non-empty infima. Indeed, if $C, C' \in \mathcal{Q}(D)$, then

$$\begin{aligned} \mathcal{Q}(f)(C \wedge C') = \mathcal{Q}(f)(C \cup C') &= \uparrow f(C \cup C') \\ &= \uparrow(f(C) \cup f(C')) \\ &= \uparrow f(C) \cup \ \uparrow f(C') \\ &= \mathcal{Q}(f)(C) \wedge \mathcal{Q}(f)(C'). \end{aligned}$$

Now, given a Scott-continuous map $f: D \to E$ with E consistently complete, we can define the mapping $\hat{f}: \mathcal{Q}(D) \to E$ by $\hat{f}(X) = \bigwedge f(X)$. Our arguments above show that this map is Scott continuous and preserves finite, non-empty infima, since it is the composition of mappings with these properties. Moreover,

$$\hat{f}(\eta_D(x)) = \bigwedge f(\uparrow x) = f(x),$$

and $\hat{f}$ is the unique map from $\mathcal{Q}(D)$ to E with this property, since each non-empty compact saturated subset of D is the infimum in $\mathcal{Q}(D)$ of sets of the form $\uparrow F$, for $F \subseteq D$ finite, and f uniquely determines $\hat{f}$ on these sets. $\square$

Our presentation of the adjunction between $\mathcal{C}_\wedge$ and $\mathcal{A}$ is in terms of the compact saturated subsets of the algebraic cpo D, but this is just another representation of the Smyth power domain. This object was originally defined as the ideal completion of the family $\mathcal{P}_{<\omega}(K(D)$ of finite, non-empty subsets of the domain D under the preorder $F \sqsubseteq G$ if and only if $(\forall y \in G)(\exists x \in F)\ x \leq y$. In fact, Smyth [15] also has given the topological representation of the Smyth power domain that we have presented; for completeness sake, we include a proof that this representation is the same as the one via the finite non-empty subsets of the compact elements.

Theorem 4.7 ([15]) *The functor $\mathcal{Q}: \mathcal{C}_\wedge \to \mathcal{A}$ is equivalent to the Smyth power domain functor.*

Proof Recall that the Smyth power domain $\mathcal{P}_S(D)$ over D is the ideal completion of the family of finite, non-empty subsets of $K(D)$ in the order given by $F \sqsubseteq G$ if and only if $(\forall y \in G)(\exists x \in F)\ x \leq y$, which is equivalent to $G \subseteq \uparrow F$. Thus, the map sending such a set F to $\uparrow F$ is an order isomorphism of $K(\mathcal{P}_S(D))$ onto $K(\mathcal{Q}(D))$, according to Corollary 4.2. It follows that the mapping extends to an order-isomorphism of $\mathcal{P}_S(D)$ onto $\mathcal{Q}(D)$. $\square$

The next adjunction we present relates a second lluf subcategory of $\mathcal{C}$. This time the morphisms are those Scott-continuous maps which preserve all non-empty infima. This is a reasonable family of morphisms to consider, since consistently complete cpo's are closed under the formation of all such infima. In fact, this observation makes clear the remark that the only difference between consistently complete algebraic cpo's and algebraic lattices is whether there is a largest element. Indeed, every consistently complete algebraic cpo gives rise to an algebraic lattice whose largest element is compact, by simply adding such a largest element to the domain. Conversely, each consistently complete algebraic cpo arises from such an algebraic lattice by deleting the largest element.

If D and E are coherent algebraic cpo's and $f: D \to E$ is Scott continuous, then we can characterize in completely topological terms when f preserves filtered infima. This may not seem so surprising when one realizes that preservation of filtered infima is the same as saying $f: D^{op} \to E^{op}$ is Scott continuous. Nonetheless, the result which we obtain does not refer to the dual Scott topology at all. We first bring some additional topological tools to bear.

Definition Let D be an algebraic cpo. The *lower topology* on D has the family $\{\uparrow x \mid x \in D\}$ as a subbasis for the family of closed sets; this topology is denoted $\omega(D)$. The *Lawson topology* on D, denoted $\lambda(D)$, is the common refinement of $\sigma(D)$ and $\omega(D)$.

It is well-known that the Lawson topology on an algebraic cpo D is Hausdorff, and it can be shown using Theorem 4.1 that the Lawson topology is compact if and only if D is coherent (cf. [5]).

Proposition 4.8 *Let $f: D \to E$ be a Scott-continuous map between algebraic cpo's. Consider the following conditions:*

1. *For each $k \in K(E)$, the set $f^{-1}(\uparrow k)$ is Scott compact.*

2. The mapping f is Lawson continuous.

Then 1) implies 2), and if D is coherent, then the conditions are equivalent. In any case, if either of these conditions are satisfied, then the mapping f preserves those filtered infima which exist.

Proof Suppose that 1) holds. If $k \in K(E)$, then $f^{-1}(\uparrow k)$ is Scott compact. But since f is Scott-continuous and k is compact, this set also is Scott-open. The only Scott-clopen subsets of an algebraic cpo are the sets of the form $\uparrow F$ for some finite subset $F \subseteq K(D)$, and so $f^{-1}(\uparrow k)$ has this form. But, this means that $f^{-1}(\uparrow k)$ is closed in the lower topology of D, and hence it is closed in the Lawson topology. Now, if $x \in E$ is an arbitrary element, then $x = \bigsqcup K(x)$ implies that $\uparrow x = \bigcap\{\uparrow k \mid k \in K(x)\}$. Thus,

$$\begin{aligned} f^{-1}(\uparrow x) &= f^{-1}(\bigcap\{\uparrow k \mid k \in K(x)\} \\ &= \bigcap\{f^{-1}(\uparrow k) \mid k \in K(x)\}, \end{aligned}$$

which means that $f^{-1}(\uparrow x)$ is the intersection of closed set, and so it also is closed in the lower, hence Lawson topology. Alexander's Subbasis Theorem then implies that f is Lawson continuous, which shows 2) holds.

Conversely, suppose that 2) holds and that D is coherent. Then D is Lawson compact. Now, given $k \in K(E)$, the set $f^{-1}(\uparrow k)$ is Scott open since f is Scott continuous, and it is Lawson closed since f is Lawson continuous. But, since D is Lawson compact and Hausdorff, the closed subset $f^{-1}(\uparrow k)$ is also compact in the Lawson topology. Since f is monotone, this set also is an upper set, and it is then an easy exercise to show that $f^{-1}(\uparrow k)$ is Scott compact. That is, $f^{-1}(\uparrow k)$ is Scott open and Scott compact. The former property means that $f^{-1}(\uparrow k) = \bigcup\{\uparrow k' \mid k' \in K(D) \cap f^{-1}(\uparrow k)\}$, and since each of these sets is Scott open, there is some finite subfamily $F \subseteq K(D) \cap f^{-1}(\uparrow k)\}$ such that $f^{-1}(\uparrow k) = \uparrow F$. Thus 1) holds.

Finally, we show that 2) also implies that f preserves filtered infima. Indeed, since f is monotone, if $F \subseteq D$ is a filtered set and $\wedge F$ exists, then $f(\wedge F) \leq f(x)$ for every $x \in F$. But, if $y \in E$ is a lower bound for $f(F)$, then $\uparrow y$ is Lawson closed in E, so the same is true of $f^{-1}(\uparrow y)$. Since f is monotone, this set also is an upper set, and it contains the set F. So, we will be done if we show that filtered sets which have an infimum converge to their infimum in the Lawson topology. And, since principal upper sets are closed, any limit point of a filtered set must be in the upper set of the infimum of the set. But, if G is filtered and $\wedge G$ exists, then any Lawson-open set containing $\wedge G$ has the form $\uparrow k \setminus \uparrow B$, where B is some finite subset of D. Since $\wedge G \notin \uparrow B$ and B is finite, it follows from the fact that G is filtered that $G \not\subseteq \uparrow B$, and this means there is some $g \in G$ such that $g' \in G$ and $g \geq g'$ imply $g' \notin \uparrow B$. But, $k \leq \wedge G$ implies that $G \subseteq \uparrow k$, and so $g' \in \uparrow k \setminus \uparrow B$ for all $g' \in G$ with $g' \leq g$. Thus G converges to $\wedge G$ in the Lawson topology, which concludes our proof. □

Corollary 4.9 *If D is a consistently complete domain, then $\wedge\colon \mathcal{Q}(D) \to D$ preserves filtered infima.*

Proof Proposition 4.5 implies that $\wedge\colon \mathcal{Q}(D) \to D$ is Scott continuous, so $\wedge^{-1}(\uparrow k)$ is Scott open for each $k \in K(D)$. And, Proposition 4.8 implies that, to complete the proof, we only need to show that for each $k \in K(D)$, there is some finite set of F compact elements of $\mathcal{Q}(D)$ such that $\wedge^{-1}(\uparrow k) = \uparrow F$. But, for each $x \in D$, the set

$$\begin{aligned} \{C \in \mathcal{Q}(D) \mid \wedge C \in \uparrow x\} &= \{C \in \mathcal{Q}(D) \mid x \leq \wedge C\} \\ &= \{C \in \mathcal{Q}(D) \mid C \subseteq \uparrow x\}. \end{aligned}$$

So, if $x \in K(D)$, then $\uparrow x \in K(\mathcal{Q}(D))$, and so this set is nothing more than $\{C \in \mathcal{Q}(D) \mid \uparrow x \sqsubseteq C\}$, which is the upper set of $\uparrow x$ in $\mathcal{Q}(D)$. □

Theorem 4.10 *If $\mathcal{C}_\sqcap$ denotes the lluf subcategory of $\mathcal{C}$ whose morphisms are those Scott-continuous maps preserving all non-empty infima, then the restriction of the functor $\mathcal{Q}$ to the subcategory $\mathcal{COH}$ of coherent algebraic cpo's and monotone, Lawson continuous maps is left adjoint to the inclusion functor from $\mathcal{C}_\sqcap$ to $\mathcal{COH}$.*

Proof We only need to show that $\mathcal{Q}$ actually restricts to the named categories. Proposition 4.8 implies this amounts to showing that $\mathcal{Q}(f)$ is Lawson continuous for each $\mathcal{COH}$-morphism f. If $f\colon D \to E$ is a monotone, Lawson continuous map between coherent domains, then $K(\mathcal{Q}(E)) = \{\uparrow F \mid \emptyset \neq F \subseteq K(E) \text{ is finite}\}$. Given such a set, $\uparrow F$, we calculate

$$\begin{aligned} f^{-1}(\{C \in \mathcal{Q}(E) \mid C \subseteq \uparrow F\}) &= \{C \in \mathcal{Q}(D) \mid f(C) \subseteq \uparrow F\} \\ &= \{C \in \mathcal{Q}(D) \mid C \subseteq f^{-1}(\uparrow F)\}. \end{aligned}$$

But, for any such F,

$$f^{-1}(\uparrow F) = f^{-1}(\cup_{k \in F} \uparrow k) = \cup_{k \in F} f^{-1}(\uparrow k),$$

and each $f^{-1}(\uparrow k)$ is of the form $\uparrow F_k$ for some finite set of compact elements of D, by Proposition 4.8. It follows that

$$f^{-1}(\uparrow F) = \{C \in \mathcal{Q}(D) \mid C \subseteq \cup_{k \in F} \uparrow F_k\} = \{C \in \mathcal{Q}(D) \mid C \subseteq \uparrow(\cup_{k \in F} F_k)\},$$

which is the upper set in $\mathcal{Q}(D)$ of a compact element of $\mathcal{Q}(D)$. Thus, Proposition 4.8 implies that $\mathcal{Q}(f)$ is Lawson continuous. □

5 A Generalization of the Hoare Power Domain

As we have pointed out in Section 3, the Hoare power domain functor is a left adjoint to the inclusion functor from the category of algebraic lattices and maps preserving all non-empty suprema to the category of algebraic cpo's. In this section, we generalize this construction by constructing a free functor whose target domain is the lluf subcategory of $\mathcal{C}$ whose morphisms preserve all existing non-empty suprema. For an algebraic cpo D, if we wish to associate to D an algebraic sub-cpo of $\Gamma(D)$, an obvious subset of $\Gamma(D)$ to try (other than D itself, which buys us nothing), is the collection of bounded, non-empty Scott-closed subsets:

$$\downarrow\eta_D(D) = \{X \in \Gamma(D) \mid (\exists x \in D)\ X \subseteq \downarrow x\}.$$

Unfortunately, this is not necessarily a cpo.

Example Let $D = (\{0,1\} \times \mathbb{N}) \cup \{\bot\}$ with the order:

$$(i,m) \sqsubseteq (j,n) \quad \text{if and only if} \quad (i = 0\ \&\ j = 1\ \&\ m \leq n) \text{ or } (i = j\ \&\ m = n),$$

with $\bot \sqsubseteq d$ $(\forall d \in D)$, of course. Then the sets

$$X_n = \downarrow\{(0,m) \mid m \leq n\},\ n > 0$$

form an increasing family of Scott-closed sets, and $X_n \subseteq \downarrow(1,n)$ $(\forall n > 0)$. Thus, $X_n \in \downarrow\eta_D(D)$ $(\forall n > 0)$, but

$$\bigsqcup_n X_n = \downarrow(\{0\} \times \mathbb{N}) \notin \downarrow\eta_D(D),$$

since there is no element of D which dominates all of $\{0\} \times \mathbb{N}$. □

But, the Scott closure of $\eta_D(D)$ is certainly a sub-cpo of $\Gamma(D)$, and it is an algebraic cpo, as we now show.

Theorem 5.1 *Let D be an algebraic cpo, Then the following hold:*

i) *The set*

$$\mathbf{C}(D) = \overline{\downarrow\eta_D(D)} = \overline{\{X \in \Gamma(D) \mid (\exists x \in D)\ X \subseteq \downarrow x\}}$$

is an algebraic sub-cpo of $(\Gamma(D), \cup)$. In particular, $\mathbf{C}(D)$ is consistently complete.

ii) $K(\mathbf{C}(D)) = \mathbf{C}(D) \cap K(\Gamma(D))$, *and there is a natural map $\iota_D: D \to \mathbf{C}(D)$ which is a homeomorphism of D onto its image in $\mathbf{C}(D)$.*

iii) $\mathbf{C}(D)$ *is $\omega-$algebraic if and only if D is.*

Proof Theorem 3.1 implies $\Gamma(D)$ is an algebraic lattice, and this implies that the Scott-closed subset (and hence, lower set) $\mathbf{C}(D)$ is an algebraic sub-cpo of $\Gamma(D)$ whose set of compact elements is $\mathbf{C}(D) \cap K(\Gamma(D))$. Thus $\mathbf{C}(D)$ is consistently complete since it is a Scott-closed subset of the complete lattice $\Gamma(D)$. This shows that i) and the first part if ii) hold.

We conclude the rest of ii) by defining ι_D to be the corestriction of η_D to $\mathbf{C}(D)$. This is a homeomorphism from D into $\mathbf{C}(D)$, since η_D is a homeomorphism and the Scott topology on $\mathbf{C}(D)$ is the inherited topology from the Scott topology of $(\Gamma(D), \cup)$ (this follows from the relation of the compact elements of $\mathbf{C}(D)$ and those of $(\Gamma(D), \cup)$). Finally, iii) follows from the description of $K(\Gamma(D), \cup)$ in Theorem 3.1, which implies this set is countable if and only if $K(D)$ is. □

Of course, we want more of the cpo $\mathbf{C}(D)$ than that it exists. It also is to be universal for D among consistently complete algebraic cpo's and Scott continuous maps preserving existing non-empty suprema. To show this, we first establish the following result:

Proposition 5.2 *If D and E are algebraic cpo's and $f: D \to E$ is a Scott-continuous map, then there is a map $\mathbf{C}(f): \mathbf{C}(D) \to \mathbf{C}(E)$ preserving all existing non-empty suprema such that $\mathbf{C}(f) \circ \iota_D = \iota_E \circ f$.*

Proof We commented in Section 3 that the map $\Gamma(f): \Gamma(D) \to \Gamma(E)$ given by $\Gamma(f)(X) = \overline{f(X)}$ for each $X \in \Gamma(D)$ preserves all non-empty suprema. We also showed that $\Gamma(f)$ has a (Scott-continuous) restriction and corestriction $\Gamma(f): \mathrm{Pr}_\cup\Gamma(D) \to \mathrm{Pr}_\cup\Gamma(E)$ such that $\Gamma(f)\circ\eta_D = \eta_E\circ f$. But, we can also restrict and corestrict $\Gamma(f)$ to a map $\mathbf{C}(f): \mathbf{C}(D) \to \mathbf{C}(E)$: indeed, $\mathrm{Pr}_\cup\Gamma(D) \subseteq \mathbf{C}(D) \subseteq \Gamma(D)$ and $\mathrm{Pr}_\cup\Gamma(E) \subseteq \mathbf{C}(E) \subseteq \Gamma(E)$. So, to show that this mapping preserves all existing non-empty suprema in $\mathbf{C}(D)$, we only need to show that $\Gamma(f)(\mathbf{C}(D)) \subseteq \mathbf{C}(E)$. But, if $X \in \mathbf{C}(D)$, then there is some $x \in D$ with $X \subseteq \downarrow x$, and then $\Gamma(f)(X) \subseteq \downarrow f(x)$. Finally, the condition $\mathbf{C}(f) \circ \iota_D = \iota_E \circ f$ follows from the corresponding condition of $\Gamma(f)$ and the fact that ι_D is a corestriction of η_D. □

In the case that D already is consistently complete, we can conclude more about the relation between D and $\mathbf{C}(D)$.

Proposition 5.3 *If D be a consistently complete algebraic cpo, then there is a retraction $\chi_D: \mathbf{C}(D) \to D$ such that $\chi_D \circ \iota_D = 1_D$ and $\iota_D \circ \chi_D \geq 1_{\mathbf{C}(D)}$. That is, D is the image of $\mathbf{C}(D)$ under a closure operator on $\mathbf{C}(D)$.*

Proof Let D be consistently complete, and let $X \in \Gamma(D)$. By definition, $\mathbf{C}(D) = \overline{\downarrow\eta_D(D)}$, and Theorem 5.1 states that $K(\mathbf{C}(D)) \subseteq \downarrow\eta_D(D)$. Now, $Y \in \downarrow\eta_D(D)$ implies there is some $x \in D$ with $Y \subseteq \downarrow x$, so, for each $Y \in K(\mathbf{C}(D))$, there is an element $x \in D$ with $Y \subseteq \downarrow x$. Since D is consistently complete, it follows that $\vee Y$ exists in D. We can therefore define a monotone map $\chi_0: K(\mathbf{C}(D)) \to D$ by $\chi_0(Y) = \vee Y$. And this map then extends to a continuous map $\chi_D: \mathbf{C}(D) \to D$. It is routine to verify that $\chi_D \circ \iota_D = 1_D$ and $\iota_D \circ \chi_D \geq 1_{\mathbf{C}(D)}$. □

Theorem 5.4 *If $\mathcal{C}_\vee$ is the category of consistently complete algebraic cpo's and maps preserving all existing non-empty suprema, then the functor $\mathbf{C}: \mathcal{A} \to \mathcal{C}_\vee$ is left adjoint to the inclusion functor from $\mathcal{C}_\vee$ to $\mathcal{A}$.*

Proof Given D an algebraic cpo, the mapping $\iota_D: D \to \mathbf{C}(D)$ is Scott continuous by Theorem 5.1. Given $f: D \to E$ from D to the Scott domain E, Propositions 5.2 and 5.3 imply the mapping $\hat{f} = \chi_E \circ \mathbf{C}(f): \mathcal{C}(D) \to E$ preserves all existing non-empty suprema, since it is a composition of maps with this property. Moreover, these results also show that $\hat{f} \circ \iota_D = f$, and $\hat{f}$ is the unique map from $\mathbf{C}(D)$ to E with this property, since $\iota_D(D)$ sup-generates $\mathbf{C}(D)$. □

If D is an algebraic cpo with a largest element, then clearly $\mathbf{C}(D) = \Gamma(D)$, and the converse also is easy to establish.

Corollary 5.5 i) *If $\mathcal{C}_{0,\vee}$ is the subcategory of $\mathcal{C}_0$ of Scott domains and maps preserving all existing non-empty suprema, then the restriction and corestriction $\mathbf{C}: \mathcal{SFP} \to \mathcal{C}_{0,\vee}$ is left adjoint to the inclusion functor from $\mathcal{C}_0$ to $\mathcal{SFP}$.*

ii) *Similarly, if $\mathcal{LD}$ is the full subcategory of $\mathcal{A}$ of algebraic L-domains (i.e., of those algebraic cpo's D such that $\downarrow x$ is a complete lattice for each $x \in D$), then the restriction and corestriction $\mathbf{C}: \mathcal{LD} \to \mathcal{C}_{0,\vee}$ is left adjoint to the inclusion functor from $\mathcal{C}_0$ to $\mathcal{LD}$.*□

Jung [5] has shown that the only maximal cartesian closed categories of $\mathcal{A}$ are the full subcategory $\mathcal{SFP}$ of $\mathcal{SFP}$-objects and Scott-continuous maps, and the full subcategory $\mathcal{LD}$ of algebraic L-domains and Scott-continuous maps.

Our next result gives an improved characterization of $\mathbf{C}(D)$ for certain algebraic cpo's D.

Definition Let D be an algebraic cpo. We say D is *conditionally bounded* if, for each set $X \subseteq D$, whenever every finite subset of X has an upper bound, then X has an upper bound.

It is not hard to show that an algebraic cpo D is conditionally bounded if and only if for each lower set $X = \downarrow X$ in D, X has an upper bound whenever every finite set of compact elements of X has an upper bound. Our interest in conditionally bounded cpo's is made clear by the following.

Proposition 5.6 *Let D be an algebraic cpo. The following are equivalent:*

i) *D is conditionally bounded.*

ii) *$\mathbf{C}(D) = \downarrow \eta_D(D)$.*

iii) *If $X \subseteq K(D)$ satisfies each finite subset of X has an upper bound in D, then there is an order-ideal I of $K(D)$ for which $X \subseteq I$.*

Proof The equivalence of the first two conditions follows from part ii) of Theorem 5.1. Indeed, that result says that a Scott-closed set $X \subseteq D$ is in $\mathbf{C}(D)$ if and only if $\downarrow F$ is in $\downarrow \eta_D(D)$ for every finite set F of compact elements from X, and this is equivalent to the property that every finite subset F of compact elements from X has an upper bound. Now, an arbitrary lower set X has an upper bound if and only if its closure does, and so the remarks just prior to this Proposition imply that D is conditionally bounded if and only if every Scott-closed set X has an upper bound whenever each of its finite subsets of compact elements has an upper bound, which is equivalent to $X \in \downarrow \eta_D(D)$.

Now, if D is conditionally bounded and $X \subseteq K(D)$ satisfies each finite subset of X has an upper bound in D, then X has an upper bound $x \in D$. Then $X \subseteq K(x)$, and $K(x)$ is the desired order-ideal of $K(D)$, so iii) holds.

Conversely, if condition iii) holds and $X \subseteq D$ satisfies the property that each finite subset of X has an upper bound, then the same is true of $Y = \downarrow X \cap K(D)$. Then there is an order-ideal $I \subseteq K(D)$ with $Y \subseteq I$, and since D is a cpo, $y = \bigsqcup Y$ exists. Since D is algebraic, $X \subseteq \downarrow \overline{Y} \subseteq \downarrow y$, so y is the desired upper bound of X. □

We now show that the class of conditionally bounded algebraic cpo's is reasonably large, and we begin with the following result gives a topological characterization of these cpo's.

Proposition 5.7 (Keimel) *An algebraic cpo D is conditionally bounded if and only if D is compact in the lower topology.*

Proof The closed subsets for the lower topology on D have the family $\{\uparrow x \mid x \in D\}$ for a subbasis, so D is compact in the lower topology if and only if every non-empty family of principal upper sets satisfying the finite intersection property has a non-empty intersection. But, this condition is clearly the same as the condition that D be conditionally bounded. $\square$

Corollary 5.8 *Any coherent domain is conditionally bounded.*

Proof A coherent domain is one which is compact in the Lawson topology. Since the Lawson topology refines the lower topology, any domain which is Lawson compact also is lower compact, so Proposition 5.7 implies such a domain also is conditionally bounded. $\square$

Not all conditionally bounded domains are coherent; for example, the L-domain $D = \{\bot, a, b\} \cup \mathbb{N}$ where $\bot < a, b < n$ for all $n \in \mathbb{N}$, and no other points are related is conditionally bounded, but not coherent. Likewise, not all L-domains are conditionally bounded; indeed, the example at the beginning of this section is an L-domain D for which $\mathbf{C}(D) \neq \downarrow\eta_D(D)$.

6 $\mathbf{C}(D)$ as an Information System

One of the advantages of working with Scott domains is their convenient representation in terms of information systems. In this representation, one can solve recursive domain equations up to equality, rather than simply up to isomorphism. In this section we give the information system representation of the Scott domain $\mathbf{C}(D)$ for any $\omega-$algebraic cpo D.

Definition An *information system* is a quadruple $\mathcal{A} = (A, \bot, Con, \vdash)$, where A is a countable set, $\bot$ is an element of A, Con is a family of non-empty finite subsets of A, and $\vdash \subseteq Con \times A$ is the entailment relation, satisfy the following properties:

i) $a \in A \;\Rightarrow\; \{a\} \in Con.$

ii) $X \in Con \;\&\; Y \subseteq X \;\Rightarrow\; Y \in Con.$

iii) $X \in Con$ implies $X \vdash \bot$.

iv) $X \vdash a \;\Rightarrow\; X \cup \{a\} \in Con.$

v) $X \in Con \;\&\; a \in X \;\Rightarrow\; X \vdash a.$

vi) $X, Y \in Con \;\&\; ((\forall b \in Y)\; X \vdash b) \;\&\; Y \vdash c \;\Rightarrow\; X \vdash c.$

A subset $x \subseteq A$ is *consistent* if $X \in Con$ for each finite subset $X \subseteq x$, and x is *deductively closed* if, for $X \subseteq x$ finite and $a \in A$ with $X \vdash a$, it follows that $a \in x$. If $X \in Con$ and $F \subseteq A$ is a finite subset satisfying $X \vdash a$ for each $a \in F$, then we write $X \vdash F$. The *elements* $|\mathcal{A}|$ of the information system $\mathcal{A}$ is the family

$$|\mathcal{A}| \;=\; \{x \subseteq A \mid x \text{ is consistent and deductively closed}\}.$$

Theorem 6.1 (Larsen and Winskel [7])

i) *If $\mathcal{A}$ is an information system, then the elements $|\mathcal{A}|$ form a Scott domain under set inclusion, and the set of compact elements of $|\mathcal{A}|$ is*

$$K(|\mathcal{A}|) = \{\overline{X} \mid X \in Con\},$$

where $\overline{X} = \{y \subseteq A \mid (\forall F \subseteq y \text{ finite})\ X \vdash F\}$ is the deductive closure *of X.*

ii) *Conversely, if D is a Scott domain, then $\mathcal{A}_D = (K(D), \perp_D, Con_D, \vdash_D)$ is an information system, where $Con_D = \{F \subseteq K(D) \mid F$ has an upper bound$\}$, $\vdash_D = \{(X, a) \mid a \sqsubseteq \vee X\}$, and $x \mapsto \vee x \colon |\mathcal{A}| \to D$ is an isomorphism of domains.* □

Theorem 6.2 *Let D be an $\omega-$algebraic cpo, and define*

i) *$\mathcal{P}_{<\omega}(K(D))$ to be the family of non-empty finite subsets of $K(D)$,*

ii) *$Con_{\mathbf{C}(D)} = \{F \in \mathcal{P}_{<\omega}(K(D)) | (\exists k \in K(D))\ F \subseteq \downarrow k\}$ to be the family of bounded, non-empty, finite subsets of $K(D)$, and*

iii) *$\vdash_{\mathbf{C}(D)} = \{(X, k) \in \mathcal{P}_{<\omega}(K(D)) \times K(D) \mid k \in \downarrow X\}$.*

Then the quadruple $\mathcal{A}_{\mathbf{C}(D)} = (\mathcal{P}_{<\omega}(K(D)), \perp, Con_{\mathbf{C}(D)}, \vdash_{\mathbf{C}(D)})$ is an information system, and the map

$$x \mapsto \overline{x}^{\sigma} \colon |\mathcal{A}_{\mathbf{C}(D)}| \to \mathbf{C}(D)$$

sending each $x \in |\mathcal{A}_{\mathbf{C}(D)}|$ to its Scott closure in D is an isomorphism.

Proof It is routine to verify that $\mathcal{A}_{\mathbf{C}(D)}$ is an information system. For each $F \in Con_{\mathbf{C}(D)}$, the subset $\downarrow F$ is Scott closed in D, and it is an element of $K(\Gamma(D))$. So, the map $F \mapsto \downarrow F \colon Con_{\mathbf{C}(D)} \to \Gamma(D)$ is a monotone map, and its image is within $\mathbf{C}(D)$ since F is bounded. If $\downarrow F = \downarrow G$ for elements $F, G \in Con$, then $\overline{F} = \overline{G}$ by the definition of $\vdash_{\mathbf{C}(D)}$. Thus the map also is one-to-one. This map is clearly a surjection onto $K(\mathbf{C}(D))$, so it is an isomorphism. Therefore the map extends to an isomorphism $x \mapsto \overline{x}^{\sigma} \colon |\mathcal{A}_{\mathbf{C}(D)}| \to \mathbf{C}(D)$. □

In the information system approach, Scott-continuous functions between Scott domains can be encoded into the information systems themselves, as *approximable relations*, rather than as functions.

Definition If $\mathcal{A} = (A, Con_{\mathcal{A}}, \vdash_{\mathcal{A}})$ and $\mathcal{B} = (B, Con_{\mathcal{B}}, \vdash_{\mathcal{B}})$ are information systems, then an *approximable relation* $\mathcal{R} \subseteq Con_{\mathcal{A}} \times Con_{\mathcal{B}}$ satisfies

i) $\emptyset\, \mathcal{R}\, \emptyset$.

ii) $X\, \mathcal{R}\, Y \ \&\ X\, \mathcal{R}\, Z \ \Rightarrow\ X\, \mathcal{R}\, (Y \cup Z) \quad (\forall X \in Con_{\mathcal{A}}\ \&\ \forall Y, Z \in Con_{\mathcal{B}})$.

iii) $X \vdash_{\mathcal{A}} Y,\ Y\, \mathcal{R}\, Z,\ \&\ Z \vdash_{\mathcal{B}} W \ \Rightarrow\ X\, \mathcal{R}\, W \quad (\forall X, Y \in Con_{\mathcal{A}}\ \&\ \forall Z, W \in Con_{\mathcal{B}})$.

Proposition 6.3 (Larsen and Winskel [7])

i) *If $\mathcal{A} = (A, \perp_A, Con_{\mathcal{A}}, \vdash_{\mathcal{A}})$ and $\mathcal{B} = (B, \perp_B, Con_{\mathcal{B}}, \vdash_{\mathcal{B}})$ are information systems and $\mathcal{R} \subseteq Con_{\mathcal{A}} \times Con_{\mathcal{B}}$ is an approximable relation, then the function*

$$f_{\mathcal{R}} \colon |\mathcal{A}| \to |\mathcal{B}| \quad by \quad f_{\mathcal{R}}(x) = \bigcup \{Y \in Con_{\mathcal{B}} \mid (\exists X \subseteq x \text{ finite})\ X\, \mathcal{R}\, Y\}$$

is a Scott-continuous map.

ii) *Conversely, given a Scott-continuous map* $f: D \to E$ *between Scott domains* D *and* E, *the relation*

$$\mathcal{R}_f = \{(X,Y) \in Con_D \times Con_E \mid Y \vdash_E f(X)\}$$

is an approximable relation which gives rise to f. □

Proposition 6.4 *Let* D *and* E *be algebraic cpo's and let* $f: D \to E$ *be a Scott-continuous map. Then the relation* $\mathcal{R}_{\hat{f}} \subseteq Con_{\mathbf{C}(D)} \times Con_{\mathbf{C}(E)}$ *is given by*

$$\mathcal{R}_{\hat{f}} = \{(X,Y) \in Con_{\mathbf{C}(D)} \times Con_{\mathbf{C}(E)} \mid Y \subseteq f(X)\}$$

is an approximable relation from $\mathcal{A}_{\mathbf{C}(D)}$ *to* $\mathcal{A}_{\mathbf{C}(E)}$ *which gives rise to the Scott-continuous extension* $\hat{f}: \mathbf{C}(D) \to \mathbf{C}(E)$.

Proof Routine. □

7 Nondeterministic Algebras and Nondeterministic Semigroups

In order to give semantic models for languages which support nondeterminism, some construct analogous to the power set operator is required. The constructs which have emerged as being the most useful are the three *power domains*: the Hoare power domain, the Smyth power domain, and the Plotkin power domain. While the Hoare and Smyth power domains over a Scott domain D are again Scott domains, this is not true of the Plotkin power domain over D. This led Plotkin [11] to investigate the category $\mathcal{SFP}$ of $\mathcal{SFP}$-objects and Scott-continuous maps, a category which is somewhat larger and a great deal more complicated (structurally) than the category $\mathcal{C}_0$. In [3], Hennessy and Plotkin characterize the three power domain constructs in terms of free functors, i.e., in terms of left adjoints to various inclusion functors.

Definition A *nondeterministic algebra* is an algebraic cpo D together with an added operation $+: D \times D \to D$ which is commutative, associative and idempotent, and continuous with respect to the Scott topologies. For nondeterministic algebras $(D, +_D)$ and $(E, +_E)$, a *nondeterministic algebra map* is a Scott-continuous map $f: D \to E$ such that $f(x +_D y) = f(x) +_E f(y)$, $(\forall x, y \in D)$. We denote by $\mathcal{ND}$ the category of nondeterministic algebras and nondeterministic algebra maps.

Theorem 7.1 (Hennessy and Plotkin [3])

i) *Let* $\mathcal{ND}_{\mathcal{H}}$ *be the full subcategory of* $\mathcal{ND}$ *of all nondeterministic algebras* $(D, +)$ *satisfying* $x \leq x + y$, $(\forall x, y \in D)$. *Then the functor* $P_{\mathcal{H}}$ *which associates to an algebraic cpo* D *the Hoare power domain* $P_{\mathcal{H}}(D)$ *is left adjoint to the inclusion functor from* $\mathcal{ND}_{\mathcal{H}}$ *to* $\mathcal{A}$.

ii) *Let* $\mathcal{ND}_{\mathcal{S}}$ *denote the full subcategory of* $\mathcal{ND}$ *of all nondeterministic algebras* $(D, +)$ *satisfying* $x + y \leq x$, $(\forall x, y \in D)$. *Then the functor* $P_{\mathcal{S}}$ *which associates to an algebraic cpo* D *the Smyth power domain* $P_{\mathcal{S}}(D)$ *is left adjoint to the inclusion functor from* $\mathcal{ND}_{\mathcal{S}}$ *to* $\mathcal{A}$.

iii) *The functor* $P_{\mathcal{P}}$ *which associates to an algebraic cpo* D *the Plotkin power domain* $P_{\mathcal{P}}(D)$ *is left adjoint to the inclusion functor from* $\mathcal{ND}$ *to* $\mathcal{A}$. □

To be a nondeterministic algebra, a domain D must have a continuous binary operation which is commutative, associative and idempotent; i.e., D must be a topological semilattice in the Scott topology. The typical example is the Hoare power domain of a domain D, which is simply the family $\Gamma(D)$ of non-empty Scott-closed subsets of D, with the semilattice operation being the union operation. The binary operation on D is to be used to give a semantic meaning to the nondeterministic choice operator in the language which D is to model. For reasons which will become apparent below, we need to generalize the concept of nondeterministic algebra somewhat.

Definition A *nondeterministic semigroup* is an algebraic cpo D together with a Scott-continuous binary operation $\oplus: D \times D \to D$ which is commutative and associative. A Scott-continuous map $f: D \to E$ between nondeterministic semigroups $(D, \oplus_D)$ and $(E, \oplus_E)$ is a *nondeterministic semigroup map* if $f(x \oplus_D y) = f(x) \oplus_E f(y)$, for all $x, y \in D$.

For a domain D, the Scott domain $\mathbf{C}(D)$ is a subfamily of the Scott-closed subsets of D, but it is not closed under union; indeed, if $x, y \in D$ have no common upper bound, then $\downarrow x, \downarrow y \in \mathbf{C}(D)$, but $\downarrow x \cup \downarrow y \notin \mathbf{C}(D)$. However, if D is a nondeterministic semigroup, then the binary operation on D does induce a binary operation on $\mathbf{C}(D)$. To understand how this works, we introduce the concept of a tensor product of Scott domains.

Definition Let D, E and F be consistently complete algebraic cpo's. A *bi-semilattice map* is a Scott-continuous map $\phi: D \times E \to F$ such that, for each $x_0 \in D$ and each $y_0 \in E$, the mappings $x \mapsto \phi(x, y_0): D \to F$ and $y \mapsto \phi(x_0, y): E \to F$ preserve existing non-empty suprema.

Definition Let D and E be consistently complete algebraic cpo's. A $\mathcal{C}_\vee$*−tensor product* of D and E is a consistently complete algebraic cpo $D \otimes E$ and a bi-semilattice map $i_{(D,E)}: D \times E \to D \otimes E$ such that, for each consistently complete algebraic cpo F and each bi-semilattice map $f: D \times E \to F$, there is a unique map $F: D \otimes E \to F$ in $\mathcal{C}$ such that $F \circ i_{(D,E)} = f$.

Theorem 7.2 *For algebraic cpo's D and E, the consistently complete algebraic cpo $\mathbf{C}(D \times E)$ together with the embedding*

$$i: \mathbf{C}(D) \times \mathbf{C}(E) \hookrightarrow \mathbf{C}(D \times E) \quad \textit{given by} \quad i(X, Y) = X \times Y$$

is a $\mathcal{C}_\vee$−tensor product of the consistently complete algebraic cpo's $\mathbf{C}(D)$ and $\mathbf{C}(E)$.

Proof Given algebraic cpo's D and E, the mapping $i: \mathbf{C}(D) \times \mathbf{C}(E) \hookrightarrow \mathbf{C}(D \times E)$ by $i(X, Y) = X \times Y$ is readily seen to preserve directed suprema, since the closed and bounded subsets of $D \times E$ which are products of closed and bounded subsets of D with those of E are closed under all intersections and closures of increasing unions. This mapping also is a bi-semilattice map, for if we restrict, say, the first coordinate to $X_0 \in \mathbf{C}(D)$, then the mapping

$$Y \mapsto X_0 \times Y: \mathbf{C}(E) \to \mathbf{C}(D \times E)$$

is easily seen to preserve existing non-empty suprema. Thus, $i: \mathbf{C}(D) \times \mathbf{C}(E) \hookrightarrow \mathbf{C}(D \times E)$ is a bi-semilattice map.

Now, let D' be an Scott domain, and let $f: \mathbf{C}(D) \times \mathbf{C}(E) \to D'$ be a bi-semilattice map. Then, the map $\iota_D \times \iota_E: D \times E \to \mathbf{C}(D) \times \mathbf{C}(E)$ defined by $(\iota_D \times \iota_E)(x, y) = (\downarrow x, \downarrow y)$ is Scott continuous, being the product of such maps. Hence,

$$f \circ (\iota_D \times \iota_E): D \times E \to D'$$

is Scott continuous, and so Theorem 5.4 implies there is a unique Scott-continuous map $F: \mathbf{C}(D \times E) \to D$ which preserves existing non-empty suprema and satisfying $F \circ \iota_{D \times E} = f \circ (\iota_D \times \iota_E)$. Since $i \circ (\iota_D \times \iota_E) = \iota_{D \times E}$ (as is readily verified), it follows that $F \circ i = f$, as required. $\square$

Proposition 7.3 *Let $(D,+)$ be a nondeterministic semigroup. Then there is a Scott-continuous binary operation $\oplus\colon \mathbf{C}(D)\times\mathbf{C}(D)\to\mathbf{C}(D)$ such that $\iota_D(x)\oplus\iota_D(y)=\iota_D(x+y)$ for all $x,y\in D$. Moreover, if D is a nondeterministic algebra and if $X\in\mathbf{C}(D)$, then $X\oplus X=X$ if and only if $X\subseteq D$ is a Scott-closed, bounded $+-$ideal of D.*

Proof If we are given a Scott-continuous binary operation $+\colon D\times D\to D$, then the previous Theorem implies there is a Scott-continuous operation $\oplus\colon \mathbf{C}(D)\times\mathbf{C}(D)\to\mathbf{C}(D)$. The fact that $\oplus$ is commutative and associative is easy to derive.

Finally, assume that D is a nondeterministic algebra, and let $X\in\mathbf{C}(D)$. Then, X is idempotent under $\oplus$ if and only if $X\oplus X=X$, which is true if and only if $\overline{\{x+y\mid x,y\in X\}}=X$. Since $x=x+x\in\overline{\{x+y\mid x,y\in X\}}$ for each $x\in X$, this is clearly amounts to the property that $x+y\in X$ for each $x,y\in X$. Thus, $X\in\mathbf{C}(D)$ is idempotent if and only if X is a Scott-closed and bounded $+-$ideal of D. □

Definition A *Scott semigroup* is a Scott domain D with a commutative, associative and Scott-continuous operation $\oplus\colon D\times D\to D$. A *Scott-semigroup map* is a Scott-continuous map $f\colon D\to E$ between Scott semigroups $(D,\oplus_D)$ and $(E,\oplus_E)$ such that $f(x\oplus_D y)=f(x)\oplus_E f(y)$ for all $x,y\in D$.

Corollary 7.4 *If D is a Scott semigroup, then the map $\chi_D\colon\mathbf{C}(D)\to D$ of Proposition 5.3 is a semigroup homomorphism, and so D is the image of a closure operator on $\mathbf{C}(D)$ which preserves the semigroup operation.*

Proof Proposition 7.3 implies that $\mathbf{C}(D)$ is a Scott semigroup for any nondeterministic semigroup D, and the inclusion $\iota_D\colon D\to\mathbf{C}(D)$ satisfies $\iota_D(x)\oplus\iota_D(y)=\iota_D(x+y)$, for all $x,y\in D$. It is routine to verify that, if D is a Scott semigroup, then the retraction $\chi_D\colon\mathbf{C}(D)\to D$ also is a semigroup homomorphism. □

Of course, this allows us to conclude the universal property of $\mathbf{C}(D)$ for Scott semigroups.

Theorem 7.5 *The functor $\mathbf{C}$ induces a left adjoint to the inclusion functor from the category $\mathcal{SS}_\vee$ of Scott semigroups and Scott-semigroup maps preserving existing non-empty suprema to the category $\mathcal{NS}$ of nondeterministic semigroups and nondeterministic semigroup maps.*

Proof The universal property for $\mathbf{C}(D)$ follows in the same manner as in the proof of Theorem 5.4. □

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