Stochastic Domain Theory

Michael Mislove

Tulane University

DOMAINS XIII
Oxford, 2018

Supported by US AFOSR

Probability In Computational Models:

- Programming language semantics
- Applications in mathematics, physics, ...
- Probabilistic programming semantics
- ...

Probability Theory

A *random variable* is a measurable map $X\colon \Omega \to S$, where $(\Omega, \Sigma_\Omega, \mu)$ is a probability space and (S, Σ_S) is a measurable space.

The *law* of X is $X_*\,\mu \in \mathrm{Prob}(Y)$ by $X_*\,\mu(A) = \mu(X^{-1}(A))$.

Note: $\mathrm{Prob}\colon \mathrm{Meas} \to \mathrm{Meas} \quad \implies \quad \mathrm{Prob}(X) = X_*$

If $X = X'$ μ-a.e., then $X_*\,\mu = X'_*\,\mu$

Goal: Develop domain-theoretic approach where random variables are Scott continuous

Gain: If X, X' are continuous on $\operatorname{supp}\mu$, then $X_*\,\mu = X'_*\,\mu \implies X = X'$.

Stochastic Processes and Skorohod's Theorem

A *stochastic process* is a time-indexed family $\{X_t \mid t \in T \subseteq \mathbb{R}_+\}$ of random variables $X_t \colon \Omega \to S$, where S is a Polish space.

Note: If S is Polish, then so is $(Prob(S), d_p)$

– d_p is the *Lévy-Prokhorov metric*; generates the *weak topology*.

Examples:

- Brownian motion, Lévy processes, Markov chains

  MCMC – Markov chain Monte Carlo
  Theme in *Probabilistic Programming Semantics*

Stochastic Processes and Skorohod's Theorem

Let λ denote Lebesgue measure on $[0,1]$.

Skorohod's Theorem

If S is a Polish space, and $\nu \in \mathrm{Prob}\, S$, then there is a random variable $X\colon [0,1] \to S$ with $X_* \lambda = \nu$; i.e., $\nu(A) = \lambda(X^{-1}(A))$ $\forall A$ measurable.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then there are random variables $X_n, X\colon [0,1] \to S$ with $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ and $X_n \to X$ λ-a.e.

So: 1) *Every* stochastic process arises as $\{X_t\colon [0,1] \to S \mid t \in T\}$.

2) Convergence in $(\mathrm{Prob}\, S, \mathit{weak})$ is equivalent to pointwise convergence λ-a.e. of the measurable maps $X_t\colon [0,1] \to S$.

Goal: Obtain domain-theoretic version of Skorohod's Theorem with Skorohod's Theorem as a Corollary.

Towards a Domain-theoretic Skorohod Theorem

First step: We can use any standard probability space for $([0,1],\lambda)$:

Let $\mathcal{C} = 2^\omega$ denote a countable product of 2-point groups, and let $\mu_\mathcal{C}$ denote Haar measure on $\mathcal{C}$.

Theorem:
If S is a Polish space, and $\nu \in \mathrm{Prob}\, S$, then there is a random variable $X\colon \mathcal{C} \to S$ with $X_*\, \mu_\mathcal{C} = \nu$.

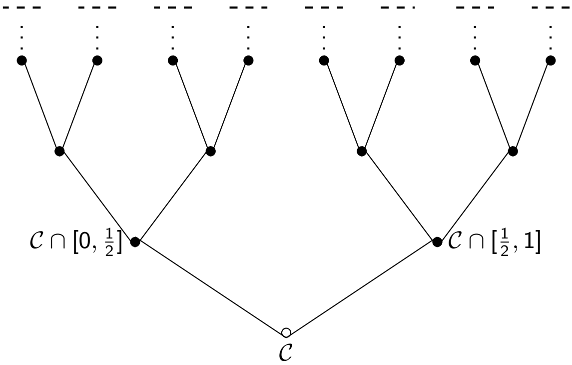
Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then there are random variables $X_n, X\colon \mathcal{C} \to S$ with $X_*\, \mu_\mathcal{C} = \nu, X_{n*}\, \mu_\mathcal{C} = \nu_n$ and $X_n \to X$ $\mu_\mathcal{C}$-a.e.

Proof: Use $\varphi\colon \mathcal{C} \twoheadrightarrow [0,1]$. $\square$

Towards a Domain-theoretic Skorohod Theorem

Second Step: Embed $\mathcal{C}$ in an appropriate domain:

$\mathbb{CT} = \{0,1\}^\infty$ is a domain in the prefix order.



$$\mathcal{C} \simeq (\{0,1\}^\omega, \Sigma(\mathbb{CT})|_{\{0,1\}^\omega}) = (\mathsf{Max}\,\mathbb{CT}, \Lambda(\mathbb{CT})|_{\mathsf{Max}\,\mathbb{CT}})$$

Towards a Domain-theoretic Skorohod Theorem

Third Step: Which domains represent Polish spaces?

BCD_ω – countably based bounded complete domains and Scott continuous maps.

- $D^\infty \simeq [D^\infty \to D^\infty]$ is in BCD_ω.
- $\mathbb{CT} = \{0,1\}^\infty$ is a bounded complete domain.

Theorem: (Lawson; Ciesielski, Flagg & Kopperman; Martin)

Each countably-based bounded complete domain D satisfies $\mathrm{Max}\, D$ is a Polish space in the inherited Scott topology.

Conversely, every Polish space can be embedded as $\mathrm{Max}\, D$ for some countably based bounded complete domain D_P.

Moreover, $\mathrm{Max}\, D_P$ is a G_δ in D.

Examples:

1) $\mathcal{C} \simeq \mathrm{Max}\, \mathbb{CT} \hookrightarrow \mathbb{CT}$.

2) $\mathbb{R} \simeq \mathrm{Max}\, \mathbb{IR} \hookrightarrow \mathbb{IR} = (\{[a,b] \mid a \leq b \in \mathbb{R}\} \cup \{\mathbb{R}\}, \supseteq)$.

Domain-theoretic Skorohod Theorem (cont'd)

Skorohod's Theorem for Domains

If D is a countably based bounded complete domain and $\nu \in \mathrm{Prob}\, D$, then there is a Scott-continuous map $X\colon \mathbb{CT} \to D$ with $X_*\, \mu_{\mathcal{C}} = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, D$ satisfy $\nu_n \to_w \nu$, then there are Scott-continuous maps $X_n, X\colon \mathbb{CT} \to D$ with $X_*\, \mu_{\mathcal{C}} = \nu$, $X_{n*}\, \mu_{\mathcal{C}} = \nu_n$ and $X_n \to X$ pointwise in Scott topology.

Note: $\{X_n\}_n$ is *not* directed: $\liminf_n X_n(x) \geq X(x)\ \forall x \in \mathbb{CT}$.

BCD_ω is Cartesian closed:

- $[D \to E] = \{f\colon D \to E \mid f \text{ Scott continuous}\}$
- $f \leq g$ iff $f(x) \leq g(x)$ $(\forall x \in D)$.

So: $X \mapsto X_*\mu_{\mathcal{C}}\colon [\mathbb{CT} \to D] \twoheadrightarrow (\mathrm{Prob}\, D, \mathit{Scott})$ continuous surjection.

Domain-theoretic Skorohod Theorem (cont'd)

Skorohod's Theorem for Domains

If D is a countably based bounded complete domain and $\nu \in \mathrm{Prob}\, D$, then there is a Scott-continuous map $X\colon \mathbb{CT} \to D$ with $X_*\, \mu_{\mathcal{C}} = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, D$ satisfy $\nu_n \to_w \nu$, then there are Scott-continuous maps $X_n, X\colon \mathbb{CT} \to D$ with $X_*\, \mu_{\mathcal{C}} = \nu$, $X_{n*}\, \mu_{\mathcal{C}} = \nu_n$ and $X_n \to X$ pointwise in Scott topology.

Note: $\{X_n\}_n$ is *not* directed: $\liminf_n X_n(x) \geq X(x)\ \forall x \in \mathbb{CT}$.

Corollary: Skorohod's Theorem

Proof: If S is Polish, then $S \simeq \mathrm{Max}\, D$ for some D in BCD_ω.
Then $(\mathrm{Prob}\, S, \mathit{weak}) \simeq (\mathrm{Max}\, \mathrm{Prob}\, D, \mathit{weak})$.
The theorem implies $\forall \nu \in \mathrm{Prob}\, S.(\exists X\colon \mathbb{CT} \to D)\, X_*\mu_{\mathcal{C}} = \nu$.

$\mathcal{C}' = X^{-1}(\mathrm{Max}\, D)$ is Borel, so $X|_{\mathcal{C}'}\colon \mathcal{C}' \to \mathrm{Max}\, D$ is measurable. $\square$

Deflations

$\phi\colon D \to D$ is a *deflation* if ϕ is Scott continuous and $\phi(D)$ is finite.

$D \in \mathrm{BCD}_\omega \implies \mathbf{1}_D = \sup_n \phi_n,\ \phi_n \ll \phi_{n+1}$, deflations

Prob functorial $\implies \mathbf{1}_{\mathrm{Prob}\,D} = \sup_n \phi_{n*}$

So: If $D \in \mathrm{BCD}_\omega$ and $\mu \in \mathrm{Prob}\,D$, then $\mu = \sup_n \phi_{n*}\,\mu$
with $\phi_{n*}\,\mu = \sum_{x \in F_n} r_x \delta_x$, where F_n finite for all n.

Example: $\pi_n\colon \mathbb{CT} \to {\downarrow}\mathcal{C}_n,$ where $\mathcal{C}_n \simeq 2^n \quad \implies \quad \mathbf{1}_{\mathbb{CT}} = \sup_n \pi_n$

So, $\mu_{\mathcal{C}} = \sup_n \pi_{n*}\,\mu_{\mathcal{C}} = \sup_n \mu_{\mathcal{C}_n}$

Finitary Mappings

$D \in \mathrm{BCD}_\omega \implies \mathbf{1}_D = \sup_n \phi_n,\ \phi_n \ll \phi_{n+1}$, deflations

But, $\phi_n \ll \phi_{n+1} \quad \not\Longrightarrow \quad \phi_{n*}\,\mu \ll \phi_{n+1*}\,\mu$.

Fix $\mu \in \mathrm{Prob}\, D$, and fix $\phi_{n*}\,\mu = \sum_{x \in F_n} r_x \delta_x$.

We *approximate* $\phi_{n*}\,\mu$:

Choose $m_n > n, |F_n|$ with $r_x - s_x < \frac{1}{2^{m_n}}$ $(\forall x \in F_n)$, where

$$s_x = \max{\downarrow}(r_x \cap \mathit{Dyad}_{m_n}), \text{ with } \mathit{Dyad}_{m_n} = \{\tfrac{k}{2^{m_n}} \mid k \leq 2^{m_n}\}.$$

Then $\nu_n = \sum_{x \in F_n} s_x \delta_x \ll \phi_{n*}\,\mu$,

Define $f_n \colon \mathcal{C}_{m_n} \to F_n \cup \{\bot\} \subseteq D$ by

$f_n^{-1}(x) = s_x\ (\forall x \in F_n)$, and $f_n^{-1}(\bot) = 1 - \sum_{x \in F_n} s_x$.

Then $f_{n*}\,\mu_{\mathcal{C}_{m_n}} = \nu_n \ll \phi_{n*}\,\mu$

Outline of Proof

Proposition: Let $\nu = \sum_{x\in F} r_x\delta_x \le \sum_{y\in G} s_y\delta_y = \nu' \in \mathrm{Prob}\, D$.

Assume r_x, s_y are dyadic rationals for each $x \in F, y \in G$.

Suppose $f_n\colon \mathcal{C}_{m_n} \to D$ satisfies $f_{n*}\,\mu_{\mathcal{C}_{m_n}} = \nu$.

Then there are $n' > n$, $m_{n'} > m_n$, and $f_{n'}: \mathcal{C}_{m_{n'}} \to D$ satisfying:

- $f_{n'*}\,\mu_{\mathcal{C}_{m_{n'}}} = \nu'$, and
- $f_n \circ \pi_{m_n m_{n'}} \le f_{n'}$, where $\pi_{m_n m_{n'}} : \mathcal{C}_{m_{n'}} \to \mathcal{C}_{m_n}$ is the canonical projection.

The proof uses the Splitting Lemma, the fact that if r_x, s_y are dyadic, then the transport numbers $t_{x,y}$ are, too, and a generalization of Hall's Marriage Problem.

Outline of Proof

Proposition: Let $\nu = \sum_{x \in F} r_x \delta_x \leq \sum_{y \in G} s_y \delta_y = \nu' \in \mathrm{Prob}\, D$.

Assume r_x, s_y are dyadic rationals for each $x \in F, y \in G$.

Suppose $f_n \colon \mathcal{C}_{m_n} \to D$ satisfies $f_{n*}\, \mu_{\mathcal{C}_{m_n}} = \nu$.

Then there are $n' > n$, $m_{n'} > m_n$, and $f_{n'} \colon \mathcal{C}_{m_{n'}} \to D$ satisfying:

- $f_{n'*}\, \mu_{\mathcal{C}_{m_{n'}}} = \nu'$, and
- $f_n \circ \pi_{m_n m_{n'}} \leq f_{n'}$, where $\pi_{m_n m_{n'}} \colon \mathcal{C}_{m_{n'}} \to \mathcal{C}_{m_n}$ is the canonical projection.

The proof of the first part of the Theorem follows by recursively defining an increasing family $\widetilde{f}_n \colon \downarrow\!\mathcal{C}_{m_n} \to D$, where $\widetilde{f}_0 \colon \mathcal{C}_0 \to D$ by $\widetilde{f}_0(\langle\ \rangle) = \perp_D$,

and $\widetilde{f}_{n+1}(x) = \begin{cases} f_{n+1}(x) & \text{if } x \in \mathcal{C}_{m_{n+1}} \\ \widetilde{f}_n \circ \pi_{nk}(x) & \text{if } x \in \mathcal{C}_k, m_n \leq k < m_{n+1}. \end{cases}$

and then letting $X = \sup_n \ \widetilde{f}_n \circ \pi_{m_n}$.

1. The same results hold for subprobability measures (*aka* valuations):

Skorohod's Theorem for Subprobability Measures

If $\nu \in \mathrm{SProb}\, D$ is a subprobability measure on a countably based BCD domain, then there is a Scott-open subset $U_\nu \subseteq \mathbb{CT}$ and a Scott-continuous map $X\colon U_\nu \to D$ satisfying $X_{\nu *}\mu_{\mathcal{C}} = \nu$.

Moreover, if $\nu_n \to_w \nu \in \mathrm{SProb}\, D$, then the Scott-continuous partial maps $X_n\colon U_{\nu_n} \to D$ satisfy $X_n \to X$ pointwise.

Proof:

1. Embed $D \hookrightarrow D_\perp \in \mathrm{BCD}$
2. Apply the theorem to $D_\perp$.
3. Restrict X, X_n to $U_\nu = \mathbb{CT} \setminus X_\nu^{-1}(\perp), U_{\nu_n} = \mathbb{CT} \setminus X_{\nu_n}^{-1}(\perp)$ $\square$

1. The same results hold for subprobability measures (*aka* valuations):

Skorohod's Theorem for Subprobability Measures

If $\nu \in \mathsf{SProb}\, D$ is a subprobability measure on a countably based BCD domain, then there is a Scott-open subset $U_\nu \subseteq \mathbb{CT}$ and a Scott-continuous map $X\colon U_\nu \to D$ satisfying $X_{\nu *}\mu_{\mathcal{C}} = \nu$.

Moreover, if $\nu_n \to_w \nu \in \mathsf{SProb}\, D$, then the Scott-continuous partial maps $X_n\colon U_{\nu_n} \to D$ satisfy $X_n \to X$ pointwise.

2. The results all hold for more general domains:

In fact, they hold for any countably based coherent domains D.

Happy Birthday, DANA!

Questions?