Semantic Models of Quantum Programming Languages: Recursion in Categorical Models

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Prototypical Quantum Computer

- Knill’s QRAM model: A classical computer with a quantum co-processor

![Diagram](image)

- Circuit: sequence of unitary operators

*How do we program such a device?*
Some Basics

Logical Foundations

*Predicate calculus:*

Predicate symbols: $P, Q, R, \ldots$ each with a fixed arity

Functions symbols: $f, g, h, \ldots$ each with a fixed arity

Terms: $t ::= x \mid c \mid f(t_1, \ldots, t_n)$ where

$x$ is a variable, $c$ a nullary function, and $f$ a function symbol with arity $n$.

Formulas: $\varphi ::= P(t_1, \ldots, t_n) \mid \bot \mid \top \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \mid \forall x \varphi \mid \exists x \varphi$.

- Sound and complete: $\vdash P$ iff $\models P$. 
Some Basics

Logical Foundations

*Predicate calculus:*

*Set Theory:*

Standard ZF axioms, including *Axiom of Infinity:*

\[ \exists S. \emptyset \in S \land (\forall T) T \in S \implies T \cup \{T\} \in S. \]

- Sound, *semantically complete.* But,

**Theorem:** [Gödel] Any system powerful enough to do arithmetic is incapable of proving its own consistency.
Some Basics

Logical Foundations

*Predicate calculus:*

*Set Theory:*

*Intuitionistic Logic:*(Brouwer, 1907)

Does not include $A \lor \neg A$, or equivalently, $\neg \neg A \rightarrow A$.

Emphasis is on *proof*, not validity.

The logic for classical computation.
Some Basics

Logical Foundations

*Predicate calculus:*

*Set Theory:*

*Intuitionistic Logic:* (Brouwer, 1907)

*Linear Logic:* (Girard, 1987)

- Regards formulas as resources.
- Each hypothesis used once and only once.
- Derivations that obey this paradigm are called *linear.*
- Logic for reasoning about computing over quantum systems.
A Primer on the Basics

Computational Structures

Lambda calculus: (Church, 1934)

Untyped $\lambda$-calculus: Terms: $t ::= x \mid \lambda x.t \mid t t$ where $x$ is a variable, $t$ a term.

Conversion Rules:

- $\lambda x.e \equiv_\alpha \lambda y.e[y/x]$ when $y \notin FV(e)$
- $(\lambda x.e)(e') \rightarrow_\beta e'[e'/x]$ if $FV(e') \cap BV(e) = \emptyset$.

Turing complete:

- Supports full recursion: $\text{rec } x.t = t[\text{rec } x.t/x]$.
- Paradoxical combinator $Y := (\lambda x.x x)(\lambda x.x x)$ produces fixed point for any term.
Computational Structures

*Lambda calculus*:(Church, 1934)

**Untyped λ-calculus:**

**Simply typed lambda calculus, λ→:**

Types: $\tau ::= 1 \mid \text{Int} \mid \text{Bool} \mid \tau \to \tau$.

Terms: $t ::= x \mid \text{null} \mid n \mid \text{true} \mid \text{false} \mid t \ t \mid \lambda x : \tau. t$.

Typing Judgements:

\[
\begin{array}{l}
\Gamma \vdash \text{null} : 1 \\
\Gamma \vdash n : \text{Int} \\
\Gamma \vdash \text{true} : \text{Bool} \\
\Gamma \vdash \text{false} : \text{Bool} \\
\Gamma(x) = \tau \\
\Gamma \vdash e : \tau \to \tau' \\
\Gamma \vdash e' : \tau \\
\Gamma, x : \tau \vdash e : \tau' \\
\Gamma \vdash (\lambda x : \tau. e) : \tau \to \tau' \\
\end{array}
\]
A Primer on the Basics

Computational Structures

Lambda calculus: (Church, 1934)

Untyped $\lambda$-calculus:

Simply typed lambda calculus, $\lambda \rightarrow$:

Curry–Howard Correspondence:

<table>
<thead>
<tr>
<th>Intuitionistic Propositional Natural Deduction</th>
<th>Simply Typed Lambda Calculus</th>
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<tr>
<td>Propositions</td>
<td>$\longleftrightarrow$</td>
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<td>$\longleftrightarrow$</td>
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Intuitionistic logic is the logic of classical computation
A Primer on the Basics

Computational Structures

Lambda calculus:(Church, 1934)

Untyped λ-calculus:

Simply typed lambda calculus, $\lambda \rightarrow$:

Linear lambda calculus:

Types $A, B ::= 0 \mid A + B \mid I \mid A \otimes B \mid A \rightarrow B \mid !A$

Terms $M, N, P ::= x \mid c \mid \text{let } x = M \text{ in } N \mid \square_A M \mid \text{left}_{A,B} M \mid \text{right}_{A,B} M$

| case $M$ of $\{ \text{left } x \rightarrow N \mid \text{right } y \rightarrow P \} \mid * \mid \langle M, N \rangle$

| \text{let } $\langle x, y \rangle = M \text{ in } N \mid \lambda x^A. M \mid MN \mid \text{lift } M \mid \text{force } M$

lift $M(=!M)$ – Allows multiple instances of resource $M$.

force $M$ – produces an instance of $M'$ when $M = !M'$. 
Semantic Models

A category $C$ consists of:

(i) a family of objects $\text{obj } C$, and

(ii) for each pair $A, B \in \text{obj } C$ a family of morphisms $C(A, B)$ satisfying:

\[ \circ : C(B, C) \times C(A, B) \to C(A, C) \text{ is associative, and } (\forall A, B \in \text{obj } C) \]

\[ 1_A : A \to A \text{ is an identity with } 1_B \circ f = f \circ 1_A. \]

Example: $\textbf{Set}$, the category of sets and functions.

A category $C$ is Cartesian closed if $C$ has

- finite products $- A \times B$,
- a terminal object, $\bot$ satisfying $|C(A, \bot)| = 1$ for all objects $A$,
- and an internal hom $[A, B]$ satisfying $C(A \times B, C) \simeq C(A, [A, B])$.

For example, $\textbf{Set}$ is Cartesian closed.
Semantic Models

Lambek’s Theorem
There is a one-to-one correspondence between Cartesian closed categories and models of the typed lambda calculus.

For example, Set is a model for $\lambda \to$

Scott’s Corollary
There is a one-to-one correspondence between reflexive objects $[X \to X] \overset{\cong}{\leftrightarrow} X$ in Cartesian closed categories and models of the untyped lambda calculus.

For example, $D_\infty \simeq [D_\infty \to D_\infty]$ is a model.

Fact
The only known non-degenerate reflexive objects in Cartesian closed categories are domains.
Semantic Models

Lambek’s Theorem
There is a one-to-one correspondence between Cartesian closed categories and models of the typed lambda calculus.

For example, \textbf{Set} is a model for \( \lambda \rightarrow \)

Scott’s Corollary
There is a one-to-one correspondence between reflexive objects \([X \rightarrow X] \leftrightarrow X\) in Cartesian closed categories and models of the untyped lambda calculus

For example, \( D_\infty \simeq [D_\infty \rightarrow D_\infty] \) is a model.

Fact
The only known \textit{non-degenerate} reflexive objects in Cartesian closed categories are domains.

\textit{What does all this have to do with quantum computing?}
Categorical Quantum Mechanics

The Hilbert space formalism provides the basic model of quantum mechanics.

The category $\text{FdHilb}$ of finite dimensional Hilbert spaces and linear maps has a number of important properties:

- $\text{FdHilb}$ has a symmetric tensor product: $H \otimes K \simeq K \otimes H$.
- $\text{FdHilb}$ has a unit object $\mathbb{C}$: $\mathbb{C} \otimes H \simeq H$.
- $\text{FdHilb}$ has a 0-object, the degenerate Hilbert space: $0 \otimes H \simeq 0$.
- $\text{FdHilb}$ has biproducts: $H \oplus K$.
- $\text{FdHilb}$ is dagger compact closed: there is an involution $H \mapsto H^*$ that extends to $f \mapsto f_*$ on linear maps satisfying $H^{**} \simeq H$, $f^{**} = f$ and $(f_*)^\dagger = (f^\dagger)_*$.
Categorical Quantum Mechanics

The Hilbert space formalism provides the basic model of quantum mechanics.

**Abramsky & Coecke:** Dagger compact closed (symmetric monoidal) categories with biproducts model finitary quantum mechanics.

These categories are a model of (multiplicative) linear logic.

They are an abstract setting in which to reason precisely about quantum protocols, such as teleportation and entanglement swapping.

There also is a diagrammatic calculus for reasoning in these categories:
Prototypical Quantum Computer

• Returning to Knill’s QRAM model:

Classical Computer \(\xrightarrow{\text{circuits}}\) Quantum Co-processor

\(\xleftarrow{\text{measurements}}\)

• Circuit: sequence of unitary operators
Prototypical Quantum Computer

- A *quantum programming language* is a classical functional language together with a linear language of *quantum circuits*:

![Diagram](Functional Language \rightarrow Linear language \leftrightarrow)

- We elide measurements and focus on a classical functional language for *constructing circuits* and a linear language for *modeling them* as linear morphisms.

- We model *circuit description languages* using Linear / Nonlinear Models.
A Linear/Non-Linear (LNL) model is given by the following data:

- A cartesian closed category $\mathbf{C}$.
- A symmetric monoidal closed category $\mathbf{L}$.
- A symmetric monoidal adjunction:

\[
\begin{align*}
F(X \times Y) &\simeq F(X) \otimes F(Y) \\
F(X + Y) &\simeq F(X) + F(Y) \\
F(\emptyset) &= 0 \\
F(1) &= I \\
F \circ G &= ! - \text{the lift comonad}
\end{align*}
\]

An LNL model is a model of Intuitionistic Linear Logic.$^1$

$^1$Nick Benton. *A mixed linear and non-linear logic: Proofs, terms and models. CSL'94*
Proto-Quipper-M (*Rios and Selinger*)

Types

\[ A, B ::= \alpha | 0 | A + B | I | A \otimes B | A \rightarrow B | !A | \text{Circ}(T, U) \]

Intuitionistic types

\[ P, R ::= 0 | P + R | I | P \otimes R | !A | \text{Circ}(T, U) \]

M-types

\[ T, U ::= \alpha | I | T \otimes U \]

Terms

\[ M, N ::= x | \ell | c | \text{let } x = M \text{ in } N \]
\[ | \Box A M | \text{left}_{A,B} M | \text{right}_{A,B} M | \text{case } M \text{ of } \{ \text{left } x \rightarrow N | \text{right } y \rightarrow P \} \]
\[ | * | M;N | \langle M, N \rangle | \text{let } \langle x, y \rangle = M \text{ in } N | \lambda x^A.M | MN \]
\[ | \text{lift } M | \text{force } M | \Box_T M | \text{apply}(M, N) | (\ell, C, \ell') \]

- All types other than Intuitionistic types are \textit{linear}
- M-types: morphisms from a symmetric monoidal category such as \( M = \text{FdHilb} \)
- Only use one (combined) form of type judgement
Example

Assume $H : Q \to Q$ is a constant representing the Hadamard gate.

**Example**

two-hadamard : Circ($Q, Q$)

two-hadamard $\equiv$ box$_Q$ lift $\lambda q^Q.HHq$

This program creates a completed circuit consisting of two $H$ gates. The term is intuitionistic (can be copied, deleted).
Circuit Model

Example

Shor's algorithm for integer factorization may be seen as an infinite family of quantum circuits – each circuit is a procedure for factoring an $n$-bit integer, for a fixed $n$.

![Quantum Fourier Transform on $n$ qubits (subroutine in Shor's algorithm).](https://commons.wikimedia.org/w/index.php?curid=14545612)

**Figure:** Quantum Fourier Transform on $n$ qubits (subroutine in Shor’s algorithm).

Proto-Quipper-M is used to describe families of morphisms in an arbitrary, but fixed, symmetric monoidal category, $M$.

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Footnote: 

2Figure source: https://commons.wikimedia.org/w/index.php?curid=14545612
Concrete model of Proto-Quipper-M

A simple Proto-Quipper-M model is given by the LNL model:

\[
\begin{array}{c}
\text{Set} \\
\downarrow \quad \downarrow \\
\overline{M}(I, -) \\
\end{array}
\]

where \( \overline{M} = [M^{\text{op}}, \text{Set}] \) is a closed, product complete category containing given SMC \( M \)

**Theorem (Rios & Selinger)**
The simple categorical model of Proto-Quipper-M is type-safe, sound, and computationally adequate
Concrete model of Proto-Quipper-M

There are two semantic models:

- For all types, $\llbracket P \rrbracket \in \overline{M}$
- For intuitionistic types, also have $\llbracket |P| \rrbracket \in \text{Set}$

**Theorem**

*For any intuitionistic type $P$, there exists a canonical isomorphism $\alpha_P : \llbracket P \rrbracket \to F(\langle P \rangle)$.*

So we can define copy and discard morphisms for each intuitionistic type $P$:

$$
\Delta_P := \llbracket P \rrbracket \xrightarrow{\alpha_P} F(\langle P \rangle) \xrightarrow{F(\langle id, id \rangle)} F(\langle P \rangle \times \langle P \rangle) \xrightarrow{\cong} F(\langle P \rangle) \otimes F(\langle P \rangle) \xrightarrow{\alpha_P^{-1} \otimes \alpha_P^{-1}} \llbracket P \rrbracket \otimes \llbracket P \rrbracket
$$

$$
\diamond_P := \llbracket P \rrbracket \xrightarrow{\alpha_P} F(\langle P \rangle) \xrightarrow{F1} F1 \xrightarrow{\cong} I
$$

where $FX = X \otimes I$
Our Work: Adding Recursion

- Focus on adding recursive *types*.
  - Term recursion follows from recursive types.
- Main difficulty is with the categorical model.
- How can we copy/discard intuitionistic recursive types?
  - A list of qubits should be *linear* – cannot copy/discard.
  - A list of natural numbers should be *intuitionistic* – can *implicitly* copy/discard.
- For the rest of the talk we focus on the linear/non-linear type structure.
- How do we design a linear/non-linear FPC \(^3\)?

---

\(^3\)FPC is an intuitionistic Fixed Point Calculus studied by Fiore and Plotkin.
Adding Recursive Datatypes

Type Variables \( X, Y \)

Types \( A, B \) ::= \( X \mid \alpha \mid A + B \mid I \mid A \otimes B \mid A \rightarrow B \mid !A \mid \text{Circ}(T, U) \mid \mu X. A \)

Intuitionistic types \( P, R \) ::= \( X \mid P + R \mid I \mid P \otimes R \mid !A \mid \text{Circ}(T, U) \mid \mu X. P \)

M-types \( T, U \) ::= \( \alpha \mid I \mid T \otimes U \)

These types are accompanied by some formation rules, which we omit.
Some useful recursive datatypes

Example
Nat ≡ \( \mu X . I + X \)  (intuitionistic)

Example
List Nat ≡ \( \mu X . I + X \otimes \text{Nat} \)  (intuitionistic)

Example
List Qubit ≡ \( \mu X . I + X \otimes \text{Qubit} \)  (linear)

Example
Stream Qubit ≡ \( \mu X . I \rightarrowo (X \otimes \text{Qubit}) \)  (linear)

Example
Stream Nat ≡ \( \mu X .!(X \otimes \text{Nat}) \)  (intuitionistic)
A $\mathbf{CPO}$-enriched model

$\mathbf{CPO}$ – $\omega$-complete partial orders and monotone maps preserving suprema of $\omega$-chains. If $\mathcal{C}$ is Cartesian closed (or even monoidal closed), then the category $\mathcal{B}$ is $\mathcal{C}$-enriched if:

- $\text{obj } \mathcal{B}$ is a set
- For each $B, B' \in \text{obj } \mathcal{B}$, the family $\mathcal{B}(B, B') \in \text{obj } \mathcal{C}$.
- The relevant morphisms – composition, etc., in $\mathcal{B}$ are $\mathcal{C}$-morphisms:

  E.g., $\circ : \mathcal{B}(B', B'') \times \mathcal{B}(B, B') \to \mathcal{B}(B, B'')$ is a $\mathcal{C}$-morphism.

Examples:

1) Since $\mathbf{Set}$ is Cartesian closed, every concrete category is $\mathbf{Set}$-enriched.

2) $\mathbf{CPO}$ is Cartesian closed, so $\mathbf{CPO}$ is self-enriched.

3) $\mathbf{CPO}_{\bot}$ is $\mathbf{CPO}$-enriched, where $\mathbf{CPO}_{\bot}$ is the subcategory of $\mathbf{CPO}$ where every object has a least element ($\bot$) and morphisms preserve $\bot$. 
A CPO-enriched model

CPO – ω-complete partial orders and monotone maps preserving suprema of ω-chains.

A CPO–enriched LNL model includes:

1. A CPO-symmetric monoidal closed category \( \mathcal{L} \) with finite CPO-coproducts.
2. A CPO-symmetric monoidal adjunction:

\[
\begin{array}{ccc}
\text{CPO} & \xrightarrow{\bot} & \mathcal{L}, \\
\mathcal{L}(I, -) & \xleftarrow{\bot} & \text{CPO}
\end{array}
\]

\( F = - \otimes I \)

3. The category \( \mathcal{L} \) is CPO\( _\bot \)-enriched and has \( \omega \)-colimits

Example: \( \mathcal{L} = \text{CPO}_{\bot} \) is the simplest example: \( I = \{ \bot \}_\bot \) and \( F(D) \simeq D_\bot \) for all CPOs \( D \).
A CPO-enriched model

CPO – ω-complete partial orders and monotone maps preserving suprema of ω-chains.

A CPO–enriched LNL model includes:

1. A CPO-symmetric monoidal closed category $\mathcal{L}$ with finite CPO-coproducts.

2. A CPO-symmetric monoidal adjunction:

\[
\begin{align*}
\mathcal{L} & \rightarrow F \dashv \oplus \\
\oplus & \Leftrightarrow \mathcal{L}(I,-)
\end{align*}
\]

3. The category $\mathcal{L}$ is CPO_{⊥!}-enriched and has ω-colimits

Remark

1. and 3. imply $\mathcal{L}$ has a zero object and we can solve recursive domain equations.
Interpretation of recursive types

Interpreting recursive types requires finding initial (final) (co)algebras of certain CPO-endofunctors.

If $T: \mathcal{C} \to \mathcal{C}$ is an endofunctor, then a $T$-algebra is an object $C \in \text{obj } \mathcal{C}$ and a map $\phi_C : TC \to C$.

$C$ is an initial $T$-algebra if for any $T$-algebra $\phi_D : TD \to D$, there is a unique morphism $f : C \to D$ satisfying $\phi_D \circ Tf = f \circ \phi_C$.

Example: If $T : \textbf{Set} \to \textbf{Set}$ is $T(S) = S \cup \{S\}$, then $\mathbb{N}$ is the initial $T$-algebra.

Dually, a final $T$-coalgebra is an object $C$ and a morphism $\psi_D : D \to TD$.

$D$ is a final $T$-coalgebra if any other $T$-coalgebra $\psi_E : E \to TE$ admits a morphism $g : E \to D$ with $\psi_D \circ g = Tg \circ \psi_E$.

Example: If $T : \textbf{Set} \to \textbf{Set}$ is $T(S) = \{\emptyset\} \cup S$, then $\{\emptyset\}$ is the final $T$-coalgebra.
Interpretation of recursive types

Interpreting recursive types requires finding initial (final) (co)algebras of certain CPO-endofunctors.

Lemma (Adámek)

Let $C$ be a category with an initial object $\emptyset$ and let $T : C \to C$ be an endofunctor. Assume further that the following $\omega$-diagram

\[
\emptyset \overset{i}{\to} T\emptyset \overset{T_i}{\to} T^2\emptyset \overset{T^2_i}{\to} \cdots
\]

has a colimit and $T$ preserves it. Then, the induced isomorphism is the initial $T$-algebra.

Corollary

In a symmetric monoidal closed category with finite coproducts and $\omega$-colimits, any endofunctor composed from constants, $\otimes$ and $+$ has an initial algebra.
Embedding-projection pairs

Problem: How do we interpret recursive types which also contain ! and \( \rightarrow \)?

The problem for \( \langle A, B \rangle \mapsto A \rightarrow B \) is that it is covariant in \( B \) and contravariant in \( A \).

Textbook Solution: CPO-enrichment and embedding-projection pairs.

Definition
Given a CPO-enriched category \( C \), an embedding-projection pair is a pair of morphisms \( e : A \rightarrow B \) and \( p : B \rightarrow A \), such that \( p \circ e = \text{id} \) and \( e \circ p \leq \text{id} \).

Theorem
If \( e \) is an embedding, then it has a unique projection, which we denote \( e^* \).

Definition
The subcategory of \( C \) with the same objects, but whose morphisms are embeddings is denoted \( C_e \).
Interpretation of recursive types (contd.)

Theorem (Smyth and Plotkin)

If \( T : C \to D \) is a CPO-enriched functor and \( C \) has \( \omega \)-colimits, then \( T \) preserves \( \omega \)-colimits of embeddings. In other words, the restriction \( T_e : C_e \to D_e \) is \( \omega \)-continuous.

Theorem

In our categorical model, any CPO-endofunctor \( T : \mathcal{L} \to \mathcal{L} \) has an initial \( T \)-algebra, whose inverse is a final \( T \)-coalgebra.

Remark

The above theorem follows directly from results in Fiore’s PhD thesis.
Main Lemma

We define $\mathbf{CPO}_{pe}$ to be the full-on-objects subcategory of $\mathbf{CPO}$ whose morphisms $f$ are those satisfying $F(f) \in \mathcal{L}_e$. We call such $f$ pre-embeddings.

Then there are two semantic models:

- For all types, $\Theta \vdash P \in \mathcal{L}$
- For intuitionistic types, also have $(\Theta \vdash P) \in \mathbf{CPO}_{pe}$

There exists a natural isomorphism

$$\alpha_{\Theta \vdash P} : \Theta \vdash P \circ F^\times \cong F \circ (\Theta \vdash P)$$

Diagrammatically:
Copy and Discard

Let $P$ be an intuitionistic object and $\alpha : P \to F(X)$ an isomorphism.

We can define three maps:

\textit{Discard:} $\diamondal_{P} := P \xrightarrow{\alpha} F(X) \xrightarrow{F(1_X)} F(1) \cong I$;

\textit{Copy:} $\Deltaal_{P} := P \xrightarrow{\alpha} F(X) \xrightarrow{F(\langle \text{id}, \text{id} \rangle)} F(X \times X) \xrightarrow{\text{id}} F(X) \otimes F(X) \xrightarrow{\alpha^{-1} \otimes \alpha^{-1}} P \otimes P$;

\textit{Lift:} $\text{lift}_{P}^{\alpha} := P \xrightarrow{\alpha} F(X) \xrightarrow{F(\eta_X)} !F(X) \xrightarrow{!\langle \alpha^{-1} \rangle} !P$.

Given two intuitionistic objects $P_1$ and $P_2$, a morphism $f : P_1 \to P_2$ is called \textit{intuitionistic}, if there exists a morphism $f' \in \text{CPO}(X, Y)$ and two isomorphisms $\alpha$ and $\beta$, such that $f = P_1 \xrightarrow{\alpha} F(X) \xrightarrow{F(f')} F(Y) \xrightarrow{\beta} P_2$.

If $f : P_1 \to P_2$ is intuitionistic, then:

- $\diamondal_{P_2} \circ f = \diamondal_{P_1}$;
- $\Deltaal_{P_2} \circ f = (f \otimes f) \circ \Deltaal_{P_1}$;
- $\text{lift}_{P_2} \circ f = !f \circ \text{lift}_{P_1}$.
Thank You!

Questions??
Syntax

\[
\frac{\Phi, \alpha : \ell \vdash \ell : \alpha}{\Phi, \ell : \alpha \vdash \ell : \alpha}\quad (\text{label})
\]
\[
\frac{\Phi, \alpha : \ell \vdash \ell : \alpha}{\Phi; \alpha \vdash \ell : \alpha}\quad (\text{var})
\]
\[
\frac{\Phi, \alpha : \ell \vdash \ell : \alpha}{\Phi; \ell \vdash c : A_c}\quad (\text{const})
\]
\[
\frac{\Phi, \ell, \Gamma_1; Q_1 \vdash m : A, \Gamma_2; Q_2 \vdash n : B}{\Phi, \ell, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{let } x = m \text{ in } n : B}\quad (\text{let})
\]
\[
\frac{\Gamma; Q \vdash m : 0}{\Gamma; Q \vdash \square C m : C}\quad (\text{initial})
\]
\[
\frac{\Gamma; Q \vdash m : A}{\Gamma; Q \vdash \text{left}_{A,B} m : A + B}\quad (\text{left})
\]
\[
\frac{\Gamma; Q \vdash m : B}{\Gamma; Q \vdash \text{right}_{A,B} m : A + B}\quad (\text{right})
\]
\[
\frac{\Phi, \Gamma_1; Q_1 \vdash m : A + B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{case } m \text{ of } (\text{left } x \to n \mid \text{right } y \to p) : C}\quad (\text{case})
\]
\[
\frac{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \langle m, n \rangle : A \otimes B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \langle m, n \rangle : A \otimes B}\quad (\text{pair})
\]
\[
\frac{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \langle m, n \rangle : A \otimes B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{let } \langle x, y \rangle = m \text{ in } n : C}\quad (\text{let-pair})
\]
\[
\frac{\Gamma; x : A; Q \vdash m : B}{\Gamma; Q \vdash \lambda x^A.m : A \to B}\quad (\text{abs})
\]
\[
\frac{\Phi, \Gamma_1; Q_1 \vdash m : A \to B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash mn : B}\quad (\text{app})
\]
\[
\frac{\Phi; \ell \vdash m : A}{\Phi; \ell \vdash \text{lift } m : !A}\quad (\text{lift})
\]
\[
\frac{\Gamma; Q \vdash m : ! (T \to U)}{\Gamma; Q \vdash \text{box}_T m : \text{Diag}(T, U)}\quad (\text{box})
\]
\[
\frac{\Phi, \Gamma_1; Q_1 \vdash m : \text{Diag}(T, U)}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{apply}(m, n) : U}\quad (\text{apply})
\]
\[
\frac{\phi; Q' \vdash \ell' : U}{\phi; Q' \vdash \ell' : U}\quad (\text{diag})
\]
\[
\frac{\phi; \ell, S \vdash \text{Diag}(T, U)}{\phi; \ell, S \vdash \text{Diag}(T, U)}\quad (\text{diag})
\]
Operational semantics

\[
\begin{align*}
(S, m) \Downarrow (S', v) & \quad (S', n) \Downarrow (S'', v') \\
(S, \langle m, n \rangle) \Downarrow (S'', \langle v, v' \rangle) \\
(S, \text{lift } m) \Downarrow (S, \text{lift } m) & \\
(S, \text{lift } n) & \quad \text{freshlabels}(T) = (Q, \tilde{\ell}) \quad (\text{id}_Q, n\tilde{\ell}) \Downarrow (D, \tilde{\ell}') \\
(S, \text{box}_T m) \Downarrow (S', (\tilde{\ell}, D, \tilde{\ell}')) & \\
(S, m) \Downarrow (S', (\tilde{\ell}, D, \tilde{\ell}')) & \quad (S', n) \Downarrow (S'', \tilde{k}) \quad \text{append}(S'', \tilde{k}, \tilde{\ell}, D, \tilde{\ell}') = (S''', \tilde{k}') & \\
(S, \text{apply}(m, n)) \Downarrow (S''', \tilde{k}') & \\
(S, m) \Downarrow (S', (\tilde{\ell}, D, \tilde{\ell}')) & \quad (S', n) \Downarrow (S'', \tilde{k}) \quad \text{append}(S'', \tilde{k}, \tilde{\ell}, D, \tilde{\ell}') \text{ undefined} & \\
(S, \text{apply}(m, n)) \Downarrow \text{Error} & \\
(S, (\tilde{\ell}, D, \tilde{\ell}')) \Downarrow (S, (\tilde{\ell}, D, \tilde{\ell}')) & \\
\end{align*}
\]