Kegelspitzen and Computational Models

In Memory of Klaus Keimel

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Overview

• Kegelspitzen are dcpos that have an additional convex structure:
  “Mixed Powerdomains for Probability and Nondeterminism,”
  Combine convex domains and dcpo cones approaches
  Present dcpos with convex structure as first class objects
  Convexity traditionally arises via the \textit{valuations monad}.

• Plan of the talk:
  Describe “classical approach” to modeling nondeterminism and probabilistic choice
  in domains.
  Outline the “Kegelspitzen approach”.
  \textit{Time permitting}: Outline potential use of Kegelspitzen as part of Linear / Nonlinear
  Models of intuitionistic logic and the lambda calculus.
Classical Approach Using Domains

• Nondeterminism and Probabilistic Choice are *computational effects*
  Modeled using monads on DCPO, $\omega$-CPO, DOM, etc. (Moggi)

• Nondeterminism has an *equational theory*:
  $x + x = x$ \hspace{1cm} $x + y = y + x$ \hspace{1cm} $x + (y + z) = (x + y) + z$

  \hspace{1cm} (L) $x, y \leq x + y$ \hspace{1cm} (U) $x + y \leq x, y$ \hspace{1cm} (C) + monotone

  so there are monads on DCPO in each case.

• *Each form of nondeterminism* has a corresponding model:

  $\mathcal{P}_L$ the lower power domain, modeled as the *monad of non-empty Scott-closed sets*,
  under inclusion. \hspace{1cm} Works for all DCPOs.

  $\mathcal{P}_U$ the upper power domain, modeled as the *monad of non-empty Scott-compact saturated (= upper) sets*, under reverse inclusion. \hspace{1cm} Works for sober DCPOs.

  $\mathcal{P}_C$ the (order-)convex power domain, modeled as the *monad of non-empty Lawson closed, order-convex subsets*, under the Egli-Milner order.
  \hspace{1cm} Works for Lawson compact (= coherent) domains.
Classical Approach Using Domains

- **Probabilistic Choice** is modeled by *valuations*.

Functions $\mu : \sigma D \to [0, 1]$ satisfying:

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Strictness</strong></td>
<td>$\mu(\emptyset) = 0$</td>
</tr>
<tr>
<td><strong>Modularity</strong></td>
<td>$\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$</td>
</tr>
<tr>
<td><strong>Scott continuity</strong></td>
<td>${U_i}_i$; directed $\implies \mu(\bigcup_i U_i) = \sup_i \mu(U_i)$</td>
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- **Pointwise Order**
  \[
  \mu \leq \nu \iff \mu(U) \leq \nu(U) \quad (\forall U \in \sigma D)
  \]

- **Probabilistic power domain**
  \[
  \forall_{\leq 1} D = \{\mu \mid \mu \text{ valuation}\}
  \]
  - subdcpo of $[D, [0, 1]]$

- **Probability measures**
  \[
  \forall_{1} D = \{\mu \in \forall D \mid \mu(D) = 1\}
  \]
  (for continuous DCPOs)
Classical Approach Using Domains

- $\forall \leq 1, \forall_1 : DCPO \to DCPO$ monads.

Have an equational theory (Jones, 1989):

$\forall r, s \in [0, 1]. \forall a, b, c \in \forall D$

Idempotence $a +_r a = a$

Identity $a +_1 b = a$

Skew commutativity $a +_r b = b +_1 1 - r a$

Skew associativity $(a +_r b) +_s c = a +_{rs} (b +_{r - \frac{n}{1 - rs}} c)$ $rs \neq 1$

*Probabilistic algebra:* dcpo $D$ satisfying the above laws with $[0, 1] \times D \times D \to D$ continuous in the usual $\times$ product Scott topologies

*Examples:* $\forall \leq_1 D, \forall_1 D$ for $D$ a DCPO.
Classical Approach Using Domains

• Combining $\mathbb{V}_{\leq 1}$ with $\mathcal{P}_*$

Given monads $T, S : C \to C$, $T \circ S$ is a monad iff there is a distributive law $S \circ T \xrightarrow{\lambda} T \circ S$. (Beck)

Theorem (Varacca) There is no distributive law for $\mathbb{V}_{\leq 1}$ and $\mathcal{P}_*$ for $* = L, U, C$.

Additional problem: $\mathbb{V}_{\leq 1} \circ \mathcal{P}_*$ leads to bizarre laws among processes.

Focus on $\mathcal{P}_* \circ \mathbb{V}_{\leq 1}$
Classical Approach Using Domains

- If $D$ is a coherent probabilistic algebra, $X \subseteq D$ Lawson closed is affine closed if $X = \langle X \rangle \equiv \{ x + r \cdot y \mid x, y \in X, r \in [0, 1] \}$ – also Lawson closed.

On coherent probabilistic algebras, there are three affine power domains:

- $\mathcal{P}_{LA}$ the lower affine power domain, modeled as the monad of non-empty affine-closed, Scott-closed sets, under inclusion.

- $\mathcal{P}_{UA}$ the upper affine power domain, modeled as the monad of non-empty Scott-compact, affine closed and saturated (= upper) sets, under reverse inclusion.

- $\mathcal{P}_{CA}$ the (order-)convex affine power domain, modeled as the monad of non-empty Lawson closed, order-convex, affine closed subsets, under the Egli-Milner order.
Classical Approach Using Domains

- All three have equational theories, hence define monads of probabilistic algebras:
  \[ \mathcal{P}_{LA} : p + q = \sup_r \{ p + r \cdot q \mid r \in [0, 1] \} \]
  \[ \mathcal{P}_{UA} : p + q = \inf_r \{ p + r \cdot q \mid r \in [0, 1] \} \]
  \[ \mathcal{P}_{CA} : p + q = \langle \{ p + r \cdot q \mid r \in [0, 1] \} \rangle \]

  Moreover, \( \mathcal{P}_{LA}, \mathcal{P}_{UA} : COH_P \rightarrow BCD \), where \( COH_P \) is coherent probabilistic algebras.

- Each free algebra \( \mathcal{P}_{\star A D} \) is a retract of the corresponding power domain \( \mathcal{P}_{\star D} \).
Alternative Approach Using d-Cones

R. Tix (1999), Keimel, Plotkin and Tix (2004/2009) developed approach using techniques from functional analysis:

- A **cone** is a set \( C \) with \( +: C \times C \rightarrow C \) and \( \cdot: \mathbb{R}_+ \times C \rightarrow C \) satisfying:
  
  \[
  a + (b + c) = (a + b) + c \quad 1 \cdot a = a; \quad 0 \cdot a = 0 \\
  a + b = b + a \quad (rs) \cdot a = r \cdot (s \cdot a) \\
  a + 0 = a \quad r \cdot (a + b) = r \cdot a + s \cdot a; \quad (r + s) \cdot a = r \cdot a + s \cdot a
  \]

  for \( a, b, c \in C \) and \( r, s \in \mathbb{R}_+ \).

- \( C \) is an **ordered cone** if \( C \) has a partial order with \( + \) and \( \cdot \) monotone.

- \( C \) is a **d-cone** if \( C \) is a dcpo and \( + \) and \( \cdot \) are Scott continuous.

**Example:** \( \overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\} \) is a continuous d-cone.

- \( f: C \rightarrow C' \) is **homogeneous** if \( f(r \cdot a) = r \cdot f(a) \); \( f \) is **linear** if \( f \) also preserves \( + \).

- Continuous d-cones are locally convex.
The Extended Probabilistic Domain

If $D$ is a DCPO, $\forall D = \{\mu: \sigma D \to \overline{\mathbb{R}}_+ \mid \mu \text{ valuation}\}$.

Properties:

- $\forall X$ is a d-cone for any topological space $X$
- $\forall D$ is a continuous d-cone for $D$ a domain. In addition:
  - $\forall D$ has an additive way-below relation, and
  - $\forall D$ is Lawson compact iff $D$ is.

**Hahn-Banach Theorem** Let $C$ be a continuous d-cone with additive way-below relation, and let $D \subseteq C$ be a d-subcone of $C$. Let $\Lambda: D \to \overline{\mathbb{R}}_+$ be linear and Scott-continuous, and let $p: C \to \overline{\mathbb{R}}_+$ be sublinear with

\[
d \leq a + c, \ d, a \in D, c \in C \implies \Lambda(d) \leq \Lambda(a) + p(c).
\]

Then there is a Scott-continuous linear extension $\hat{\Lambda}: C \to \overline{\mathbb{R}}_+$ with $\hat{\Lambda} \leq p$. 
The Extended Probabilistic Domain

- The Three Powercones

Each affine power domain operator defines a related powercone on CONE, the category of d-cones and Scott-continuous linear maps:

\( \mathcal{H} \): The lower powercone of non-empty Scott-closed, affine subsets of \( C \), ordered by inclusion.

If \( A, B \in \mathcal{H}(C) \), then \( A \cup_{\mathcal{H}} B = \overline{A + B} \), \( r \cdot_{\mathcal{H}} A = r \cdot A \), and \( A \vee B = \langle A \cup B \rangle \).

\( \mathcal{H}(C) \) is a continuous lattice if \( C \) is a domain.

\( S \): The upper powercone of non-empty Scott-compact saturated affine subsets of \( C \), ordered by reverse inclusion. \( S(C) \) is a continuous inf-semilattice if \( C \) is a domain.

\( B \): The biconvex powercone of non-empty Lawson compact, order- and affine-convex subsets of \( C \). \( B(C) \) is Lawson compact iff \( C \) is a coherent domain.

Each of these is a left adjoint to an obvious inclusion functor.
Kegelspitzen: Combining Approaches

- **Goal:** Integrate domain-theoretic models into d-cone approach

- A *barycentric algebra* $A$ is equipped with a family $+_r : A \times A \to A$, $r \in [0, 1]$ satisfying
  
  Idempotence $a +_r a = a$
  
  Identity $a +_1 b = a$
  
  Skew commutativity $a +_r b = b +_1 -r a$
  
  Skew associativity $(a +_r b) +_s c = a +_r s (b +_{rac{r-rs}{1-rs}} c)$ \( rs \neq 1 \)

  These are the same laws that characterize the equational theory for the valuations monad; they define *abstract convex sets*.

  A map $f : A \to B$ between barycentric algebras is *affine* if $f(a +_r a') = f(a) +_r f(a')$.

- If $I$ is a set, then $\bigoplus_{i \in I} \mathbb{R}_+$ is the free cone over $I$, and
  $P_I = \{(x_i)_{i \in I} \mid \sum_{i \in I} x_i = 1\}$ is the free barycentric algebra over $I$. 
Kegelspitzen: Combining Approaches

One adds:

A point 0 to ordered barycentric algebras to form *pointed barycentric algebras* and a partial order to ordered barycentric algebras so that

\[ a \leq a' \implies a + r \cdot b \leq a' + r \cdot b, \text{ for } a, a', b \in A \text{ and } r \in [0, 1]. \]

Maps between such are 0-affine or *linear* if they are affine and preserve 0.

- If \( I \) is a set, the family

\[ S_I = \{(x_i)_{i \in I} \in \bigoplus_{i \in I} \mathbb{R}_+ \mid \sum_{i \in I} x_i \leq 1\} \]

of finitely supported \((x_i)_{i \in I}\) is the free pointed barycentric algebra over \( I \).
Kegelspitzen: Combining Approaches

An \emph{b-cone} is an ordered cone $C$ for which
\[ +: C \times C \rightarrow C \text{ and } \cdot: [0, 1] \times C \rightarrow C \]
are Scott-continuous, and in which every bounded directed set has a supremum.

A \emph{d-cone} is a b-cone that is a dcpo.

\textit{Examples}: $\mathbb{R}_+$ is a b-cone and $\overline{\mathbb{R}}_+$ is a d-cone.

A \emph{Kegelspitze} is a pointed barycentric algebra $K$ that is a dcpo in which
\[ +_r: K \times K \rightarrow K \text{ and } \cdot: [0, 1] \times K \rightarrow K \]
are Scott continuous.
Kegelspitzen: Combining Approaches

- **Next goal:** Embed a Kegelspitze in a b-cone, and the b-cone in a d-cone.

  Every Kegelspitze satisfies: $r \cdot x \leq r \cdot y \implies x \leq y$ for $0 < r < 1$.

  A Kegelspitze is **full** if $a \leq r \cdot b \implies (\exists a' \in K) a = r \cdot a'$ for $0 < r < 1$.

  **Lemma** A Kegelspitze can be embedded as a lower set in a cone iff it is full.

  **Theorem** Every full Kegelspitze can be order-embedded in a free b-cone $b$-$\text{Cone}(K)$, and $b$-$\text{Cone}(K)$ has a universal dcpo-completion $\overline{\text{Cone}(K)} = d$-$\text{Cone}(K)$ that is a d-cone.

  So we get: $K \hookrightarrow b$-$\text{Cone}(K) \hookrightarrow d$-$\text{Cone}(K)$.

  The free cone $b$-$\text{Cone}(K)$ is a quotient of the cone $\bigoplus_{i \in I} \mathbb{R}_+$ for a set $I$ dependent on $K$.

  **Example:** $[0, 1]^n \hookrightarrow \mathbb{R}_+^n \hookrightarrow \overline{\mathbb{R}_+^n}$. 
Kegelspitzen: Combining Approaches

- **Continuity Results:** If $K$ is a full Kegelspitze, then $K$ is continuous iff $d$-$Cone(K)$ is continuous, in this case $\{r \cdot K \mid r \geq 0\}$ forms a basis for $d$-$Cone(K)$. Moreover, $K$ is coherent iff $d$-$Cone(K)$ is coherent.

- **Valuations:** If $P$ is a domain, then $\forall_{\leq 1}$ is a Kegelspitze with $d$-$Cone(\forall P) \simeq \forall P$. 
Kegelspitzen: Combining Approaches

- **Kraftkegelspitzen**: As before, there are three power Kegelspitzen:

  **H**: If $K$ is a full Kegelspitze, then so is $(\mathcal{H}K, +_\mathcal{H}, \{0\})$, where $\mathcal{H}K$ is the family of non-empty Scott-closed convex subsets of $K$, and $X +_\mathcal{H} Y = \overline{X + rY}$. Moreover, $\mathcal{H}K$ is the universal join-semilattice Kegelspitze over $K$.

  **S**: If $K$ is a full continuous Kegelspitze, then so is $(\mathcal{S}K, +_\mathcal{S}, K)$, where $\mathcal{S}K$ is the family of non-empty Scott-compact saturated and convex subsets of $K$, and $X \land Y = \uparrow \langle X \cup Y \rangle$. If convex combinations in $K$ preserve way-below, then $\mathcal{S}K$ is the universal inf-semilattice continuous Kegelspitze over $K$.

  **P**: If $K$ is a continuous coherent full Kegelspitze, then so is $(\mathcal{P}K, +_\mathcal{P}, \{0\})$, the family of non-empty Lawson-compact, order- and affine convex subsets of $K$ in the Egli-Milner order: $X \sqsubseteq Y$ iff $X \subseteq \downarrow Y$ & $Y \subseteq \uparrow X$. If way-below on $K$ is closed under convex combinations, then the same is true in $\mathcal{P}K$, and in this case, $\mathcal{P}K$ is the universal Kegelspitze semilattice over $K$. 
Kegelspitzen and Linear / Nonlinear Modals

An abstract model of linear logic is induced by a Linear/Non-Linear (LNL) model\(^1\):

- A cartesian closed category \(\mathbf{V}\).
- A symmetric monoidal closed category \(\mathbf{C}\).
- A symmetric monoidal adjunction:

\[
\begin{array}{c}
\mathbf{V} \\
\downarrow \quad \downarrow \\
\mathbf{C}
\end{array}
\]

\[F \quad G\]

\[
\begin{array}{c}
\mathbf{V} \\
\downarrow \quad \downarrow \\
\mathbf{C}
\end{array}
\]

\[F \quad G\]

... together with some additional data which is irrelevant for this talk.

An LNL model is a model of Intuitionistic Linear Logic.

\(^{1}\)Nick Benton. A mixed linear and non-linear logic: Proofs, terms and models. CSL’94
A concrete model of linear logic is induced by a Linear/Non-Linear (LNL) model:

• The cartesian closed category $\omega CPO$ of posets having sups of countable chains and Scott-continuous maps.

• The symmetric monoidal closed category $KS$ of Kegelspitzen and affine Scott-continuous maps.\(^1\)

• A symmetric monoidal adjunction:\(^1\)

\[ \omega CPO \xymatrix@C-1.5pc{\ar@{<-}[rr]^F & & KS \ar@{<-}[ll]_G} \]

\(^1\)This all has to be validated.