Probability In Computational Models:

- Programming language semantics
- Applications in mathematics, physics, ... 
- Probabilistic programming semantics
- ...
Probability Theory

A random variable is a measurable map $X : \Omega \rightarrow S$, where $(\Omega, \Sigma_\Omega, \mu)$ is a probability space and $(S, \Sigma_S)$ is a measurable space.

The law of $X$ is $X_* \mu \in \text{Prob}(Y)$ by $X_* \mu(A) = \mu(X^{-1}(A))$.

Note: $\text{Prob}: \text{Meas} \rightarrow \text{Meas} \implies \text{Prob}(X) = X_*$

If $X = X' \mu$-a.e., then $X_* \mu = X'_* \mu$

Goal: Develop domain-theoretic approach where random variables are Scott continuous

Gain: If $X, X'$ are continuous on supp $\mu$, then $X_* \mu = X'_* \mu \implies X = X'$. 
A stochastic process is a time-indexed family \( \{X_t \mid t \in T \subseteq \mathbb{R}_+\} \) of random variables \( X_t : \Omega \to S \), where \( S \) is a Polish space.

**Note:** If \( S \) is Polish, then so is \((\text{Prob}(S), d_p)\)

\[ d_p \text{ is the Lévy-Prokhorov metric; generates the weak topology.} \]

**Examples:**

- Brownian motion, Lévy processes, Markov chains
- MCMC – Markov chain Monte Carlo
- Theme in *Probabilistic Programming Semantics*
Stochastic Processes and Skorohod’s Theorem

Let $\lambda$ denote Lebesgue measure on $[0, 1]$.

**Skorohod’s Theorem**
If $S$ is a Polish space, and $\nu \in \text{Prob } S$, then there is a random variable $X : [0, 1] \to S$ with $X_* \lambda = \nu$; i.e., $\nu(A) = \lambda(X^{-1}(A)) \ \forall A$ measurable.

Moreover, if $\nu_n, \nu \in \text{Prob } S$ satisfy $\nu_n \to_w \nu$, then there are random variables $X_n, X : [0, 1] \to S$ with $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ and $X_n \to X \ \lambda$-a.e.

**So:**  
1) Every stochastic process arises as $\{X_t : [0, 1] \to S \mid t \in T\}$.

2) Convergence in $(\text{Prob } S, \text{weak})$ is equivalent to pointwise convergence $\lambda$-a.e. of the measurable maps $X_t : [0, 1] \to S$.

**Goal:** Obtain domain-theoretic version of Skorohod’s Theorem with Skorohod’s Theorem as a Corollary.
Towards a Domain-theoretic Skorohod Theorem

First step: We can use any standard probability space for \([0, 1], \lambda\):

Let \(C = 2^\omega\) denote a countable product of 2-point groups, and let \(\mu_C\) denote Haar measure on \(C\).

Theorem:
If \(S\) is a Polish space, and \(\nu \in \text{Prob} S\), then there is a random variable \(X : C \to S\) with \(X_* \mu_C = \nu\).

Moreover, if \(\nu_n, \nu \in \text{Prob} S\) satisfy \(\nu_n \to_w \nu\), then there are random variables \(X_n, X : C \to S\) with \(X_* \mu_C = \nu, X_{n*} \mu_C = \nu_n\) and \(X_n \to X\) \(\mu_C\)-a.e.

Proof: Use \(\varphi : C \to [0, 1]\). \(\square\)
Towards a Domain-theoretic Skorohod Theorem

Second Step: Embed $\mathcal{C}$ in an appropriate domain:

$$\mathcal{C}_T = \{0, 1\}^\infty$$ is a domain in the prefix order.

$$\mathcal{C} \cong (\{0, 1\}^\omega, \Sigma(\mathcal{C}_T)\mid_{\{0, 1\}^\omega}) = (\text{Max } \mathcal{C}_T, \Lambda(\mathcal{C}_T)\mid_{\text{Max } \mathcal{C}_T})$$
Third Step: Which domains represent Polish spaces?

BCD_ω – countably based bounded complete domains and Scott continuous maps.

- \( D^\infty \simeq [D^\infty \rightarrow D^\infty] \) is in BCD_ω.
- \( \mathbb{CT} = \{0, 1\}^\infty \) is a bounded complete domain.

Theorem: (Lawson; Ciesielski, Flagg & Kopperman; Martin)
Each countably-based bounded complete domain \( D \) satisfies \( \text{Max } D \) is a Polish space in the inherited Scott topology.

Conversely, every Polish space can be embedded as \( \text{Max } D \) for some countably based bounded complete domain \( D_P \).

Moreover, \( \text{Max } D_P \) is a \( G_\delta \) in \( D \).

Examples:

1) \( \mathcal{C} \simeq \text{Max } \mathbb{CT} \hookrightarrow \mathbb{CT} \).

2) \( \mathbb{R} \simeq \text{Max } \mathbb{IR} \hookrightarrow \mathbb{IR} = (\{[a, b] \mid a \leq b \in \mathbb{R}\} \cup \{\mathbb{R}\}, \supseteq) \).
**Domain-theoretic Skorohod Theorem (cont’d)**

**Skorohod’s Theorem for Domains**

If $D$ is a countably based bounded complete domain and $\nu \in \text{Prob} \, D$, then there is a Scott-continuous map $X : \mathbb{C} \mathbb{T} \to D$ with $X_* \mu_C = \nu$.

Moreover, if $\nu_n, \nu \in \text{Prob} \, D$ satisfy $\nu_n \to_w \nu$, then there are Scott-continuous maps $X_n, X : \mathbb{C} \mathbb{T} \to D$ with $X_* \mu_C = \nu$, $X_{n*} \mu_C = \nu_n$ and $X_n \to X$ pointwise in Scott topology.

*Note:* \{X_n\}_n is not directed: $\lim \inf_n X_n(x) \geq X(x) \ \forall x \in \mathbb{C} \mathbb{T}$.

**BCD$_\omega$** is Cartesian closed:

- $[D \to E] = \{f : D \to E \mid f \text{ Scott continuous}\}$
- $f \leq g \iff f(x) \leq g(x) \ (\forall x \in D)$.

**So:** $X \mapsto X_* \mu_C : [\mathbb{C} \mathbb{T} \to D] \to (\text{Prob} \, D, \text{Scott})$ continuous surjection.
Domain-theoretic Skorohod Theorem (cont’d)

**Skorohod’s Theorem for Domains**

If \( D \) is a countably based bounded complete domain and \( \nu \in \text{Prob} \ D \), then there is a Scott-continuous map \( X : \mathbb{C}T \to D \) with \( X_* \mu_C = \nu \).

Moreover, if \( \nu_n, \nu \in \text{Prob} \ D \) satisfy \( \nu_n \to_w \nu \), then there are Scott-continuous maps \( X_n, X : \mathbb{C}T \to D \) with \( X_* \mu_C = \nu \), \( X_{n*} \mu_C = \nu_n \) and \( X_n \to X \) pointwise in Scott topology.

*Note:* \( \{X_n\}_n \) is *not* directed: \( \lim \inf_n X_n(x) \geq X(x) \ \forall x \in \mathbb{C}T \).

**Corollary: Skorohod’s Theorem**

*Proof:* If \( S \) is Polish, then \( S \simeq \text{Max} \ D \) for some \( D \) in \( \text{BCD}_\omega \).

Then \( (\text{Prob} \ S, \text{weak}) \simeq (\text{Max} \text{Prob} \ D, \text{weak}) \).

The theorem implies \( \forall \nu \in \text{Prob} \ S. (\exists X : \mathbb{C}T \to D) X_* \mu_C = \nu \).

\( C' = X^{-1}(\text{Max} \ D) \) is Borel, so \( X|_{C'} : C' \to \text{Max} \ D \) is measurable. \( \square \)
Deflations

\( \phi : D \to D \) is a deflation if \( \phi \) is Scott continuous and \( \phi(D) \) is finite.

\( D \in \text{BCD}_\omega \implies 1_D = \sup_n \phi_n, \quad \phi_n \ll \phi_{n+1}, \) deflations

Prob functorial \( \implies 1_{\text{Prob} D} = \sup_n \phi_n^* \)

So: If \( D \in \text{BCD}_\omega \) and \( \mu \in \text{Prob} D \), then \( \mu = \sup_n \phi_n^* \mu \)

with \( \phi_n^* \mu = \sum_{x \in F_n} r_x \delta_x \), where \( F_n \) finite for all \( n \).

Example: \( \pi_n : \mathcal{C} \to \downarrow C_n \), where \( C_n \simeq 2^n \implies 1_{\mathcal{C}} = \sup_n \pi_n \)

So, \( \mu_C = \sup_n \pi_n^* \mu_C = \sup_n \mu_{C_n} \)
Outline of Proof

**Finitary Mappings**

\[ D \in \text{BCD}_\omega \implies 1_D = \sup_n \phi_n, \ \phi_n \ll \phi_{n+1}, \text{ deflations} \]

But, \( \phi_n \ll \phi_{n+1} \iff \phi_{n*}\mu \ll \phi_{n+1*}\mu. \)

Fix \( \mu \in \text{Prob } D, \) and fix \( \phi_{n*}\mu = \sum_{x \in F_n} r_x \delta_x. \)

We approximate \( \phi_{n*}\mu: \)

Choose \( m_n > n, |F_n| \) with \( r_x - s_x < \frac{1}{2^{m_n}} \) (\( \forall x \in F_n \)), where

\[ s_x = \max_{D} (r_x \cap \text{Dyad}_{m_n}), \text{ with } \text{Dyad}_{m_n} = \left\{ \frac{k}{2^{m_n}} \mid k \leq 2^{m_n} \right\}. \]

Then \( \nu_n = \sum_{x \in F_n} s_x \delta_x \ll \phi_{n*}\mu, \)

Define \( f_n: C_{m_n} \to F_n \cup \{\perp\} \subseteq D \) by

\[ f_n^{-1}(x) = s_x \ (\forall x \in F_n), \ \text{and } f_n^{-1}(\perp) = 1 - \sum_{x \in F_n} s_x. \]

Then \( f_{n*}\mu_{C_{m_n}} = \nu_n \ll \phi_{n*}\mu \)
Proposition: Let \( \nu = \sum_{x \in F} r_x \delta_x \leq \sum_{y \in G} s_y \delta_y = \nu' \in \text{Prob} \ D \).

Assume \( r_x, s_y \) are dyadic rationals for each \( x \in F, y \in G \).

Suppose \( f_n : C_{m_n} \to D \) satisfies \( f_n \ast \mu_{C_{m_n}} = \nu \).

Then there are \( n' > n, m_{n'} > m_n \), and \( f_{n'} : C_{m_{n'}} \to D \) satisfying:

1. \( f_{n'} \ast \mu_{C_{m_{n'}}} = \nu' \), and
2. \( f_n \circ \pi_{m_n m_{n'}} \leq f_{n'} \), where \( \pi_{m_n m_{n'}} : C_{m_{n'}} \to C_{m_n} \) is the canonical projection.

The proof uses the Splitting Lemma, the fact that if \( r_x, s_y \) are dyadic, then the transport numbers \( t_{x,y} \) are, too, and a generalization of Hall’s Marriage Problem.
Outline of Proof

Proposition: Let \( \nu = \sum_{x \in F} r_x \delta_x \leq \sum_{y \in G} s_y \delta_y = \nu' \in \text{Prob } D \).
Assume \( r_x, s_y \) are dyadic rationals for each \( x \in F, y \in G \).
Suppose \( f_n : C_{m_n} \to D \) satisfies \( f_n^* \mu_{C_{m_n}} = \nu \).
Then there are \( n' > n, m_{n'} > m_n, \) and \( f_{n'} : C_{m_{n'}} \to D \) satisfying:

- \( f_{n'}^* \mu_{C_{m_{n'}}} = \nu' \), and
- \( f_n \circ \pi_{m_n m_{n'}} \leq f_{n'} \), where \( \pi_{m_n m_{n'}} : C_{m_{n'}} \to C_{m_n} \) is the canonical projection.

The proof of the first part of the Theorem follows by recursively defining an increasing family \( \tilde{f}_n : \downarrow C_{m_n} \to D \), where \( \tilde{f}_0 : C_0 \to D \) by \( \tilde{f}_0(\langle \rangle) = \bot_D \), and

\[
\tilde{f}_{n+1}(x) = \begin{cases} 
  f_{n+1}(x) & \text{if } x \in C_{m_{n+1}} \\
  \tilde{f}_n \circ \pi_{nk}(x) & \text{if } x \in C_k, m_n \leq k < m_{n+1}.
\end{cases}
\]
and then letting \( X = \sup_n \tilde{f}_n \circ \pi_{m_n} \).
Generalizations

1. The same results hold for subprobability measures (aka valuations):

**Skorohod’s Theorem for Subprobability Measures**

If $\nu \in \text{SProb } D$ is a subprobability measure on a countably based BCD domain, then there is a Scott-open subset $U_\nu \subseteq \mathcal{C} \mathcal{T}$ and a Scott-continuous map $X : U_\nu \to D$ satisfying $X_\nu \ast \mu_C = \nu$.

Moreover, if $\nu_n \to_w \nu \in \text{SProb } D$, then the Scott-continuous partial maps $X_n : U_{\nu_n} \to D$ satisfy $X_n \to X$ pointwise.

**Proof:**

1. Embed $D \hookrightarrow D_\perp \in \text{BCD}$

2. Apply the theorem to $D_\perp$.

3. Restrict $X, X_n$ to $U_\nu = \mathcal{C} \mathcal{T} \setminus X_\nu^{-1}(\perp), U_{\nu_n} = \mathcal{C} \mathcal{T} \setminus X_{\nu_n}^{-1}(\perp)$ \(\square\)
1. The same results hold for subprobability measures (aka valuations):

**Skorohod’s Theorem for Subprobability Measures**

If $\nu \in \text{SProb } D$ is a subprobability measure on a countably based BCD domain, then there is a Scott-open subset $U_\nu \subseteq \mathbb{C}T$ and a Scott-continuous map $X : U_\nu \rightarrow D$ satisfying $X_{\nu^*}\mu_C = \nu$.

Moreover, if $\nu_n \rightarrow_w \nu \in \text{SProb } D$, then the Scott-continuous partial maps $X_n : U_{\nu_n} \rightarrow D$ satisfy $X_n \rightarrow X$ pointwise.

2. The results all hold for more general domains:

In fact, they hold for any countably based coherent domains $D$. 
Happy Birthday, DANA!
Questions?