Semantic Models of Quantum Programming Languages:

Recursion in Categorical Models

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Prototypical Quantum Computer

- Knill's QRAM model: A classical computer with a quantum co-processor

$$
\begin{tikzcd}
\text{Classical Computer} \arrow[rr, bend left, "\text{circuits}"] & & \text{Quantum Co-processor} \arrow[ll, bend left, "\text{measurements}"]
\end{tikzcd}
$$

- Circuit: sequence of unitary operators

How do we program such a device?

Some Basics

Logical Foundations

Predicate calculus:

Predicate symbols: $P, Q, R, \ldots$ each with a fixed arity

Functions symbols: $f, g, h, \ldots$ each with a fixed arity

Terms: $t ::= x \mid c \mid f(t_1, \ldots, t_n)$ where

x is a variable, c a nullary function, and f a function symbol with arity n.

Formulas: $\varphi ::= P(t_1, \ldots, t_n) \mid \perp \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \forall x\, \varphi \mid \exists x\, \varphi.$

- Sound and complete: $\vdash P$ iff $\models P$.

Some Basics

Logical Foundations

Predicate calculus:

Set Theory:

Standard ZF axioms, including *Axiom of Infinity:*
$\exists S.\emptyset \in S \ \wedge (\forall T) T \in S \implies T \cup \{T\} \in S.$

- Sound, *semantically complete*. But,

  Theorem:[Gödel] Any system powerful enough to do arithmetic is incapable of proving its own consistency.

Some Basics

Logical Foundations

Predicate calculus:

Set Theory:

Intuitionistic Logic:(Brouwer, 1907)

Does not include $A \vee \neg A$, or equivalently, $\neg\neg A \to A$.

Emphasis is on *proof*, not validity.

The logic for classical computation.

Some Basics

Logical Foundations

Predicate calculus:

Set Theory:

Intuitionistic Logic:(Brouwer, 1907)

Linear Logic:(Girard, 1987)

- Regards formulas as resources.
- Each hypothesis used once and only once.
- Derivations that obey this paradigm are called *linear*.
- Logic for reasoning about computing over quantum systems.

A Primer on the Basics

Computational Structures

Lambda calculus:(Church, 1934)

- *Untyped λ-calculus:* Terms: $t ::= x \mid \lambda x.t \mid t\,t$ where x is a variable, t a term.
- *Conversion Rules:*
  - $\lambda x.e \equiv_\alpha \lambda y.e[y/x]$ when $y \notin FV(e)$
  - $(\lambda x.e)(e') \to_\beta e[e'/x]$ if $FV(e') \cap BV(e) = \emptyset$.
- Turing complete:
  - Supports full recursion: $\mathrm{rec}\, x.t = t[\mathrm{rec}\, x.t/x]$.
  - Paradoxical combinator $Y := (\lambda x.x\,x)(\lambda x.x\,x)$ produces fixed point for any term

A Primer on the Basics

Computational Structures

Lambda calculus:(Church, 1934)

Untyped λ-calculus:

Simply typed lambda calculus, $\lambda^{\rightarrow}$:

Types: $\tau ::= 1 \mid Int \mid Bool \mid \tau \rightarrow \tau.$
Terms: $t ::= x \mid null \mid n \mid true \mid false \mid t\ t \mid \lambda^{\rightarrow} x : \tau . t.$
Typing Judgements:

$$\frac{}{\Gamma \vdash null{:}1} \qquad \frac{}{\Gamma \vdash n{:}Int} \qquad \frac{}{\Gamma \vdash true{:}Bool} \qquad \frac{}{\Gamma \vdash false{:}Bool}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x{:}\tau} \qquad \frac{\Gamma \vdash e{:}\ \tau \rightarrow \tau' \qquad \Gamma \vdash e'{:}\tau}{\Gamma \vdash e\ e'{:}\tau'} \qquad \frac{\Gamma, x{:}\tau \vdash e{:}\tau'}{\Gamma \vdash (\lambda^{\rightarrow} x{:}\tau . e){:}\tau \rightarrow \tau'}$$

A Primer on the Basics

Computational Structures

Lambda calculus:(Church, 1934)

Untyped λ-calculus:

Simply typed lambda calculus, $\lambda^{\rightarrow}$:

<u>Curry–Howard Correspondence:</u>

Intuitionistic Propositional Natural Deduction		Simply Typed Lambda Calculus
Propositions	$\longleftrightarrow$	*Types*
Proofs	$\longleftrightarrow$	*Terms*

Intuitionistic logic is the logic of classical computation

Computational Structures

Lambda calculus:(Church, 1934)

Untyped λ-calculus:

Simply typed lambda calculus, $\lambda^{\rightarrow}$:

Linear lambda calculus:

$$
\begin{array}{lll}
\text{Types} \quad A, B & ::= & 0 \mid A + B \mid \mathrm{I} \mid A \otimes B \mid A \multimap B \mid !A \\
\text{Terms} \quad M, N, P & ::= & x \mid c \mid \text{let } x = M \text{ in } N \mid \Box_A M \mid \text{left}_{A,B} M \mid \text{right}_{A,B} M \\
 & & \mid \text{case } M \text{ of } \{\text{left } x \to N \mid \text{right } y \to P\} \mid * \mid \langle M, N \rangle \\
 & & \mid \text{let } \langle x, y \rangle = M \text{ in } N \mid \lambda x^A.M \mid MN \mid \text{lift } M \mid \text{force } M
\end{array}
$$

lift $M (= !M)$ – Allows multiple instances of resource M.

force M – produces an instance of M' when $M = !M'$.

Semantic Models

A *category* $\mathcal{C}$ consists of:

(i) a family of objects $obj\ \mathcal{C}$, and

(ii) for each pair $A, B \in obj\ \mathcal{C}$ a family of *morphisms* $\mathcal{C}(A, B)$ satisfying:

$\circ\colon \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ is associative, and $(\forall A, B \in obj\ \mathcal{C})$

$1_A\colon A \to A$ is an identity with $1_B \circ f = f \circ 1_A$.

Example: **Set**, the category of sets and functions.

A category $\mathcal{C}$ is *Cartesian closed* if $\mathcal{C}$ has

finite products – $A \times B$,

a terminal object, $\perp$ satisfying $|\mathcal{C}(A, \perp)| = 1$ for all objects A,

and an internal hom $[A, B]$ satisfying $\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, [A, B])$.

For example, **Set** is Cartesian closed.

Semantic Models

Lambek's Theorem

There is a one-to-one correspondence between Cartesian closed categories and models of the typed lambda calculus.

For example, **Set** is a model for $\lambda^{\to}$

Scott's Corollary

There is a one-to-one correspondence between *reflexive objects* $[X \to X] \leftrightarrows X$ in Cartesian closed categories and models of the untyped lambda calculus

For example, $D_\infty \simeq [D_\infty \to D_\infty]$ is a model.

Fact

The only known *non-degenerate* reflexive objects in Cartesian closed categories are domains.

Semantic Models

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Fact

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What does all this have to do with quantum computing?

Categorical Quantum Mechanics

The Hilbert space formalism provides the basic model of quantum mechanics.

The category **FdHilb** of finite dimensional Hilbert spaces and linear maps has a number of important properties:

FdHilb has a *symmetric tensor product:* $H \otimes K \simeq K \otimes H$.

FdHilb has a unit object $\mathbb{C}$: $\mathbb{C} \otimes H \simeq H$.

FdHilb has a 0-object, the degenerate Hilbert space: $0 \otimes H \simeq 0$.

FdHilb has biproducts: $H \oplus K$.

FdHilb is *dagger compact closed:* there is an involution $H \mapsto H^*$ that extends to $f \mapsto f_*$ on linear maps satisfying $H^{**} \simeq H$, $f_{**} = f$ and $(f_*)^\dagger = (f^\dagger)_*$.

Categorical Quantum Mechanics

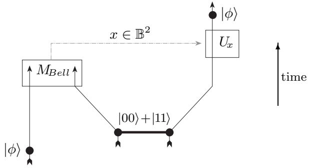
The Hilbert space formalism provides the basic model of quantum mechanics.

Abramsky & Coecke: Dagger compact closed (symmetric monoidal) categories with biproducts model finitary quantum mechanics.

These categories are a model of (multiplicative) linear logic.

They are an abstract setting in which to reason precisely about quantum protocols, such as teleportation and entanglement swapping.

There also is a diagrammatic calculus for reasoning in these categories:



Prototypical Quantum Computer

- Returning to Knill's QRAM model:

$$
\begin{tikzcd}
\text{Classical Computer} \arrow[rr, bend left, "\text{circuits}"] & & \text{Quantum Co-processor} \arrow[ll, bend left, "\text{measurements}"]
\end{tikzcd}
$$

- Circuit: sequence of unitary operators

Prototypical Quantum Computer

- A *quantum programming language* is a classical functional language together with a linear language of *quantum circuits*:

$$\begin{tikzcd} \mathit{Functional\ Language} \arrow[rr, bend left] & & \mathit{Linear\ language} \arrow[ll, bend left] \end{tikzcd}$$

- We elide measurements and focus on a classical functional language for *constructing circuits* and a linear language for *modeling them* as linear morphisms.

- We model *circuit description languages* using Linear / Nonlinear Models

Linear/Non-Linear models

A Linear/Non-Linear (LNL) model is given by the following data:

- A cartesian closed category **C**.
- A symmetric monoidal closed category **L**.
- A symmetric monoidal adjunction:

$$
\begin{tikzcd}
\mathbf{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathbf{L} \arrow[ll, bend left, "G"]
\end{tikzcd}
$$

$$
\begin{array}{rcl}
F(X \times Y) & \simeq & F(X) \otimes F(Y) \\
F(X + Y) & \simeq & F(X) + F(Y) \\
F(\emptyset) = 0 & & F(1) = I \\
F \circ G & = & ! \text{ -- the lift comonad}
\end{array}
$$

An LNL model is a model of Intuitionistic Linear Logic.[1]

[1] Nick Benton. *A mixed linear and non-linear logic: Proofs, terms and models.* CSL'94

Proto-Quipper-M (*Rios and Selinger*)

Types	A, B	$::=$	$\alpha \mid 0 \mid A + B \mid \mathrm{I} \mid A \otimes B \mid A \multimap B \mid !A \mid \mathrm{Circ}(T, U)$
Intuitionistic types	P, R	$::=$	$0 \mid P + R \mid \mathrm{I} \mid P \otimes R \mid !A \mid \mathrm{Circ}(T, U)$
M-types	T, U	$::=$	$\alpha \mid \mathrm{I} \mid T \otimes U$

$$
\begin{array}{lll}
\text{Terms} \quad M, N & ::= & x \mid \ell \mid c \mid \text{let } x = M \text{ in } N \\
& & \mid \Box_A M \mid \text{left}_{A,B} M \mid \text{right}_{A,B} M \mid \text{case } M \text{ of } \{\text{left } x \to N \mid \text{right } y \to P\} \\
& & \mid * \mid M; N \mid \langle M, N \rangle \mid \text{let } \langle x, y \rangle = M \text{ in } N \mid \lambda x^A.M \mid MN \\
& & \mid \text{lift } M \mid \text{force } M \mid \mathbf{box}_T M \mid \text{apply}(M, N) \mid (\vec{\ell}, C, \vec{\ell'})
\end{array}
$$

- All types other than Intuitionistic types are *linear*
- M-types: morphisms from a symmetric monoidal category such as M = **FdHilb**
- Only use one (combined) form of type judgement

Example

Assume $H : Q \multimap Q$ is a constant representing the Hadamard gate.

Example

two-hadamard : $\mathrm{Circ}(Q, Q)$
two-hadamard $\equiv \mathrm{box}_Q$ lift $\lambda q^Q.HHq$

This program creates a completed circuit consisting of two H gates. The term is intuitionistic (can be copied, deleted).

Example

Shor's algorithm for integer factorization may be seen as an infinite family of quantum circuits – each circuit is a procedure for factoring an *n*-bit integer, for a fixed *n*.

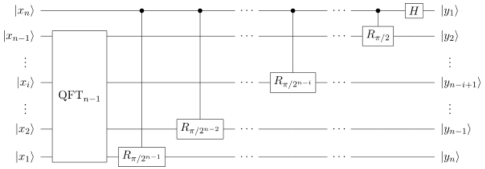


Figure: Quantum Fourier Transform on *n* qubits (subroutine in Shor's algorithm).[2]

Proto-Quipper-M is used to describe *families* of morphisms in an arbitrary, but fixed, symmetric monoidal category, **M**.

[2]Figure source: `https://commons.wikimedia.org/w/index.php?curid=14545612`

Concrete model of Proto-Quipper-M

A simple Proto-Quipper-M model is given by the LNL model:

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] & \overline{\mathbf{M}} \arrow[l, bend left, "\overline{\mathbf{M}}(I, -)"] \end{tikzcd}$$

where $\overline{\mathbf{M}} = [\mathbf{M}^{\mathrm{op}}, \mathbf{Set}]$ is a closed, product complete category containing given SMC $\mathbf{M}$

Theorem (Rios & Selinger)

The simple categorical model of Proto-Quipper-M is type-safe, sound, and computationally adequate

Concrete model of Proto-Quipper-M

There are two semantic models:

- For all types, $[\![P]\!] \in \overline{\mathbf{M}}$
- For intuitionistic types, also have $(\!|P|\!) \in \mathbf{Set}$

Theorem

For any intuitionistic type P, there exists a canonical isomorphism $\alpha_P : [\![P]\!] \to F(\!|P|\!)$.

So we can define copy and discard morphisms for each intuitionistic type P:

$$\Delta_P := [\![P]\!] \xrightarrow{\alpha_P} F(\!|P|\!) \xrightarrow{F\langle id, id\rangle} F((\!|P|\!) \times (\!|P|\!)) \xrightarrow{\cong} F(\!|P|\!) \otimes F(\!|P|\!) \xrightarrow{\alpha_P^{-1} \otimes \alpha_P^{-1}} [\![P]\!] \otimes [\![P]\!]$$

$$\diamond_P := [\![P]\!] \xrightarrow{\alpha_P} F(\!|P|\!) \xrightarrow{F1} F1 \xrightarrow{\cong} I$$

where $FX = X \odot I$

Our Work: Adding Recursion

- Focus on adding recursive *types*.
  - Term recursion follows from recursive types.
- Main difficulty is with the categorical model.
- How can we copy/discard intuitionistic recursive types?
  - A list of qubits should be *linear* – cannot copy/discard.
  - A list of natural numbers should be *intuitionistic* – can *implicitly* copy/discard.
- For the rest of the talk we focus on the linear/non-linear type structure.
- How do we design a linear/non-linear FPC [3] ?

[3] FPC is an intuitionistic Fixed Point Calculus studied by Fiore and Plotkin.

Adding Recursive Datatypes

Type Variables	X, Y		
Types	A, B	$::=$	$X \mid \alpha \mid A + B \mid \mathrm{I} \mid A \otimes B \mid A \multimap B \mid !A \mid \mathrm{Circ}(T, U)$
			$\mid \mu X.A$
Intuitionistic types	P, R	$::=$	$X \mid P + R \mid \mathrm{I} \mid P \otimes R \mid !A \mid \mathrm{Circ}(T, U) \mid \mu X.P$
M-types	T, U	$::=$	$\alpha \mid \mathrm{I} \mid T \otimes U$

These types are accompanied by some formation rules, which we omit.

Some useful recursive datatypes

Example

$\mathsf{Nat} \equiv \mu X.I + X \qquad$ (intuitionistic)

Example

$\mathsf{List\ Nat} \equiv \mu X.I + X \otimes \mathsf{Nat} \qquad$ (intuitionistic)

Example

$\mathsf{List\ Qubit} \equiv \mu X.I + X \otimes \mathsf{Qubit} \qquad$ (linear)

Example

$\mathsf{Stream\ Qubit} \equiv \mu X.I \multimap (X \otimes \mathsf{Qubit}) \qquad$ (linear)

Example

$\mathsf{Stream\ Nat} \equiv \mu X.!(X \otimes \mathsf{Nat}) \qquad$ (intuitionistic)

A CPO-enriched model

CPO – ω-complete partial orders and monotone maps preserving suprema of ω-chains.

If $\mathcal{C}$ is Cartesian closed (or even monoidal closed), then the category $\mathcal{B}$ is $\mathbb{C}$-enriched if:

obj $\mathcal{B}$ is a set

For each $B, B' \in \textit{obj } \mathcal{B}$, the family $\mathcal{B}(B, B') \in \textit{obj } \mathcal{C}$.

The relevant morphisms – composition, etc., in $\mathcal{B}$ are $\mathcal{C}$-morphisms:

E.g., $\circ\colon \mathcal{B}(B', B") \times \mathcal{B}(B, B') \to \mathcal{B}(B, B")$ is a $\mathcal{C}$-morphism.

Examples: 1) Since **Set** is Cartesian closed, every *concrete* category is **Set**-enriched.

2) **CPO** is Cartesian closed, so **CPO** is self-enriched.

3) $\mathbf{CPO}_{\perp!}$ is **CPO**-enriched, where $\mathbf{CPO}_{\perp!}$ is the subcategory of **CPO** where every object has a least element ($\perp$) and morphisms preserve $\perp$.

A **CPO**-enriched model

CPO – ω-complete partial orders and monotone maps preserving suprema of ω-chains.

A **CPO**–enriched LNL model includes:

1. A **CPO**-symmetric monoidal closed category $\mathcal{L}$ with finite **CPO**-coproducts.
2. A **CPO**-symmetric monoidal adjunction:

$$
\begin{tikzcd}
\mathbf{CPO} \arrow[rr, bend left, "F = - \odot I"] \arrow[rr, phantom, "\perp"] & & \mathcal{L} \arrow[ll, bend left, "\mathcal{L}(I, -)"]
\end{tikzcd},
$$

3. The category $\mathcal{L}$ is $\mathbf{CPO}_{\perp !}$-enriched and has ω-colimits

Example: $\mathcal{L} = \mathbf{CPO}_{\perp !}$ is the simplest example: $I = \{\perp\}_\perp$ and $F(D) \simeq D_\perp$ for all **CPO**s D.

A **CPO**-enriched model

CPO – ω-complete partial orders and monotone maps preserving suprema of ω-chains.

A **CPO**–enriched LNL model includes:

1. A **CPO**-symmetric monoidal closed category $\mathcal{L}$ with finite **CPO**-coproducts.
2. A **CPO**-symmetric monoidal adjunction:

$$\begin{tikzcd} \mathbf{CPO} \arrow[rr, bend left, "F = - \odot I"] \arrow[rr, phantom, "\perp"] && \mathcal{L}, \arrow[ll, bend left, "\mathcal{L}(I, -)"] \end{tikzcd}$$

3. The category $\mathcal{L}$ is $\mathbf{CPO}_{\perp!}$-enriched and has ω-colimits

Remark

1. and 3. imply $\mathcal{L}$ has a zero object and we can solve recursive domain equations.

Interpretation of recursive types

Interpreting recursive types requires finding initial (final) (co)algebras of certain **CPO**-endofunctors.

If $T: \mathcal{C} \to \mathcal{C}$ is an endofunctor, then a T-algebra is an object $C \in obj\ \mathbf{C}$ and a map $\phi_C: TC \to C$.

C is an *initial T-algebra* if for any T-algebra $\phi_D: TD \to D$, there is a unique morphism $f: C \to D$ satisfying $\phi_D \circ Tf = f \circ \phi_C$.

Example: If $T: \mathbf{Set} \to \mathbf{Set}$ is $T(S) = S \cup \{S\}$, then $\mathbb{N}$ is the initial T-algebra.

Dually, a final T-coalgebra is an object C and a morphism $\psi_D: D \to TD$.

D is a *final T-coalgebra* if any other T-coalgebra $\psi_E: E \to TE$ admits a morphism $g: E \to D$ with $\psi_D \circ g = Tg \circ \psi_E$.

Example: If $T: \mathbf{Set} \to \mathbf{Set}$ is $T(S) = \{\emptyset\} \cup S$, then $\{\emptyset\}$ is the final T-coalgebra.

Interpretation of recursive types

Interpreting recursive types requires finding initial (final) (co)algebras of certain **CPO**-endofunctors.

Lemma (Adámek)

Let $\mathbf{C}$ *be a category with an initial object* $\emptyset$ *and let* $T : \mathbf{C} \to \mathbf{C}$ *be an endofunctor. Assume further that the following* ω*-diagram*

$$\emptyset \xrightarrow{\iota} T\emptyset \xrightarrow{T\iota} T^2\emptyset \xrightarrow{T^2\iota} \cdots$$

has a colimit and T *preserves it. Then, the induced isomorphism is the initial* T*-algebra.*

Corollary

In a symmetric monoidal closed category with finite coproducts and ω*-colimits, any endofunctor composed from constants,* $\otimes$ *and* $+$ *has an initial algebra.*

Embedding-projection pairs

Problem: How do we interpret recursive types which also contain ! and $\multimap$?

The problem for $\langle A, B\rangle \mapsto A \multimap B$ is that it is covariant in B and contravariant in A.

Textbook Solution: CPO-enrichment and embedding-projection pairs.

Definition

Given a **CPO**-enriched category $\mathbf{C}$, an *embedding-projection* pair is a pair of morphisms $e : A \to B$ and $p : B \to A$, such that $p \circ e = \mathrm{id}$ and $e \circ p \leq \mathrm{id}$.

Theorem

If e is an embedding, then it has a unique projection, which we denote e^*.

Definition

The subcategory of $\mathbf{C}$ with the same objects, but whose morphisms are embeddings is denoted $\mathbf{C}_e$.

Interpretation of recursive types (contd.)

Theorem (Smyth and Plotkin)

*If $T : \mathbf{C} \to \mathbf{D}$ is a **CPO**-enriched functor and $\mathbf{C}$ has ω-colimits, then T preserves ω-colimits of embeddings. In other words, the restriction $T_e : \mathbf{C}_e \to \mathbf{D}_e$ is ω-continuous.*

Theorem

*In our categorical model, any **CPO**-endofunctor $T : \mathcal{L} \to \mathcal{L}$ has an initial T-algebra, whose inverse is a final T-coalgebra.*

Remark

The above theorem follows directly from results in Fiore's PhD thesis.

Main Lemma

We define $\mathbf{CPO}_{pe}$ to be the full-on-objects subcategory of $\mathbf{CPO}$ whose morphisms f are those satisfying $F(f) \in \mathcal{L}_e$. We call such f *pre-embeddings*.

Then there are two semantic models:

- For all types, $[\![\Theta \vdash P]\!] \in \mathcal{L}$
- For intuitionistic types, also have $(\!|\Theta \vdash P|\!) \in \mathbf{CPO}_{pe}$

There exists a natural isomorphism

$$\alpha_{\Theta \vdash P} : [\![\Theta \vdash P]\!]_s \circ F^{\times n} \Longrightarrow F \circ (\!|\Theta \vdash P|\!)$$

Diagrammatically:

$$\begin{tikzcd}
\mathcal{L}_e^{|\Theta|} \arrow[rr, "{[\![\Theta \vdash P]\!]_e}"] & & \mathcal{L}_e \\
& \arrow[r, phantom, "\overset{\alpha}{\Longrightarrow}"] & \\
\mathbf{CPO}_{pe}^{|\Theta|} \arrow[uu, "F^{\times|\Theta|}"] \arrow[rr, "{(\!|\Theta \vdash P|\!)}"'] & & \mathbf{CPO}_{pe} \arrow[uu, "F"']
\end{tikzcd}$$

Copy and Discard

Let P be an intuitionistic object and $\alpha : P \to F(X)$ an isomorphism.

We can define three maps:

$$\textit{Discard:}\quad \diamond_P^\alpha := P \xrightarrow{\alpha} F(X) \xrightarrow{F(1_X)} F(1) \xrightarrow{\cong} I;$$

$$\textit{Copy:}\quad \Delta_P^\alpha := P \xrightarrow{\alpha} F(X) \xrightarrow{F(\langle \mathrm{id}, \mathrm{id}\rangle)} F(X \times X) \xrightarrow{\cong} F(X) \otimes F(X) \xrightarrow{\alpha^{-1} \otimes \alpha^{-1}} P \otimes P;$$

$$\textit{Lift:}\quad \mathbf{lift}_P^\alpha := P \xrightarrow{\alpha} F(X) \xrightarrow{F(\eta_X)} !F(X) \xrightarrow{!(\alpha^{-1})} !P.$$

Given two intuitionistic objects P_1 and P_2, a morphism $f : P_1 \to P_2$ is called *intuitionistic*, if there exists a morphism $f' \in \mathbf{CPO}(X, Y)$ and two isomorphisms α and β, such that $f = P_1 \xrightarrow{\alpha} F(X) \xrightarrow{F(f')} F(Y) \xrightarrow{\beta} P_2$.

If $f : P_1 \to P_2$ is intuitionistic, then:

- $\diamond_{P_2} \circ f = \diamond_{P_1}$;
- $\Delta_{P_2} \circ f = (f \otimes f) \circ \Delta_{P_1}$;
- $\mathbf{lift}_{P_2} \circ f = !f \circ \mathbf{lift}_{P_1}$.

Thank You!

Questions??

Syntax

$$\frac{}{\Phi, x : A; \emptyset \vdash x : A}\ (\text{var}) \qquad \frac{}{\Phi; \ell : \alpha \vdash \ell : \alpha}\ (\text{label}) \qquad \frac{}{\Phi; \emptyset \vdash c : A_c}\ (\text{const}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : A \qquad \Phi, \Gamma_2, x : A; Q_2 \vdash n : B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{let } x = m \text{ in } n : B}\ (\text{let})$$

$$\frac{\Gamma; Q \vdash m : 0}{\Gamma; Q \vdash \Box_C m : C}\ (\text{initial}) \qquad \frac{\Gamma; Q \vdash m : A}{\Gamma; Q \vdash \text{left}_{A,B} m : A + B}\ (\text{left}) \qquad \frac{\Gamma; Q \vdash m : B}{\Gamma; Q \vdash \text{right}_{A,B} m : A + B}\ (\text{right}) \qquad \frac{}{\Phi; \emptyset \vdash * : I}\ (^*)$$

$$\frac{\Phi, \Gamma_1; Q_1 \vdash m : A + B \qquad \Phi, \Gamma_2, x : A; Q_2 \vdash n : C \qquad \Phi, \Gamma_2, y : B; Q_2 \vdash p : C}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{case } m \text{ of } \{\text{left } x \to n \mid \text{right } y \to p\} : C}\ (\text{case}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : I \qquad \Phi, \Gamma_2; Q_2 \vdash n : C}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash m; n : C}\ (\text{seq})$$

$$\frac{\Phi, \Gamma_1; Q_1 \vdash m : A \qquad \Phi, \Gamma_2; Q_2 \vdash n : B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \langle m, n \rangle : A \otimes B}\ (\text{pair}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : A \otimes B \qquad \Phi, \Gamma_2, x : A, y : B; Q_2 \vdash n : C}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{let } \langle x, y \rangle = m \text{ in } n : C}\ (\text{let-pair})$$

$$\frac{\Gamma, x : A; Q \vdash m : B}{\Gamma; Q \vdash \lambda x^A . m : A \multimap B}\ (\text{abs}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : A \multimap B \qquad \Phi, \Gamma_2; Q_2 \vdash n : A}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash mn : B}\ (\text{app}) \qquad \frac{\Phi; \emptyset \vdash m : A}{\Phi; \emptyset \vdash \text{lift } m : !A}\ (\text{lift}) \qquad \frac{\Gamma; Q \vdash m : !A}{\Gamma; Q \vdash \text{force } m : A}\ (\text{force})$$

$$\frac{\Gamma; Q \vdash m : !(T \multimap U)}{\Gamma; Q \vdash \text{box}_T m : \text{Diag}(T, U)}\ (\text{box}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : \text{Diag}(T, U) \quad \Phi, \Gamma_2; Q_2 \vdash n : T}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{apply}(m, n) : U}\ (\text{apply}) \qquad \frac{\emptyset; Q \vdash \vec{\ell} : T \quad \emptyset; Q' \vdash \vec{\ell}' : U \quad S \in \mathbf{M}_L(Q, Q')}{\Phi; \emptyset \vdash (\vec{\ell}, S, \vec{\ell}') : \text{Diag}(T, U)}\ (\text{diag})$$

Operational semantics

$$\frac{(S, m) \Downarrow (S', v) \quad (S', n) \Downarrow (S'', v')}{(S, \langle m, n\rangle) \Downarrow (S'', \langle v, v'\rangle)} \qquad \frac{(S, m) \Downarrow (S', \langle v, v'\rangle) \quad (S', n[v / x, v' / y]) \Downarrow (S'', w)}{(S, \mathrm{let}\ \langle x, y\rangle = m\ \mathrm{in}\ n) \Downarrow (S'', w)}$$

$$\frac{}{(S, \mathrm{lift}\ m) \Downarrow (S, \mathrm{lift}\ m)} \qquad \frac{(S, m) \Downarrow (S', \mathrm{lift}\ m') \quad (S', m') \Downarrow (S'', v)}{(S, \mathrm{force}\ m) \Downarrow (S'', v)}$$

$$\frac{(S, m) \Downarrow (S', \mathrm{lift}\ n) \quad \mathrm{freshlabels}(T) = (Q, \vec{\ell}) \quad (\mathrm{id}_Q, n\vec{\ell}) \Downarrow (D, \vec{\ell}')}{(S, \mathrm{box}_T m) \Downarrow (S', (\vec{\ell}, D, \vec{\ell}'))}$$

$$\frac{(S, m) \Downarrow (S', (\vec{\ell}, D, \vec{\ell}')) \quad (S', n) \Downarrow (S'', \vec{k}) \quad \mathrm{append}(S'', \vec{k}, \vec{\ell}, D, \vec{\ell}') = (S''', \vec{k}')}{(S, \mathrm{apply}(m, n)) \Downarrow (S''', \vec{k}')}$$

$$\frac{(S, m) \Downarrow (S', (\vec{\ell}, D, \vec{\ell}')) \quad (S', n) \Downarrow (S'', \vec{k}) \quad \mathrm{append}(S'', \vec{k}, \vec{\ell}, D, \vec{\ell}')\ \mathrm{undefined}}{(S, \mathrm{apply}(m, n)) \Downarrow \mathrm{Error}} \qquad \frac{}{(S, (\vec{\ell}, D, \vec{\ell}')) \Downarrow (S, (\vec{\ell}, D, \vec{\ell}'))}$$