A Primer on Domains and Measure Theory

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Goals

I. Primer on Domain Theory

II. Classical Approaches to Measures and Probability

III. Domain Theory Approach to Probability Measures

IV. Applications

Informatic partial order

$p \sqsubseteq q$ if q contains more information than p.

Example: Zero finding

$[a, b] \sqsubseteq [c, d] \in \mathbb{IR}$ iff $[c, d] \subseteq [a, b]$.

Directed completeness

$\emptyset \neq D \subseteq P$ *directed* if $x, y \in D \;\Rightarrow\; (\exists z \in D)\, x, y \leq z$.
P *directed complete:* D directed $\;\Rightarrow\;$ $\sup D$ exists.

$D \subseteq \mathbb{IR}$ directed $\Rightarrow\;$ $\sup D = \bigcap D$.

Approximation

$x \ll y$ iff $y \leq \sup D \;\Rightarrow\; (\exists d \in D)\, x \leq d$.
Domain: $\twoheaddownarrow y = \{x \mid x \ll y\}$ directed and $y = \sup \twoheaddownarrow y$

$[a, b] \ll [c, d]$ iff $[c, d] \subseteq (a, b)$;

$[c, d] = \bigcap \{[a, b] \mid [c, d] \subseteq (a, b)\}$.

Morphisms

$f : P \to Q$ *D-continuous* if :

- f monotone, and
- D directed $\Rightarrow$ $f(\sup D) = \sup f(D)$.

DCPO – directed complete partial orders and D-continuous maps

Theorem: Tarski, Knaster, Scott
$D \in$ DCPO with least element, $\bot$, $f : D \to D$ monotone. Then:

- $\mathtt{Fix}\, f = \sup_{\alpha \in Ord} f^{\alpha}(\bot)$ is the least fixed point of f.
- f D-continuous $\Longrightarrow \mathtt{Fix}\, f = \sup_{n \geq 0} f^{n}(\bot)$.

Least fixed point semantics:

$\mathtt{rec}\, x.p \longrightarrow p[\mathtt{rec}\, x.p/x] \quad \Longrightarrow \quad [\![\mathtt{rec}\, x.p]\!] = \mathtt{Fix}\, [\![p]\!]$.

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Properties:

- $f : P \times Q \to R$ jointly D-continuous iff f is separately D-continuous.
- $[P \to Q]$ ordered *pointwise:* $f \sqsubseteq g$ iff $f(x) \leq g(x)$ $(\forall x \in P)$.

  $[P \to Q]$ is a DCPO if P, Q are DCPOs.
- Cartesian closed categories of domains: BCD $\subseteq$ RB $\subseteq$ FS.

Morphisms

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Scott Topology

U *Scott open* if:

- $U = \uparrow U = \{x \in P \mid (\exists u \in U)\, u \leq x\}$ and
- D directed, $\sup D \in U$ $\Rightarrow$ $D \cap U \neq \emptyset$.

Always T_0, in fact, *sober;* T_1 $\Rightarrow$ flat order.

$\lim D = \sup D$ for D directed.

$f : P \to Q$ D-continuous iff f is Scott continuous.

D domain $\Rightarrow$ $\mathcal{B}_D = \{\twoheaduparrow x \mid x \in D\}$ basis for $\sigma_D = \{U \mid U \text{ Scott open}\}$.

Transitivity: $x \leq y \ll y' \leq z \Rightarrow x \ll z$; Implies $\uparrow(\twoheaduparrow x) = \twoheaduparrow x$.

Interpolation: $x \ll z \Rightarrow (\exists y)\, x \ll y \ll z$. Implies $\twoheaduparrow x$ Scott open.

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Lawson Topology

Basis: $\{\twoheaduparrow x \setminus \uparrow F \mid F \in \mathcal{P}_{<\omega} D\}$

Hausdorff refinement of Scott topology.

D is *coherent* if Lawson topology is compact.

All CCCs of domains consist of coherent domains, but Coh is not a CCC.

Basic Models

Unit interval: $([0,1],\leq)$, $x \ll y$ iff $x = 0$ or $x < y$

Interval domain: $(\mathbb{I}[0,1],\supseteq)$ – restriction of $(\mathbb{IR},\supseteq)$

Topology

Upper space: X – locally compact Hausdorff space
$\Gamma(X)$ – nonempty compact subsets of X under reverse inclusion:
$A \sqsubseteq B$ iff $B \subseteq A$. $A \ll B$ iff $B \subseteq A^\circ$.

EDALAT: Used $\Gamma(X)$ to model fractals, weakly hyperbolic Iterated Function Systems, neural nets...

Generalizes to the *upper power domain:*
$\mathcal{P}_U(D) = (\{X \subseteq D \mid \emptyset \neq X = \uparrow X \textit{ Scott compact}\}, \supseteq)$.

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HOFMANN-M.:
Scott-open filters in $\sigma(D) \simeq$ Scott-compact upper sets.

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \bigcap \mathcal{F} \\ U(C) & \longleftarrow & C \end{array}$$

It follows that $X \ll Y \in \mathcal{P}_U(D)$ iff $Y \subseteq X^\circ$.

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Domain Environments

Maximal Point Spaces: (Lawson)

Domain P is *bounded complete* if each pair of elements has an infimum.

Example: $(\Gamma(X), \supseteq)$; $X \simeq \mathrm{Max}\,\Gamma(X)$ by $x \mapsto \{x\}$.

Countably-based bounded complete domains are *computational models*.

Basic Models

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Domain Environments

X – metrizable space;

M – countably-based bounded complete domain.

Lawson; Ciesielski, Flagg & Kopperman:

$$(\exists M)\ (X, \tau_M) \simeq (\mathrm{Max}\, M, \sigma_M|_{\mathrm{Max}\, M}) \quad \text{iff} \quad X \text{ is a Polish space.}$$

Banach (1933)

X complete metric space

$C_b(X, \mathbb{R})$ - Banach space; $C_b(X, \mathbb{R})^*$ – dual space

Riesz Representation Theorem implies $\mathcal{M}(X) \simeq C_b(X, \mathbb{R})^*$

$\operatorname{Prob} X$ – unit sphere of $C_b(X, \mathbb{R})^*$ in weak $*$-topology.

Banach-Alaoglu: Unit ball is weak $*$-compact.

Weak $*$-topology is same as weak topology, so:

$\mu_n \to \mu$ *weakly* if $\int f \, d\mu_n \to \int f \, d\mu$ for $f : X \to \mathbb{R}$ bounded, continuous

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$\mu_n \to \mu$ *weakly* if $\int f\, d\mu_n \to \int f\, d\mu$ for $f\colon X \to \mathbb{R}$ bounded, continuous

Simple Measures Weak $*$-dense

$\{\sum_{x \in F} r_x \delta_x \mid r_x \geq 0, \sum r_x = 1, F \subseteq X \text{ finite}\}$ weak $*$-dense in $\mathrm{Prob}\, X$.

Kolmogorov (1936)

Developed *abstract theory* of *measure spaces* and *probability*:

$(\Omega, \Sigma_\Omega, \mu)$ – Probability space; $X \colon \Omega \to \mathbb{R}$ random variable

Probability measures on infinite product spaces; 0–1 Laws

Probability measure as a set function: $\mu \colon \Sigma_\Omega \to [0,1]$ satisfying:

(i) $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$;

(ii) $\mu(\dot{\bigcup}_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ if $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma_\Omega$ pairwise disjoint.

Note: Condition (ii) implies:

- $\mu(A) \leq \mu(B)$ if $A \subseteq B$, and
- $\mu(\bigcup_n A_n) = \sup_n \mu_n(A_n)$ if $m \leq n \Rightarrow A_m \subseteq A_n$.

Portmanteau Theorem

Let $\mu_n, \mu \in \operatorname{Prob} X$ for X complete metric space. TAE:

- $\mu_n \to \mu$ in the weak topology
- $\int f \, d\mu_n \to \int f \, d\mu$ for all $f \colon X \to \mathbb{R}$ bounded, uniformly continuous
- $\limsup_n \mu_n(F) \leq \mu(F)$ for all $F \subseteq X$ closed
- $\liminf_n \mu_n(O) \geq \mu(O)$ for all $O \subseteq X$ open
- $\lim_n \mu_n(A) = \mu(A)$ for all $A \subseteq X$ μ-continuity sets

Valuations

Let D be a domain and let σ_D denote its family of Scott-open sets.
A *continuous valuation* is a mapping $\mu\colon \sigma_D \to [0,1]$ satisfying:

Strictness $\mu(\emptyset) = 0$

Modularity $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$

Monotonicity $U \subseteq V \quad \Longrightarrow \quad \mu(U) \leq \mu(V)$

Continuity $\{U_i\} \subseteq \sigma_D$ directed implies $\mu(\bigcup_i U_i) = \sup_i \mu(U_i)$.

Clearly every Borel subprobability measure induces a valuation on σ_D;

The converse was shown by LAWSON for countably-based bounded complete domains, and by ALVAREZ-MANILLA, EDALAT AND SAHEB-DJARHOMI for general domains.

Probabilistic power domain:

$\mathbb{V}D$ – valuations on D, ordered pointwise:

$\mu \sqsubseteq \nu$ iff $\mu(U) \leq \nu(U)$ $(\forall U \in \sigma_D)$.

$\mathbb{V}D \subseteq [D \to [0,1]]$ is a subdcpo, but domain structure is elusive.

The Domain Order from the Classical Approach

Recall for a compact space X and $\mu, \nu \in \mathrm{Prob}\, X$,

$$\int f\, d\mu \leq \int f\, d\nu\ (\forall f\colon X \to \mathbb{R}) \iff \mu = \nu.$$

Theorem: If D is a coherent domain and $\mu, \nu \in \mathbb{V}D$, then TAE:

- $\mu \sqsubseteq \nu$, i.e., $\mu(U) \leq \nu(U)\ (\forall U \in \sigma(D))$.
- $\int f\, d\mu \leq \int f\, d\nu$ for all $f\colon D \to \mathbb{R}_+$ Scott continuous.
- $\int f\, d\mu \leq \int f\, d\nu$ for all $f\colon D \to \mathbb{R}_+$ monotone Lawson continuous.

The Splitting Lemma and Simple Measures

Splitting Lemma (Jones 1989)

Let $\mu = \sum_{x \in F} r_x \delta_x, \nu = \sum_{y \in G} s_y \delta_y$ in $\mathbb{V}D$. Then

$\mu \leq \nu$ iff there are *transport numbers* $\{t_{x,y}\}_{(x,y) \in F \times G} \subseteq \mathbb{R}_+$ satisfying:

1. $r_x = \sum_y t_{x,y}$ $(\forall x \in F)$
2. $\sum_x t_{x,y} \leq s_y$ $(\forall y \in G)$
3. $t_{x,y} > 0 \ \Rightarrow \ x \leq y$.

   Moreover, $\mu \ll \nu$ iff

4. $t_{x,y} > 0 \implies \sum_x t_{x,y} < s_y$ and $x \ll y$ $(\forall x, y)$.

The proof is an application of the Max Flow – Min Cut Theorem.

In addition to being a useful tool for proving results about subprobability measures on domains, the expectation was that the Splitting Lemma would provide insights into the domain structure of $\mathbb{V}D$.

The Splitting Lemma and Simple Measures

$B_D \subseteq D$ is a *basis* if

- $\twoheaddownarrow x \cap B_D$ is directed, and
- $x = \sup(\twoheaddownarrow x \cap B_D)$

for all $x \in D$.

Simple Measures are Dense

Let D be a domain with basis B_D, and let $\mathcal{B}$ be a basis for $[0,1]$. Then:

$$B_{\mathbb{V}D} = \{\sum_{x \in F} r_x \delta_x \mid r_x \in \mathcal{B}, \sum_x r_x \leq 1 \ \& \ F \subseteq B_D \text{ finite}\}$$

is a basis for $\mathbb{V}D$.

As a consequence, $\mu = \sup(\twoheaddownarrow \mu \cap B_{\mathbb{V}D})$ for all $\mu \in \mathbb{V}D$.

When Scott is Weak on the Top (Edalat 1996)

If D is a countably-based domain and $\mu_n, \mu \in \mathbb{V}D$, then TAE:

1. $\mu_n \to \mu$ in the Scott topology on $\mathbb{V}D$.
2. $\liminf_n \mu_n(U) \geq \mu(U)$ $(\forall U \in \sigma_D)$.

Proof of (i) ⇒ (ii): We know $\mu = \sup_m \nu_m$, $\nu_m \ll \mu$.

Fix U open, $\epsilon > 0$; then $\mu = \sup_m \nu_m \;\Rightarrow\; (\exists m > 0)\, \nu_m(U) > \mu(U) - \epsilon$.

Then $\Uparrow\!\nu_m$ open and $\mu_n \to \mu$ implies $(\exists N)\, n \geq N \;\Rightarrow\; \mu_n \in \Uparrow\!\nu_m$.

Then $\mu(U) - \epsilon < \nu_m(U) \leq \mu_n(U)$, so $\liminf_n \mu_n(U) \geq \mu(U) - \epsilon$. □

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Corollary: If X is a separable metric space and $e\colon X \hookrightarrow \mathrm{Max}\, D$ is a topological embedding of X as a G_δ in the relative Scott topology, then $\mathrm{Prob}\, e\colon \mathrm{Prob}\, X \to \mathrm{Max}\, \mathbb{V}D$ is a topological embedding wrt the weak topology.

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Application: Iterated Function Systems.

An *IFS with probabilities* consists of:

- X – compact metric space,
- $f_i \colon X \to X,\ i = i, \ldots, N$.
- $p_i > 0$ with $\sum_{i \leq N} p_i = 1$.

The system is *hyperbolic* if f_i is a contraction for each i.

Markov operator $T \colon \mathrm{Prob}\, X \to \mathrm{Prob}\, X$:

$$T(\mu)(B) = \sum_{i \leq N} p_i \cdot \mu(f_i^{-1}(B)) = \sum_{i \leq N} p_i \cdot f_{i*}\, \mu(B).$$

Hutchinson: T is a contraction in the Kantorovich-Wasserstein metric, so it has a unique fixed point $\mu \in \mathrm{Prob}\, X$.

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The IFS is *weakly hyperbolic* if

$(\forall i_1, i_2, \ldots \in \{1, \ldots, N\}^\omega)(\exists x \in X) \implies \bigcap_{n \geq 1} f_{i_1} f_{i_2} \cdots f_{i_n}(X) = \{x\}$.

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Theorem: For any weakly hyperbolic IFS with probabilities, there is a unique $\mu^* \in \mathrm{Prob}\, X$ satisfying the sequence $T^n(\mu) \to \mu^*$ weakly for every $\mu \in \mathrm{Prob}\, X$.

Proof: $\mathrm{Prob}\, \Gamma(X)$ is a domain with least element δ_X. Then show

$$\mathtt{Fix}\, T = \sup_n T^n(\delta_X) = \mu^* \in \mathrm{Max}\, \mathrm{Prob}\, (\Gamma(X), \supseteq),$$

so μ^* is unique fixed point of T. Then

$$X \hookrightarrow (\Gamma(X), \supseteq) \;\Rightarrow\; \mathrm{Prob}\, X \hookrightarrow \mathrm{Max}\, \mathrm{Prob}(\Gamma(X), \supseteq)$$

implies $T^n(\delta_X) \sqsubseteq T^n(\mu)$, so $T^n(\mu) \to_w \mu^*$ for each $\mu \in \mathrm{Prob}\, X$. $\square$

Testing LPMs (van Breugel, M., Ouaknine & Worrell 2003)

Theorem: If D is a countably-based coherent domain, and $\mu_n, \mu \in \mathbb{V}D$, then $\mu_n \to \mu$ in the Lawson topology on $\mathbb{V}D$ iff:

- $\liminf_n \mu_n(U) \geq \mu(U)$ $(\forall U \in \sigma_D)$, and
- $\limsup_n \mu_n(\uparrow F) \leq \mu(\uparrow F)$ ($\forall F \subseteq D$ finite).

Corollary: If D is coherent and countably-based, then $\mathbb{V}D$ is coherent and the Lawson topology on $\mathbb{V}D$ agrees with the weak topology.

Proof: In light of the Theorem, the Portmanteau Theorem implies the Lawson topology is coarser than the weak topology, but both are compact Hausdorff. $\square$

This provides an alternative to Jung & Tix's proof that $\mathbb{V}D$ is coherent if D is.

Applications in Domain Theory

$\mathbb{V}$ extends to a monad on DCPO by $f\colon P \to Q \;\mapsto \mathbb{V}f\colon VP \to \mathbb{V}Q$ by $\mathbb{V}f\nu(U) = \nu(f^{-1}(U))$, the push forward of ν by f.

Denote $\mathbb{V}f(\nu)$ by $f_* \nu$

Our Knowledge of $\mathbb{V}$ (Jung & Tix 1988)

- $\mathbb{V}\colon \mathit{Coh} \to \mathit{Coh}$ is a monad.
- $\mathbb{V}T \in \mathit{BCD}$ for any rooted tree T.
- $\mathbb{V}T^{rev} \in \mathit{RB}$ for any finite reverse tree T.

$\mathbb{V}D$ was devised to model *probabilistic choice:* $p +_r q$, which chooses p with probability r and q with probability $1 - r$.

$\mathbb{V}D$ has seen limited success, because:

- $\mathbb{V}$ is not known to leave any CCC of domains invariant.
- $\mathbb{V}$ doesn't satisfy a distributive law wrt any of the models of nondeterminism.

The Cantor Tree

$CT := \{0,1\}^* \cup \{0,1\}^\omega$ – use prefix order.

$s \ll t$ iff $s \leq t$ & $s \in \{0,1\}^*$.

$C := \{0,1\}^\omega$ – Cantor set of infinite words, with inherited Scott topology.

$C_m = \{0,1\}^m$ – m-bit words. Outcomes of m-flips of a coin.

$\pi_m \colon CT \to \downarrow C_m$; $\pi_{mn} \colon C_n \to C_m$ projections.

$FAC(CT) = \{M \subseteq CT \mid M$ is a *full* Lawson-closed antichain$\}$;

M *full* iff $C \subseteq \uparrow M$

$M \sqsubseteq N$ iff $M \sqsubseteq_{EM} N$ iff $\exists\, \pi_{MN} \colon N \to M$.

For a domain D, we define:

$$\begin{array}{rcl} RC(D) & := & \{(M,X) \in FAC(CT) \times [M \to D]\} \\ (M,X) \sqsubseteq (N,Y) & \text{iff} & M \sqsubseteq_{EM} N \ \& \ X \circ \pi_{MN} \leq Y. \end{array}$$

***RC* is a monad (T. Barker 2016)**

RC defines a monad on BCD, the category of bounded complete domains. Moreover, *RC* enjoys a distributive law wrt the upper power domain.

Random Variable Monads

DANA'S model of the *stochastic lambda calculus* uses a random variable $X\colon [0,1] \to \mathcal{P}(\mathbb{N})$ to model randomness in the lambda calculus.

TYLER BARKER's monad provides a general approach:

Randomized PCF

Simply typed lambda calculus with ground types *Nat* and *Bool*, and probabilistic choice:

$$
\begin{array}{rcl}
t & ::= & Nat \mid Bool \mid t \to t \\
M & ::= & 0 \mid \texttt{true} \mid \texttt{false} \mid \\
 & & \mathrm{succ}(M) \mid \mathrm{pred}(M) \mid \mathrm{zero?}(M) \mid \mathrm{if}\ M\ \mathrm{then}\ M\ \mathrm{else}\ M \\
 & & \mid x \mid \lambda x : t.M \mid MM \mid \mu x : t.M \mid M \oplus M
\end{array}
$$

Standard semantics in a domain $D \in \mathrm{BCD}$ for PCF, but with additional *tree structure* to replicate branching of nested choices in $M \oplus M$.

A random variable $(M, X) \in RC(D)$ models probabilistic choice.

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 & & \mid x \mid \lambda x : t.M \mid MM \mid \mu x : t.M \mid M \oplus M
\end{array}
$$

Construction models randomized algorithms; feeds results of coin tosses to both instances:

E.g., in Miller-Rabin, test $p \vee q$ prime with $X(t) \vee Y(t)$

Implementation available on Github.

Domain environments $X \hookrightarrow M_X$ seek to approximate continuous maps $f: X \to Y$ with Scott-continuous approximants $f_n: M_X \to M_Y$.

Random variables are measurable maps. We illustrate how to approximate measurable maps using domain-theoretic techniques.

A *stochastic process* is a family $\{X_t \mid t \in T \subseteq \mathbb{R}_+\}$ of random variables $X_t: \Omega \to S$, where $(\Omega, \Sigma_\Omega, \mu)$ is a probability space, and S is a Polish space.

Skorohod's Theorem
Let S be a Polish space, let $\nu \in \mathrm{Prob}\, S$, and let λ denote Lebesgue measure on $[0, 1]$. Then there is a random variable $X: [0, 1] \to S$ satisfying $X_* \lambda = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X: [0, 1] \to S$ with $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ satisfy $X_n \to X$ λ-a.e.

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Proof Outline: Basic set up:

- $S \hookrightarrow M_S$ – countably-based bounded complete domain environment.
- $\mathrm{Prob}\, S \hookrightarrow \mathrm{Max}\,\mathrm{Prob}\, M_S \subseteq \mathbb{V} M_S$; weak topology is the inherited Scott topology.
- $B_S \subseteq M_S$ – countable basis
  $\mathcal{B} = \{\sum_{x \in F} r_x \delta_x \mid r_x \text{ dyadic}, \sum_x r_x = 1, F \subseteq B_S\}$ countable basis for $\mathrm{Prob}\, M_S$

Skorohod's Theorem

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X\colon [0,1] \to S$ with $X_*\,\lambda = \nu, X_{n*}\,\lambda = \nu_n$ satisfy $X_n \to X$ λ-a.e.

Proof Outline: Now, given $\nu \in \mathrm{Prob}\, S$:

- Choose $\nu_n \ll \nu_{n+1} \ll \nu$ with $\nu = \sup_n \nu_n$ & $\nu_n \in \mathcal{B}$.
- Let $\nu_n = \sum_{x \in F_n} r_x \delta_x \ll \sum_{y \in F_{n+1}} s_y \delta_y = \nu_{n+1}$.
- Since r_x, s_y are dyadic, the transport numbers $\{t_{x,y}\}_{(x,y) \in F_n \times F_{n+1}}$ also are dyadic. Because ν_n, ν_{n+1} are probability measures,
    - $r_x = \sum_y t_{x,y}$ for each $x \in F_n$;
    - $s_y = \sum_x t_{x,y}$ for each $y \in F_{n+1}$.

Domains and Random Variables

Skorohod's Theorem

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X\colon [0,1] \to S$ with $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ satisfy $X_n \to X$ λ-a.e.

Proof Outline:

To start, let $\nu_1 = \sum_{x \in F_1} r_x \delta_x \ll \sum_{y \in F_2} s_y \delta_y = \nu_2;\ |F_i| = k_i;$
$2^{m_1} = \gcd\{r_x\};\ 2^{m_2} = \gcd\{s_y\}$, wlog $m_1 \leq m_2$.

$$C_{m_2} = \ [t_{x_1 y_1}][t_{x_1 y_2}] \cdots [t_{x_1 y_{k_2}}] \quad \cdots \quad [t_{x_{k_1} y_1}]\,[t_{x_{k_1} y_2}] \cdots [t_{x_{k_1} y_{k_2}}] \ \xrightarrow[t_{x_i y_j} \mapsto y_j]{f_{n_2}} S$$

$$\downarrow \pi_{m_1 m_2} \qquad\qquad \sqsubseteq$$

$$C_{m_1} = \ [r_{x_1}] \ \cdots \ [r_{x_{k_1}}] \quad \xrightarrow[r_{x_i} \mapsto x_i]{f_{n_1}} S$$

- $f_{m_1}\colon C_{m_1} = \{0,1\}^{m_1} \to S$ satisfies $f_{m_1}\, \mu_{m_1} = \nu_1$;
- $f_{m_2}\colon C_{m_2} = \{0,1\}^{m_2} \to S$ satisfies $f_{m_2}\, \mu_{m_2} = \nu_2$.

Skorohod's Theorem

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X \colon [0,1] \to S$ with $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ satisfy $X_n \to X$ λ-a.e.

Proof Outline:

The inductive step is the same idea, complicated by the repeated subdivisions of subintervals. In the end, we have

$f_{m_i} \colon \{0,1\}^{m_i} \to S$ with $f_{m_i} \circ \pi_{m_i} \sqsubseteq f_{m_{i+1}} \circ \pi_{m_{i+1}}$, and

$f_{m_i} \circ \pi_{m_1} \colon \{0,1\}^{\mathbb{N}} \to S$ with $(f_{m_i} \circ \pi_{m_1})\, \mu_{\{0,1\}^{\mathbb{N}}} = \nu_{n_i}$.

Then, $f := \lim_i (f_{m_i} \circ \pi_{m_1}) \colon \{0,1\}^{\mathbb{N}} \to S$ satisfies $f\, \mu_{\{0,1\}^{\mathbb{N}}} = \nu$.

Finally, if λ denotes Lebesgue measure on $[0,1]$:

$\iota \colon [0,1] \leftrightarrows \{0,1\}^{\mathbb{N}} \colon \pi \;\Rightarrow\; \iota\, \lambda = \mu_{\{0,1\}^{\mathbb{N}}}$, so

$(f \circ \iota) \colon [0,1] \to S$ satisfies $(f \circ \iota)\, \lambda = \nu$.

Skorohod's Theorem

Let S be a Polish space, let $\nu \in \mathrm{Prob}\, S$, and let λ denote Lebesgue measure on $[0,1]$. Then there is a random variable $X\colon [0,1] \to S$ satisfying $X_* \lambda = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X\colon [0,1] \to S$ with $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ satisfy $X_n \to X$ λ-a.e.

Proof Outline:
The last statement follows by an argument showing

$\lambda(\{x \in [0,1] \mid X_n(x) \not\to_\Lambda X(x)\}) = 0.$ □

Actually, the Theorem also holds for $\nu_n, \nu \in \mathbb{V}S$. In fact,

Corollary: (*to the Proof:*)
Let $S \hookrightarrow M_S$ be a Polish space with domain environment M_S, and let $f\colon [0,1] \to S$ be a measurable map. Then:

there is a measurable map $g\colon [0,1] \to S$ with $f = g$ λ-a.e. satisfying

$g = \sup_n g_n$, with $g_n\colon [0,1] \to M_S$ piecewise constant.

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