Completions, $\mathbb{K}$-categories and Commutative Probability Monads

To Dana Scott on his 90th Birthday

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Joint work with Xiaodong Jia, Bert Lindenhovius and Vladimir Zamdzhiev

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Some History

1972 *Continuous lattices* by D. S. Scott, in which the injective T_0-spaces X are characterized as retracts of $\mathbb{S}^{\mathcal{O}(X)}$ under maps preserving directed suprema, where $\mathbb{S}$ denotes *Sierpinski space*.

- Implies continuous lattices are *sober spaces* in the Scott topology.

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1992 (H. Zhang) Same result holds for *continuous posets*.

2008 (*DCPO-completion of posets*, Zhao & Fan) Described a DCPO-completion of a poset that is finer than the sobrification.

2009 (*D-completions and the d-topology*, Keimel and Lawson) Gave topological account of Zhao & Fan's construction, and introduced $\mathtt{K}$-categories, which provide further examples.

In this talk, we'll apply these results to produce new *commutative* probabilistic monads over DCPO.

We begin by explaining the problem that inspired our work.

What's the Problem?

- The application of interest is *statistical programming:* functional languages that sample from probability distributions.

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- *Natural model:* Sub-probabilistic power domain $\mathcal{V}_{\leq 1}(P)$ over a dcpo P, the family of *Scott-continuous valuations:* maps $\mu\colon \sigma(P) \to [0,1]$ satisfying:
  - $\mu(\emptyset) = 0$
  - $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
  - $\mu(\bigcup_{U \in D} U) = \sup_{U \in D} \mu(U)$, if $D \subseteq \sigma(P)$ is a directed family.

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- *Natural model:* Sub-probabilistic power domain $\mathcal{V}_{\leq 1}(P)$ over a dcpo P, the family of *Scott-continuous valuations:* maps $\mu\colon \sigma(P) \to [0,1]$

- $\mathcal{V}_{\leq 1}$ defines a monad over DOM – category of domains and Scott continuous maps. In fact, $\mathcal{V}_{\leq 1}$ is *commutative*; i.e., the Fubini-like equation

$$\int_{x\in P}\int_{y\in Q}\chi_U(x,y)d\mu d\nu = \int_{y\in Q}\int_{x\in P}\chi_U(x,y)d\nu d\mu,$$

  holds, for domains P, Q, valuations $\mu \in \mathcal{V}_{\leq 1}(P), \nu \in \mathcal{V}_{\leq 1}(Q)$ and $U \subseteq P \times Q$ Scott open. *But,* DOM is not Cartesian closed.

  - *Jung-Tix Problem:* There is no known Cartesian closed category of domains on which $\mathcal{V}_{\leq 1}$ defines a monad.

What's the Problem?

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  These are viewed as the simply-typed lambda calculus extended with a *probability monad*.

- *Natural model:* Sub-probabilistic power domain $\mathcal{V}_{\leq 1}(P)$ over a dcpo P, the family of *Scott-continuous valuations:* maps $\mu\colon \sigma(P) \to [0, 1]$

- However, $\mathcal{V}_{\leq 1}$ also defines a monad on DCPO, the category of dcpos and Scott continuous maps, which is Cartesian closed.

- *But:* $\mathcal{V}_{\leq 1}(P)$ is not known to be commutative over DCPO, so there is no proof that Fubini holds.

- So we search for *submonads of* $\mathcal{V}_{\leq 1}$ that are commutative over DCPO.
  - Xiaodong will describe some of the monads we found in his talk on Wednesday.
  - I want to outline the mathematical results that underpin such monads.

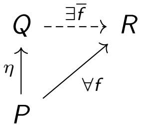
The DCPO-completion of a Poset

- *Zhao & Fan:* Defined the *d-topology* on a poset and the associated *d-completion*.[1]
  - A subset $A \subseteq P$ is *d-closed* if A is closed under existing suprema of directed sets: i.e., if $D \subseteq A$ is directed and $\sup D \in P$ exists, then $\sup D \in A$.
  - This defines the closed sets of the *d-topology:* the union of finitely many d-closed sets is d-closed (by short argument), and any intersection of d-closed sets is d-closed.
  - Scott-closed subsets are d-closed, so the d-topology refines the Scott topology.
  - The d-closed subsets of a dcpo are exactly the sub-dcpos.
  - Any lower set is d-open, so $\downarrow x$ is d-clopen for each $x \in P$.

[1] Zhao & Fan use the notation D-topology, etc., but that clashes with further results we discuss next.

The DCPO-completion of a Poset

- *Zhao & Fan:* Defined the *d-topology* on a poset and the associated *d-completion*.
- A dcpo Q is a *d-completion* of a poset P if there is $\eta\colon P \to Q$ Scott continuous satisfying



  where R is a dcpo and $f, \overline{f}$ are Scott continuous.
  Any two d-completions of P are isomorphic. Denote this by $\overline{P}$.

- Can be formed as follows:
  1) Embed $(P, \sigma(P))$ in a dcpo Q. E.g., take $Q = \Gamma(P)$.

  2) Take the intersection of all sub-dcpos of Q containing P. This is $\overline{P}$.

  Theorem: $\overline{P}$ is the smallest dcpo satisfying $P \hookrightarrow \overline{P}$ is an embedding in the Scott topology.

Monotone Convergence Spaces

2009 *D-completions and the d-topology*, Keimel and Lawson.

The d-topology is not order-theoretic: $Id\colon \mathbb{S} \to \{0,1\}^\flat$ is d-continuous.

A *monotone convergence space* is a T_0-space $(X, \mathcal{O}(X))$ in which each directed subset $D \subseteq X$ in the specialization order ($x \leq_s y$ iff $x \in \overline{\{y\}}$) converges to its supremum, $\sup D$.

Equivalently, $\Omega(X) = (X, \leq_s)$ is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X, \leq_s)$.

Examples: 1) Any sober space.

2) $\Sigma(P) = (P, \sigma(P))$ – Scott space of dcpo P.

Initially studied as *d-spaces* by Wyler (1981) and later by Ershov (1999).

D - category of monotone convergence spaces and continuous maps.

$$\mathrm{DCPO} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathrm{D} \qquad (\bot)$$

Monotone Convergence Spaces

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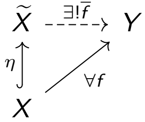
D-completion of a T_0-space X: a d-dense embedding $X \hookrightarrow \widetilde{X}$ into a monotone convergence space $\widetilde{X}$. They exist because sober spaces are monotone convergence spaces. (Topological version of Zhao & Fan's d-completion.)

- *D-completions are universal:* for every $\eta\colon X \hookrightarrow \widetilde{X}$, for every $f : X \to Y (\in \mathrm{D})$:



This implies any two D-completions are homeomorphic. D_c denotes the D-completion.

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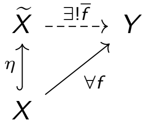
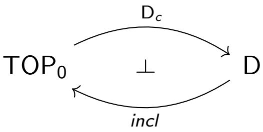


  This implies any two D-completions are homeomorphic. D_c denotes the D-completion.

- D is a full reflective subcategory of TOP_0 – T_0-spaces and continuous maps:



  D_c defines an idempotent monad on TOP_0:

$$f\colon X \to Y \quad \mapsto \quad \mathrm{D}_c(f) = \overline{\eta_Y \circ f}\colon \mathrm{D}_c(X) \to \mathrm{D}_c(Y)$$

Monotone Convergence Spaces

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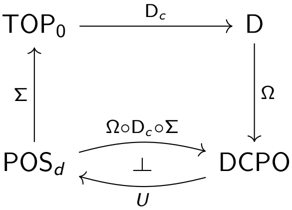
- The d-completion of a poset in D; POS_d - posets and Scott continuous maps.



So, $\Omega \circ \mathrm{D}_c \circ \Sigma(P)$ is the d-completion of a poset P in the Scott topology.

K-categories

2009 *D-completions and the d-topology*, Keimel and Lawson.

- K-category: a subcategory K of TOP_0 (objects are called "K spaces") satisfying
  - SOB is a subcategory of K
  - K is closed under homeomorphic images
  - If X is a sober space, the intersection of any family of K-subspaces of X is a K-space
  - If $f : X \to Y$ in SOB, then f is K-continuous:
    i.e., if $Z \subseteq Y$ is a K-subspace, then $f^{-1}(Z)$ is a K-subspace of X;
    equivalently, if $W \subseteq X$ is a subspace, then $f(cl_K(W)) \subseteq cl_K(f(W))$.

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  Example: D is a K-category.

  We can form a K-completion $K_c(X)$ of any T_0-space: use embedding $\eta: X \hookrightarrow X^s$ into its sobrification and then take

$$\eta_X : X \hookrightarrow K_c(X) = \bigcap \{X' \subseteq X^s \mid X \subseteq X' \text{ a K space}\}.$$

  The conditions assure that $K_c(X)$ is a K-space and any continuous $f: X \to Y$ into a K-space admits a unique $\overline{f}: K_c(X) \to Y$ with $\overline{f} \circ \eta_X = f$

$\mathbb{K}$-categories

2009 *D-completions and the d-topology*, Keimel and Lawson.

Theorem: Each $\mathbb{K}$-category is a full reflective subcategory of TOP_0:

$$\mathsf{TOP}_0 \underset{I}{\overset{\mathsf{K}_c}{\rightleftarrows}} \mathsf{K} \qquad (\mathsf{K}_c \dashv I)$$

where K_c is the K-completion functor; this defines an idempotent monad on TOP_0:

$$f\colon X \to Y \quad \mapsto \quad \mathsf{K}_c(f) = \overline{\eta_Y \circ f}\colon \mathsf{K}_c(X) \to \mathsf{K}_c(Y)$$

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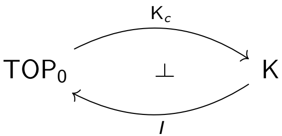
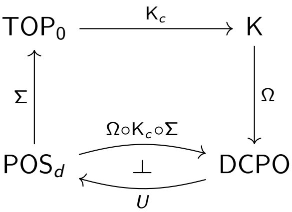


- If $\mathsf{K} \subseteq \mathsf{D}$, then $\Omega \circ \mathsf{K}_c \circ \Sigma(P)$ defines the K-completion of a poset P in the Scott topology:



Topological View of Valuations

Definition The *(extended) probabilistic power domain* $\mathcal{V}(X)$ over a topological space $(X, \mathcal{O}(X))$, is the family of maps $\mu \colon \mathcal{O}(X) \to \overline{\mathbb{R}_+}$ satisfying:

- $\mu(\emptyset) = 0$
- $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
- $\mu(\bigcup_{U \in D} U) = \sup_{U \in D} \mu(U)$, if $D \subseteq \mathcal{O}(X)$ is directed.

We endow $\mathcal{V}(X)$ with the *stochastic order:* $\mu \leq \nu$ iff $\mu(U) \leq \nu(U)$ $(\forall U \in \mathcal{O}(X))$.

- $\mathcal{V}(X)$ is a DCPO – in fact, $\mathcal{V} \colon \mathsf{TOP}_0 \to \mathsf{DCPO}$ is a functor, where

$$f \in \mathsf{TOP}_0(X, Y) \quad \mapsto \quad \mu \mapsto \mathcal{V}f(\mu) = \mu \circ f^{-1} \in \mathsf{DCPO}(\mathcal{V}(X), \mathcal{V}(Y)).$$

(Jones) $\mathcal{V} \circ \Sigma \colon \mathsf{DCPO} \to \mathsf{DCPO}$ defines a strong monad;

$(\mathcal{V} \circ \Sigma)|_{\mathsf{DOM}} \colon \mathsf{DOM} \to \mathsf{DOM}$ is a commutative monad.

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$\mathcal{V}_w X$ denotes $\mathcal{V}X$ in the *weak topology:* $[U > r] = \{\mu \mid \mu(U) > r\} = \{\mu \mid eval(\mu, U) > r\}$ forms a sub-basis, where $eval\colon \mathcal{V}X \times \mathcal{O}(X) \to \mathbb{R}_+$. Heckmann showed $\mathcal{V}_w(X)$ is sober.

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(Goubault-Larrecq & Jia) $\mathcal{V}_w$ defines a monad on TOP_0:
$\mathcal{V}_w f\colon \mathcal{V}_w X \to \mathcal{V}_w Y$ is $\mathcal{V}_w f(\mu) = \mu \circ f^{-1}$ is continuous, for $f\colon X \to Y$.

The *unit* is the point valuation $x \mapsto \delta_x = \lambda U. \begin{cases} 1 & x \in U, \\ 0 & \text{otherwise.} \end{cases}$

We can define the Choquet-like integral $\int_X h(x) d\mu = \int_0^\infty \mu(h^{-1}(r, \infty]) dr$ (the Riemann integral) for $h\colon X \to \overline{\mathbb{R}_+}$ lower semicontinuous and any space X.

Then the *multiplication* $m\colon \mathcal{V}_w^2 X \to \mathcal{V}_w X$ is defined as

$$m(\varpi) = \lambda U. \int_{\mathcal{V}_w X} \mu(U) d\varpi \; = \lambda U. \int_0^\infty \varpi(eval(-, U)^{-1}(r, \infty]) dr = \lambda U. \int_0^\infty \varpi([U > r]) dr.$$

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- $\mu(\bigcup_{U \in D} U) = \sup_{U \in D} \mu(U)$, if $D \subseteq \mathcal{O}(X)$ is directed.

Theorem For any space X, $\mathcal{V}_w X$ has a canonical cone structure making it a locally linear topological cone. Moreover, $\mathcal{V}_w f\colon \mathcal{V}_w X \to \mathcal{V}_w Y$ is continuous and linear, for each $f\colon X \to Y$.

$+\colon \mathcal{V}_w X \times \mathcal{V}_w X \to \mathcal{V}_w X$ is $(\mu + \nu)(U) = \mu(U) + \nu(U)$ for each open set $U \subseteq X$;
$r \cdot -\colon \mathcal{V}_w X \to \mathcal{V}_w X$ is $(r \cdot \mu)(U) = r \cdot \mu(U)$ for each $r \in \mathbb{R}_+$; and
$U \mapsto 0 \in \overline{\mathbb{R}_+}$ is the zero valuation.
The sub-basis $[U > r]$ consists of (open) half-spaces (i.e., convex subsets whose complement also is convex), which is what locally linear means. The last claim is straightforward from the definitions of $\mathcal{V}_w f(\mu) = \mu \circ f^{-1}$ and of the sub-bases $[U > r]$ for $\mathcal{V}_w X$ and $\mathcal{V}_w Y$.

K-completions of Simple Valuations

Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x \in F} r_x \delta_x \mid r_x \in \mathbb{R}_+ \ \& \ F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

$\mathcal{V}_s$ is a submonad of $\mathcal{V}_w$

- If $f : X \to Y$, then $\mathcal{V}_s f(\sum_{x \in F} r_x \cdot \delta_x) = \sum_{x \in F} r_x \cdot \delta_{f(x)}$.
- The unit $x \mapsto \delta_x : X \to \mathcal{V}_w X$ is simple.
- The multiplication $m \colon \mathcal{V}_w^2 X \to \mathcal{V}_w X$ restricts to $\mathcal{V}_s X^2 \to \mathcal{V}_s X$ via

$$m(\sum_{x \in F} r_x (\sum_{y \in G_x} s_{x,y} \delta_{x,y})) = \sum_{x,y} r_x s_{x,y} \delta_{x,y}.$$

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In particular, there is an embedding $\eta_{\mathcal{V}_s X} \colon \mathcal{V}_s X \hookrightarrow \mathcal{V}_w X$, which is sober.

If K is a $\mathtt{K}$-category, it follows that the K-completion $\mathcal{V}_{\mathsf{K}} X \stackrel{\text{def}}{=} \mathsf{K}_c(\mathcal{V}_s X)$ is a sub-dcpo of $\mathcal{V}_w X$.

Theorem For each $\mathtt{K}$-category K, $\mathcal{V}_{\mathsf{K}}$ is a monad on TOP_0.

Proof: $\mathcal{V}_{\mathsf{K}} X$ is a subcone of $\mathcal{V}_w X$, so it is a locally linear topological cone, and each continuous linear map $f \colon \mathcal{V}_s X \to \mathcal{V}_s Y$ satisfies $\eta_{\mathcal{V}_{\mathsf{K}}} \circ f(\mathcal{V}_s X) \subseteq \mathcal{V}_{\mathsf{K}} X$, so $\mathsf{K}_c(f) \colon \mathcal{V}_{\mathsf{K}} X \to \mathcal{V}_{\mathsf{K}} Y$. The linearity of $\mathsf{K}_c(f)$ follows from the density of $\mathcal{V}_s X$ in $\mathcal{V}_{\mathsf{K}} X$ and the continuity of addition on $\mathcal{V}_{\mathsf{K}} Y$. This shows $\mathcal{V}_{\mathsf{K}}$ is an endofunctor on TOP_0.

The unit is the point valuation $x \mapsto \delta_x$, and if $f \colon X \to \mathcal{V}_{\mathsf{K}} Y$ is continuous, then

$$\mathsf{K}_c(f) = \mu \mapsto (U \mapsto \int_X f(x)(U) d\mu) \colon \mathcal{V}_{\mathsf{K}} X \to \mathcal{V}_{\mathsf{K}} Y$$

is well-defined and continuous. This shows $\mathcal{V}_K$ defines a submonad of $\mathcal{V}_w$.

K-completions of Simple Valuations

Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x \in F} r_x \delta_x \mid r_x \in \mathbb{R}_+ \;\&\; F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

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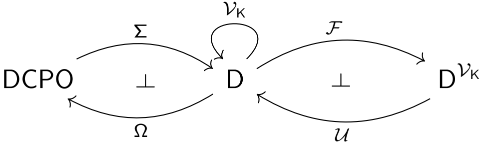
Theorem For each $\mathtt{K}$-category K, $\mathcal{V}_\mathsf{K}$ is a monad on TOP_0.

Theorem For each $\mathtt{K}$-category $\mathsf{K} \subseteq \mathsf{D}$, $\mathcal{V}_{\mathsf{K},\leq} \stackrel{\text{def}}{=} \Omega \circ \mathcal{V}_K \circ \Sigma$ is a monad on DCPO.

Proof: $\mathcal{V}_\mathsf{K}$ is a monad on TOP_0, and since K is a full subcategory of D, it follows that $\mathcal{V}_\mathsf{K}$ restricts to a monad on D. Writing $\mathcal{V}_\mathsf{K} = \mathcal{U} \circ \mathcal{F}$, we have $\Omega \circ \mathcal{V}_\mathsf{K} \circ \Sigma = (\Omega \circ \mathcal{U}) \circ (\mathcal{F} \circ \Sigma)$, and adjoints compose:



It's straightforward to see that the unit and multiplication of $\mathcal{V}_\mathsf{K}$ are transported to DCPO.

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Theorem For each $\mathbb{K}$-category K $\subseteq$ D, $\mathcal{V}_{\mathrm{K},\leq} \stackrel{\text{def}}{=} \Omega \circ \mathcal{V}_K \circ \Sigma$ is a monad on DCPO.

In particular, this applies to the D-completion $\mathcal{V}_{\mathrm{D}} X$.

Theorem The monad $\mathcal{M} \stackrel{\text{def}}{=} \mathcal{V}_{\mathrm{D},\leq 1}$ is commutative on DCPO.

Proof: If $\mu = \sum_{x \in F} r_x \cdot \delta_x \in \mathcal{M}X$ and $\nu \in \mathcal{M}Y$, then

$$\int_X \int_Y \chi_U(x,y) d\nu d\mu = \sum_{x \in F} r_x \cdot \int_Y \chi_U(x,y) d\nu = \int_Y \sum_{x \in F} r_x \cdot \chi_U(x,y) d\nu = \int_Y \int_X \chi_U(x,y) d\mu d\nu,$$

and since $\mathcal{V}_s X$ is dense in $\mathcal{M}X$ and integration is continuous, the equation holds for all $\mu \in \mathcal{M}X$.

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Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x\in F} r_x\delta_x \mid r_x \in \mathbb{R}_+ \ \&\ F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

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Theorem The monad $\mathcal{M} \stackrel{\text{def}}{=} \mathcal{V}_{\mathrm{D},\leq 1}$ is commutative on DCPO.

This also applies to any full $\mathtt{K}$-subcategory K of D; in fact, for such a K, we have

$\mathcal{V}_{s,\leq 1} \subseteq \mathcal{M} \subseteq \mathcal{V}_{\mathrm{K},\leq} \subseteq \mathcal{P}$ (but $\mathcal{V}_{s,\leq 1}$ isn't a subcategory of DCPO), where $\mathcal{P} = \mathcal{V}_{\mathrm{SOB},\leq 1}$.

Heckmann showed that $\mathcal{P}$ consists of *point continuous valuations*; hence the name.

Another example we know of is the full $\mathtt{K}$-subcategory W of D consisting of *well-filtered spaces*. So the commutative monads we know so far are:

$\mathcal{V}_{s,\leq 1} \subseteq \mathcal{M} \subseteq \mathcal{W} \subseteq \mathcal{P} \subseteq \mathcal{Z}$

The last – $\mathcal{Z}$ – denotes the category of *central valuations:* those for which integration satisfies the Fubini equation with any valuation in $\mathcal{V}$ for the other component. This category exists by abstract reasoning, and doesn't rely on $\mathcal{V}_s$ being a dense subcategory relative to some completion operation. It also is the only one we know that contains the pushforward of Lebesgue measure by a lower semicontinuous map to a DCPO.

Some References

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