Probabilistic Automata, Probabilistic Models and Crypto-protocols

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Joint Work With
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Perspective

- Dolev-Yau model is unrealistic
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  - Result is probability distribution over action–state pairs, or over next state
- Many approaches are arcane, especially in application of mathematical / statistical results to computational models.

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Different equations result in different models, but common factor is the need to combine probabilistic and computational reasoning.

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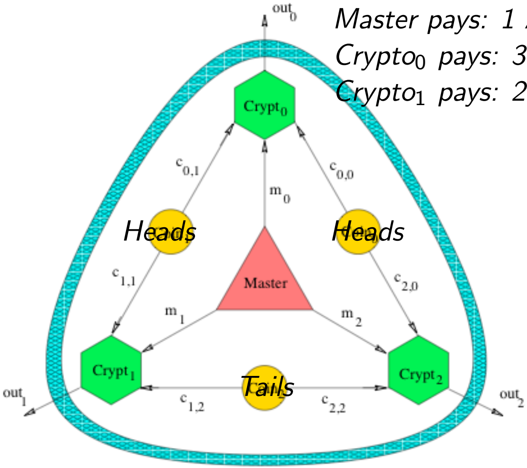
5. Apply results to model

Dining Cryptographers



Master pays: 1 Agree, 2 Disagree
$Crypto_0$ pays: 3 Disagree
$Crypto_1$ pays: 2 Agree, 1 Disagree

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- Canetti, Lynch, et al (2006) Probabilistic input / output automata augmented with *tasks*

  Used to model oblivious transfer under universal composability

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High-level vs. Low-level Nondeterminism

- *Low level:* Doesn't affect outcome
- *High level:* Can affect outcome

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Schedulers:

- Set order of Low-level events
  Choose payer, Inform cryptographers, Flip coins, Compare, Announce
- Can't see outcome of High-level (Internal) choices

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$$D \colon S \times \text{Act} \rightharpoonup \text{Disc}(S)$$

How to compose PIOAs

$\mathcal{A}_i = (S_i, s_{\mathcal{A}_i}, I_i, O_i, H_i, D_i),\ \ i = 1, 2$ are *compatible* if

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- $S_{\mathcal{A}_1\|\mathcal{A}_2} = S_1 \times S_2$
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$\mathcal{A}$ is *closed* if $I_{\mathcal{A}} = \emptyset$

Tasks

- **Tasks:** Equivalence relation $\mathcal{R} \subseteq (O \cup H) \times (O \cup H)$.

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PIOA Semantics

$\mathcal{A} = (S, s_0, I, O, H, D)$

Execution fragment:

$\alpha = s_1 a_1 s_2 a_2 \cdots$ with $s_{i+1} \in \mathrm{supp}(\mu_i)$ & $(s_i, a_i, \mu_i) \in D$

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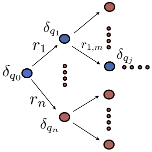

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Which α should be used?

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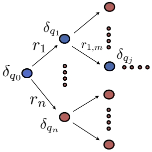

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Apply task schedule $\rho = T_1 T_2 \cdots$

$$\delta_{s_0} \stackrel{a_{T_1}}{\longrightarrow} \sum_s \mu_{s_0 a_{T_1}}(s) \delta_{s_0 a_{T_1} s}$$

$$\stackrel{a_{T_2}}{\longrightarrow} \sum_s \mu_{s_0 a_{T_1}}(s) \left(\sum_{s'} \mu_{s, a_{T_2}}(s') \delta_{s_0 a_{T_1} s a_{T_2} s'} \right) \cdots$$

Domains

Frags($\mathcal{A}$) is a language from $(S \times \mathrm{Act})^\omega$

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$\text{Frags}(\mathcal{A})$ is a language from $(S \times \text{Act})^\omega$

Chain-complete partial order under the prefix order.

$U \subseteq \text{Frags}(\mathcal{A})$ is *Scott open* if
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In particular, $\sum_{i \leq m} r_i \delta_{\alpha_i} \sqsubseteq \sum_{j \leq n} s_j \delta_{\beta_j}$ iff $(\exists t_{i,j} \geq 0)$ with

- $t_{i,j} > 0 \ \Rightarrow \ \alpha_i \sqsubseteq \beta_j$.
- $r_i = \sum_j t_{i,j}\ (\forall i)$.
- $\sum_i t_{i,j} \leq s_j\ (\forall j)$.

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(X, Ω) measure space $\Rightarrow$ Prob(X, Ω) is a measure space.

So, Prob defines an endofunctor on Meas. This is in fact a monad.

$\eta\colon X \to \text{Prob}(X)$ by $\eta(x) = \delta_x$

$m\colon \text{Prob}(\text{Prob}(X)) \to \text{Prob}(X)$ is just integration.

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$m\colon \mathrm{Prob}(\mathrm{Prob}(X)) \to \mathrm{Prob}(X)$ is just integration. For example,

$$m\left(\sum_s \mu_{s_0,a_{T_1}}(s)\left(\sum_{s'} \mu_{s,a_{T_2}}(s')\delta_{s_0 a_{T_1} s a_{T_2} s'}\right)\right) = \sum_{s,s'} \mu_{s_0,a_{T_1}}(s)\mu_{s,a_{T_2}}(s')\delta_{s_0 a_{T_1} s a_{T_2} s'}$$

Defining Apply

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$$\begin{aligned}\mathsf{Apply}(\mu, T) &= \sum_{\alpha \notin A_T} \mu(\alpha)\delta_\alpha \\ &\quad + \sum_{\alpha \in A_T} \mu(\alpha) \left(\sum_s \mu_{\mathsf{ls}(\alpha), a_T}(s)\delta_{\alpha a s} \right)\end{aligned}$$

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Incomparable Fragments

μ has *incomparable fragments* if $\alpha, \beta \in \operatorname{supp} \mu$ implies α and β are incomparable initial segments in $\mathsf{Frags}^*(\mathcal{A})$.

μ has incomparable fragments $\Rightarrow$ $\mathsf{Apply}(\mu, T)$ has incomparable fragments.

Defining Apply (cont'd)

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$\mu \leq \nu \in \mathsf{Disc}(\mathsf{Frags}^*(\mathcal{A})) \;\Leftrightarrow\; \mu(\uparrow\alpha) \leq \nu(\uparrow\alpha)\ (\forall\alpha).$

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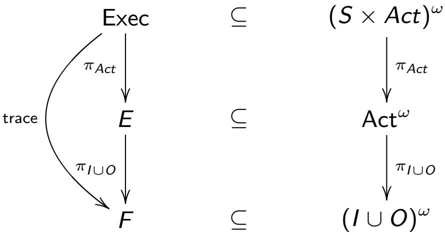
Hence $\mathsf{Apply}(\mu, \rho) = \sup_n \mathsf{Apply}(\mu, T_1 \cdots T_n)$ is well-defined.

Semantics of Observable Events



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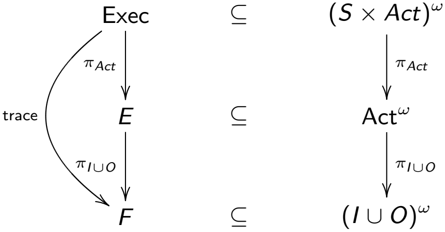

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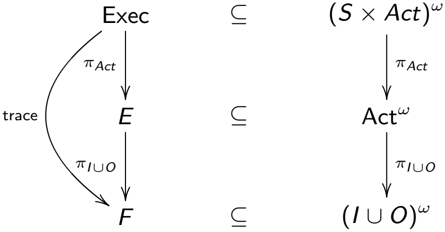

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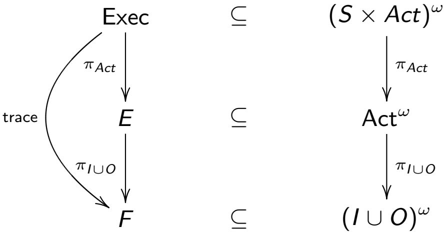

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Schedulers

A *task scheduler* is a map

$$\sigma \colon \mathsf{Frags}^*(\mathcal{A}) \to \mathbb{V}(\mathsf{Act}) = \{\mu \mid \mu \text{ subprobability measure}\}$$

Measures from Schedulers

Let $\mathcal{A}$ be a task PIOA and let $\sigma\colon \mathsf{Frags}^*(\mathcal{A}) \to \mathbb{V}(\mathsf{Act})$ be a task scheduler. If $\alpha \in \mathsf{Frags}^*(\mathcal{A})$, we define

$$\epsilon_{\sigma,\alpha} = (1 - ||\sigma(\alpha)||)\delta_\alpha + \sum_{a \in \mathsf{Act}} \sigma(\alpha)(a) \left(\sum_{s} \mu_{\alpha,a}(s) \epsilon_{\sigma,\alpha a s} \right)$$

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Fact:

$$\begin{aligned} \epsilon_{\sigma,\alpha,0} &= \delta_\alpha \\ \epsilon_{\sigma,\alpha,n+1} &= (1 - ||\sigma(\alpha)||)\delta_\alpha + \sum_{a\in \mathsf{Act}} \sigma(\alpha)(a) \left(\sum_s \mu_{\alpha,a}(s)\epsilon_{\sigma,\alpha a s,n} \right) \end{aligned}$$

implies $\epsilon_{\sigma,\alpha} = \sup_n \epsilon_{\sigma,\alpha,n}$ extends to infinite α.

Schedulers vs. Task Schedules

Theorem

Let $\mu \in \mathsf{Disc}(\mathsf{Frags}^*(A))$ have support consisting of incomparable fragments, and let ρ be a task schedule. Then there is a scheduler $\sigma_\rho \colon \mathsf{Frags}^*(\mathcal{A}) \to \mathbb{V}(\mathsf{Act})$ such that $\mathsf{Apply}(\mu, \rho) = \epsilon_{\sigma_\rho, \mu}$.

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In fact, for $\mu = \sum_\alpha \mu(\alpha)\delta_\alpha$ and $\rho = T\rho'$, the scheduler σ_ρ is deterministic:

$$\sigma_\rho(\alpha) = \begin{cases} \delta_{a_{\mathsf{ls}(\alpha),T}} & \text{if } \alpha \in A_T, \\ \sigma_{\rho'}(\alpha) & \text{if } \sigma_{\rho'}(\alpha) \neq 0 \ \& \ \alpha \notin A_T, \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho = T_1 T_2 \cdots \text{ infinite implies } \sigma_\rho = \cup_n \sigma_{T_1 \cdots T_n}.$$

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Corollary Let $\mathcal{A}$ be a Task PIOA.

- For each task schedule ρ, there is a deterministic task scheduler σ_ρ satisfying
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- $\mathsf{tdist}(\mathcal{A}) \subseteq \{\mathsf{trace}(\epsilon_{\sigma, \delta_{s_0}}) \mid \sigma \text{ deterministic scheduler}\}$.

Expansions and Monads

$(X, \Sigma_X), (Y, \Sigma_Y)$ measure spaces, $R \subseteq X \times Y$. The *lift of* R is $\widehat{R} \subseteq \mathbb{V}X \times \mathbb{V}Y$ defined by

$$(\sum_x r_x \delta_x, \sum_y s_y \delta_y) \in \widehat{R} \;\Leftrightarrow\; \exists t\colon X \times Y \to [0,1] \text{ with}$$

- $r_x = \sum_y t(x,y)\ (\forall x)$
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But, (X, Σ_X) a measure space implies $\mathbb{V}X$ is a measure space
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$\mathbb{V}$: Meas $\to$ Meas is a *monad*, and $R \mapsto \mathcal{E}(R)$ utilizes the *lifting* and *multiplication* of the monad. In particular, if $R \subseteq \text{Disc}(\text{Frags}(\mathcal{A})) \times \text{Disc}(\text{Frags}(\mathcal{A}))$, define $\mathcal{E}(R) \subseteq \text{Disc}(\text{Frags}(\mathcal{A})) \times \text{Disc}(\text{Frags}(\mathcal{A}))$ by

$$\begin{aligned}\mathcal{E}(R) &= (m \times m)(\widehat{R})\\ &= \{\langle \mu.\nu\rangle \mid (\exists p_i)(\exists \langle \mu_i, \nu_i\rangle \in R)\ \mu = \sum p_i\mu_i \ \&\ \nu = \sum p_i\nu_i\}\end{aligned}$$

Simulations

Let $(\mathcal{A}_i, \mathcal{R}_i)$ be two task PIOAs and let $f\colon \mathcal{R}_1^* \times \mathcal{R}_1 \to \mathcal{R}_2^*$ be a function. We define $\mathrm{full}(f)\colon \mathcal{R}_1^* \to \mathcal{R}_2^*$ by

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$R \subseteq \mathrm{Disc}(\mathrm{Exec}(\mathcal{A}_1)) \times \mathrm{Disc}(\mathrm{Exec}(\mathcal{A}_2))$ is a *simulation* if

- $(\mu_1, \mu_2) \in R \;\Rightarrow\; \mathrm{tdist}(\mu_1) \subseteq \mathrm{tdist}(\mu_2)$
- $(\delta_{s_{0,1}}, \delta_{s_{0,2}}) \in R$
- $(\exists f : \mathcal{R}_1^* \times \mathcal{R}_1 \to \mathcal{R}_2^*)(\forall \rho \in \mathcal{R}_1^*)(\forall T \in \mathcal{R}_1)$

$$\begin{aligned} &(\mu_1, \mu_2) \in R \;\wedge\; \mathrm{supp}(\mu_1) \subseteq \rho \;\wedge\; \mathrm{supp}(\mu_2) \subseteq \mathrm{full}(f)(\rho) \\ &\quad\Rightarrow\; \langle \mathrm{Apply}(\mu_1, T), \mathrm{Apply}(\mu_2, \mathrm{full}(f)(\rho, T)) \rangle \in \mathcal{E}(R) \end{aligned}$$

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Theorem (Canetti, et al)
Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be comparable task-PIOAs that are closed and action-deterministic. If there exists a simulation relation from $\mathcal{A}_1$ to $\mathcal{A}_2$, then $\text{tdist}(\mathcal{A}_1) \subseteq \text{tdist}(\mathcal{A}_2)$.

Remember

$$\mathcal{M} \parallel \mathcal{C}rypt_0 \parallel \mathcal{C}rypt_1 \parallel \mathcal{C}rypt_2$$

$$\mathcal{M} \;::\; Choose\_payer \parallel Inform\_Crypt_0 \parallel Inform\_Crypt_1 \parallel Inform\_Crypt_2$$

$$\mathcal{C}rypt_i \;::\; Flip\_coin_i \parallel Tell\_Crypt_{i+1} \parallel Input\_coin_{i-1} \parallel Compare_i \parallel Announce_i \parallel Read\_Announce_{i-1} \parallel Read\_Announce_{i+1}$$

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Task Schedule:

Choose_payer; *Inform_Crypt*$_0$; *Inform_Crypt*$_1$; *Inform_Crypt*$_2$;
Flip_coin$_0$; *Flip_coin*$_1$; *Flip_coin*$_2$; *Tell_crypt*$_1$; *Tell_crypt*$_2$; *Tell_crypt*$_0$;
Compare$_0$; *Compare*$_1$; *Compare*$_2$; *Announce*$_0$; *Announce*$_1$; *Announce*$_2$

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$Flip\_coin_0; Flip\_coin_1; Flip\_coin_2; Tell\_crypt_1; Tell\_crypt_2; Tell\_crypt_0;$
$Compare_0; Compare_1; Compare_2; Announce_0; Announce_1; Announce_2$

$\mathcal{M}_i$ ::= Master who chooses $Crypt_i$

$$\mathcal{M} := \sum_i r_i \delta_{\mathcal{M}_i}, \qquad \sum_i r_i = 1$$

Dining Cryptographers

One approach:

$\mathcal{M}_i$ ::= Master who chooses Crypt_i

Show $(\forall i)(\forall \sigma)(\forall O)$

$$\mu_\sigma(\mathcal{M}_{i+1} \parallel \mathit{Crypto}_0 \parallel \mathit{Crypt}_1 \parallel \mathit{Crypt}_2))(O)|_{\mathit{Crypt}_i} =$$
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Definition: (Barghava & Palamidessi)

Protocol satisfies *strong probabilistic anonymity* if

$$(\forall \sigma)(\forall O)\ \mu_\sigma(O \mid \mathcal{M}_i) = \mu_\sigma(O \mid \mathcal{M}_{i+1}) = \mu_\sigma(O \mid \mathcal{M}_{i+2})$$

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Definition: (Barghava & Palamidessi)
Protocol satisfies *strong probabilistic anonymity* if

$$(\forall \sigma)(\forall O)\ \mu_\sigma(O \mid \mathcal{M}_i) = \mu_\sigma(O \mid \mathcal{M}_{i+1}) = \mu_\sigma(O \mid \mathcal{M}_{i+2})$$

But

$$\mu_\sigma(O \mid \mathcal{M}_i) = \frac{\mu_\sigma(O \wedge \mathcal{M}_i)}{\mu(\mathcal{M}_i)} = \mu_\sigma(O)$$

since the observation O is independent of which cryptographer $\mathcal{M}$ chooses.