# Approximating Measurable Maps

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# Outline

## Stochastic Processes and Skorohod's Theorem

A *stochastic process* is a time-indexed family $\{X_t \mid t \in T \subseteq \mathbb{R}_+\}$ of random variables / elements $X_t \colon \Omega \to S$, where $(\Omega, \Sigma_\Omega, \mu)$ is a probability space, and $S$ is a Polish space.

**Fact:** If $S$ is Polish, then so is $(Prob(S), d_p)$, where $d_p$ is the *Prokhorov metric*. In fact, $d_p$ generates the weak topology.

### Skorohod's Theorem

Let $S$ be a Polish space, let $\nu \in \mathrm{Prob}\, S$, and let $\lambda$ denote Lebesgue measure on $[0,1]$. Then there is a random variable $X \colon [0,1] \to S$ satisfying $X_*(\lambda) = \nu$.[1]

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X \colon [0,1] \to S$ with $X_*(\lambda) = \nu, X_{n*}(\lambda) = \nu_n$ satisfy $X_n \to X$ $\lambda$-a.e.

**Fact:** We could use any *standard probability space* $(S, \Sigma_S, m)$ instead of $([0,1], \mathcal{B}_{[0,1]}, \lambda)$.

[1] $X_*(\lambda)(A) = \lambda(X^{-1}(A))$ is the *push forward of* $X$, also called the *law of* $X$.

*Domains* are partially ordered sets with additional properties;

**Informatic partial order**

$p \sqsubseteq q$ if $q$ contains more information than $p$.

*Example: The Upper Space of $Y$ locally compact sober:*

$$U(Y) = (\{K \subseteq Y \mid \emptyset \neq K \text{ compact, saturated}\} \cup \{Y\}, \supseteq).$$

**Directed completeness**

$\emptyset \neq D \subseteq P$ *directed* if $x, y \in D \;\Rightarrow\; (\exists z \in D)\, x, y \leq z$.
$P$ is *directed complete* if $D \subseteq P$ directed $\;\Rightarrow\;$ $\sup D$ exists.

$$\mathcal{F} \subseteq U(Y) \text{ directed} \Rightarrow \sup \mathcal{F} = \bigcap \mathcal{F}.$$

**Approximation**

$x \ll y$ iff $y \leq \sup D \;\Rightarrow\; (\exists d \in D)\, x \leq d$.
*Domain:* $\mathord{\downarrow\!\!\downarrow} y = \{x \mid x \ll y\}$ directed and $y = \sup \mathord{\downarrow\!\!\downarrow} y$

$$K \ll L \text{ iff } L \subseteq K^\circ; \qquad L = \bigcap \{K \mid L \subseteq K^\circ\} = \sup\{K \mid K \ll L\}.$$

## Scott Topology

$U \subseteq P$ *Scott open* if:

- $U = \uparrow U = \{x \in P \mid (\exists u \in U)\, u \leq x\}$ and
- $D$ directed, $\sup D \in U \;\Rightarrow\; D \cap U \neq \emptyset$.

Upper Vietoris topology on $U(Y)$.

$\Uparrow x = \{y \mid x \ll y\}$ is Scott open.

## Morphisms

$f\colon P \to Q$ is *Scott continuous* if:

- $f$ is monotone, and
- $D$ directed $\Rightarrow\; f(\sup D) = \sup f(D)$.

Upper semicontinuous maps between $U(Y)$ and $U(Z)$:

$f\colon Y \to Z \;\Rightarrow U(f)\colon U(Y) \to U(Z)$ by $U(f)(K) = f(K)$

is monotone and $f(\bigcap \mathcal{F}) = \bigcap f(\mathcal{F})$.

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## Lawson Topology

*Basis:* $\{\Uparrow x \setminus \uparrow F \mid F \subseteq P \text{ finite}\}$

Hausdorff refinement of Scott topology.

All the domains we discuss are Lawson compact.

Vietoris topology on $U(Y)$.

## Embedding $S$ in $U(\overline{S})$

$S$ Polish $\Rightarrow$ $S \hookrightarrow [0,1]^\omega$ $\Rightarrow$ $S \subseteq_{G_\delta} \overline{S}$ compact Polish

$\{K_n \mid K_n \in U(\overline{S}), n > 0\}$: neighborhood basis of compact subsets of $\overline{S}$.
Then:

1° $S \subseteq \overline{S} \hookrightarrow \operatorname{Max} U(\overline{S}) \subseteq U(\overline{S})$ by $x \mapsto \{x\} = \bigcap_n \{K_n \mid x \in K_n^\circ\}$

2° $S$ inherits the Scott topology = Lawson topology on $\operatorname{Max} U(\overline{S})$.

3° Each cover $\mathbb{K}_n \stackrel{\text{def}}{=} \{K_1, \ldots, K_{m_n}\}$ with $\overline{S} \subseteq \bigcup_{i \le m_n} K_i^\circ$ defines a Scott-continuous projection $\psi_n\colon U(\overline{S}) \to U(\overline{S})$ with finite image $\mathbb{L}_n \stackrel{\text{def}}{=} \langle \mathbb{K}_n \rangle$. This implies $\psi_n \ll \mathbf{1}_{U(\overline{S})}$.

4° Ordering covers by refinement yields $\mathbf{1}_{U(\overline{S})} = \sup_n \psi_n$.

## Prob$(P)$ is a Domain

$P$ (Lawson compact) domain $\Rightarrow$ Prob$(P)$ (Lawson compact) domain:

1° $\mu \leq \nu$ iff $\int f d\mu \leq \int f d\nu$ ($\forall f : P \to \mathbb{R}_+$ Scott continuous)

2° $P$ Lawson compact $\Rightarrow$ $(\mathrm{Prob}(P), \mathit{weak}) = (\mathrm{Prob}(P), \mathit{Lawson})$.

3° $P = U(\overline{S}), S$ Polish $\Rightarrow$ $\mathbf{1}_{\mathrm{Prob}(P)} = \sup_n \psi_{n*}$, so

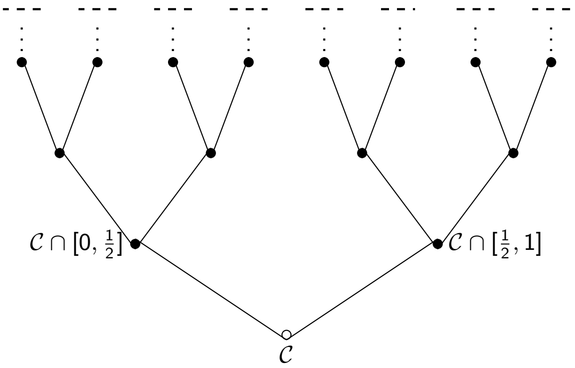
$\mu = \sup_n \psi_{n*}\, \mu$, with $\psi_{n*}\, \mu = \sum_{K \in \mathbb{L}_n} \mu(K)\delta_K$ $(\forall \mu)$.

4° By construction

$$\psi_{1*}(\mu) \ll \cdots \ll \psi_{n*}(\mu) \ll \cdots \ll \mu = \sup_n \psi_{n*}(\mu).$$

*Note:* $\psi_{n*}(\mu) \ll \mu \Rightarrow \uparrow\psi_{n*}(\mu)$ compact neighborhood of $\mu$.

## A Domain Environment for the Cantor set, $\mathcal{C}$




$$\mathcal{C} \hookrightarrow \mathsf{Max}\,\mathbb{CT} \subseteq \mathbb{CT} ::= \{0,1\}^* \cup \{0,1\}^\omega$$

$$\mathcal{C}_p = \{0,1\}^p \Rightarrow \exists \pi_p \colon \mathcal{C} \to \mathcal{C}_p \text{ retraction}$$

$$\iota_p \colon \mathcal{C}_p \hookrightarrow \mathcal{C} \;\Rightarrow\; \iota_n \circ \pi_p \colon \mathcal{C} \to \mathcal{C} \text{ lower semicontinuous,}$$
$$\text{and } \mathbf{1}_{\mathcal{C}} = \sup_p \iota_p \circ \pi_p|_{\mathcal{C}}.$$

## Some Random Variables

- Given $\mu \in \mathrm{Prob}U(\overline{S})$, $\psi_{n*}(\mu) = \sum_{K \in \mathbb{L}_n} \mu(K)\delta_K$, and any $p \geq \log_2(|\mathbb{L}_n| \cdot \min_{K \in \mathbb{L}_n} \mu(K))$, there are $r_K \in D_p = \{\frac{s}{2^p} \mid 0 \leq s \leq 2^p\}$ with $\mu(K) - \frac{r_K}{2^p} \leq \frac{1}{|\mathbb{L}_n| \cdot 2^p}$.
- Then

$$\nu_{n,p} \stackrel{\text{def}}{=} (1 - \sum_{K \in \mathbb{L}_n} r_K)\delta_S + \sum_{K \in \mathbb{L}_n} r_K \delta_K \ll \sum_{K \in \mathbb{L}_n} \mu(K)\delta_K = \psi_{n*}(\mu),$$

$||\nu_{n,p} - \psi_{n*}(\mu)|| \leq \frac{1}{2^p}$ and $\sup_p \nu_{n,p} = \psi_{n*}(\mu)$.

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$||\nu_{n,p} - \psi_{n*}(\mu)|| \leq \frac{1}{2^p}$ and $\sup_p \nu_{n,p} = \psi_{n*}(\mu)$.

- Define $f_{n,p}\colon \mathcal{C}_p \to U(\overline{S})$ by $f_{n,p}(j) = \begin{cases} K_n & \text{if } 1 \leq j \leq r_1 \\ K_2 & \text{if } r_1 < j \leq r_1 + r_2 \\ \vdots & \\ S & \text{if } \sum_{K\in\mathbb{L}_n} r_K < j \end{cases}$

Then $f_{n,p*}(\frac{1}{2^p}\sum_{i\leq 2^p} \delta_{\frac{i}{2^p}}) = (1 - \sum_i r_i)\delta_S + \sum_{K\in\mathbb{L}_n} r_K\delta_K = \nu_{n,p}$.

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Then $f_{n,p*}(\frac{1}{2^p} \sum_{i \leq 2^p} \delta_{\frac{i}{2^p}}) = (1 - \sum_i r_i)\delta_S + \sum_{K \in \mathbb{L}_n} r_K \delta_K = \nu_{n,p}$.

- And, $f_{n,p} \circ \pi_p \colon \mathcal{C} \to U(\overline{S})$ is Lawson continuous with

$$(f_{n,p} \circ \pi_p)_* (\mu_{\mathcal{C}}) = f_{n,p*}\left(\frac{1}{2^p} \sum_{i \leq 2^p} \delta_{\frac{i}{2^p}}\right) = \nu_{n,p} \ll \psi_{n*}(\mu).$$

## Recursively Defining Random Variables

To bring order to the family $\{f_{n,p} \mid n, p\}$, we apply domain theory and recursion:

$1^\circ$ Let $f_{1,p_1}\colon \mathcal{C}_{p_1} \to U(\overline{S})$ with $(f_{1,p_i} \circ \pi_{p_1})(\mu_{\mathcal{C}}) = \nu_{1,p_1} \ll \psi_{1*}(\mu)$.

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$2^\circ$ Since $\psi_1(\mu) \ll \psi_2(\mu) = \sup_p \nu_{2,p}$, we can find $p_2 > p_1$ with $\nu_{1,p_1} \ll \nu_{2,p_2}$.

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2° Since $\psi_1(\mu) \ll \psi_2(\mu) = \sup_p \nu_{2,p}$, we can find $p_2 > p_1$ with $\nu_{1,p_1} \ll \nu_{2,p_2}$.

3° Using the Splitting Lemma, this implies $f_{1,p_1} \circ \pi_{p_1 p_2} \leq f_{n_2,p_2}$, from which it follows that $f_{1,p_1} \circ \pi_{p_1} \leq f_{2,p_2} \circ \pi_{p_2}$.

4° We obtain an *increasing sequence* $f_{n,p_n}$ with $f_{n,p_n} \circ \pi_{p_n p_m} \leq f_{m,p_m}$ for $n \leq m$.

## Recursively Defining Random Variables

- By construction $f_{n,p_n}\colon \uparrow\mathcal{C}_{p_n} \to U(\overline{S})$ satisfies $f_{n,p_n} \circ \pi_{p_n p_m} \leq f_{m,p_m}$, so $X_n = (\sup_n f_{n,p_n} \circ \pi_{p_n})|_{\mathcal{C}} \colon \mathcal{C} \to U(\overline{S})$ measurable with $X_{n*}(\mu_{\mathcal{C}}) = \mu$.
- If $\mu_m \to_w \mu \in \mathrm{Prob}(S)$, define $\nu_{m,n} \ll \psi_{n*}(\mu_m)$ and $f_{m,n,p_{m,n}}\colon \uparrow\mathcal{C}_{p_{m,n}} \to U(\overline{S})$ as above.
- Then $X_m = (\sup_n f_{m,n,p_{m,n}})|_{\mathcal{C}}$ satisfies $X_m\colon \mathcal{C} \to U(\overline{S})$ is measurable with $X_{m*}(\mu_{\mathcal{C}}) = \mu_m$.
- $\mu_m \to_w \mu = \sup_n \nu_{n,p_n}, \mu_m = \sup_{m,n} \nu_{m,n} \ \Rightarrow\ \nu_{n,p_n} \ll \nu_{m,n}$ eventually.
- This implies $f_{n,p_n} \circ \pi_{p_n p_{m',n'}} \leq f_{m',n',p_{m',n'}}$ eventually.
- This is used to show $X_m \to X$ a.s. $\mu_{\mathcal{C}}$.

## Recursively Defining Random Variables

- By construction $f_{n,p_n}\colon {\uparrow}\mathcal{C}_{p_n} \to U(\overline{S})$ satisfies $f_{n,p_n} \circ \pi_{p_n p_m} \leq f_{m,p_m}$, so $X_n = (\sup_n f_{n,p_n} \circ \pi_{p_n})|_{\mathcal{C}}\colon \mathcal{C} \to U(\overline{S})$ measurable with $X_{n*}(\mu_{\mathcal{C}}) = \mu$.
- If $\mu_m \to_w \mu \in \mathrm{Prob}(S)$, define $\nu_{m,n} \ll \psi_{n*}(\mu_m)$ and $f_{m,n,p_{m,n}}\colon {\uparrow}\mathcal{C}_{p_{m,n}} \to U(\overline{S})$ as above.
- Then $X_m = (\sup_n f_{m,n,p_{m,n}})|_{\mathcal{C}}$ satisfies $X_m\colon \mathcal{C} \to U(\overline{S})$ is measurable with $X_{m*}(\mu_{\mathcal{C}}) = \mu_m$.
- $\mu_m \to_w \mu = \sup_n \nu_{n,p_n}, \mu_m = \sup_{m,n} \nu_{m,n} \;\Rightarrow\; \nu_{n,p_n} \ll \nu_{m,n}$ eventually.
- This implies $f_{n,p_n} \circ \pi_{p_n p_{m',n'}} \leq f_{m',n',p_{m',n'}}$ eventually.
- This is used to show $X_m \to X$ a.s. $\mu_{\mathcal{C}}$.
- We know $\varphi\colon \mathcal{C} \overset{\rightarrow}{\leftarrow} [0,1]\colon \iota$ is a projection-embedding pair, with $\varphi_*(\mu_{\mathcal{C}}) = \lambda$ and $\iota_*(\lambda) = \mu_{\mathcal{C}}$. Composing $X\colon \mathcal{C} \to U(\overline{S})$ with $\iota$ yields random variable $X \circ \iota\colon [0,1] \to U(\overline{S})$ with law $(\iota \circ X)_*(\lambda) = \mu$, etc.

*Questions?*