Kegelspitzen and Computational Models

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Overview

- Kegelspitzen are dcpos that have an additional convex structure:

  "Mixed Powerdomains for Probability and Nondeterminism,"
  Klaus Keimel and Gordon Plotkin, LMCS 2017.

  Combine convex domains and dcpo cones approaches

  Present dcpos with convex structure as first class objects

  Convexity traditionally arises via the *valuations monad*.

- Plan of the talk:

  Describe "classical approach" to modeling nondeterminism and probabilistic choice in domains.

  Outline the "Kegelspitzen approach".

  Time permitting: Outline potential use of Kegelspitzen as part of Linear / Nonlinear Models of intuitionistic logic and the lambda calculus.

Classical Approach Using Domains

- Nondeterminism and Probabilistic Choice are *computational effects*

  Modeled using monads on DCPO, ω-CPO, DOM, etc. (Moggi)

- Nondeterminism has an *equational theory:*

  $x + x = x \qquad x + y = y + x \qquad x + (y + z) = (x + y) + z$

  (L) $x, y \leq x + y \qquad$ (U) $x + y \leq x, y \qquad$ (C) $+$ monotone

  so there are monads on DCPO in each case.

- *Each form of nondeterminism* has a corresponding model:

  $\mathcal{P}_L$ the lower power domain, modeled as the *monad of non-empty Scott-closed sets*, under inclusion. Works for all DCPOs.

  $\mathcal{P}_U$ the upper power domain, modeled as the *monad of non-empty Scott-compact saturated (= upper) sets*, under reverse inclusion. Works for sober DCPOs.

  $\mathcal{P}_C$ the (order-)convex power domain, modeled as the *monad of non-empty Lawson closed, order-convex subsets*, under the Egli-Milner order. Works for Lawson compact (= coherent) domains.

Classical Approach Using Domains

- *Probabilistic Choice* is modeled by *valuations.*

  Functions $\mu\colon \sigma D \to [0,1]$ satisfying:

Strictness:	$\mu(\emptyset) = 0$
Modularity:	$\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
Scott continuity	$\{U_i\}_i$ directed $\implies \mu(\bigcup_i U_i) = \sup_i \mu(U_i)$

Pointwise Order	$\mu \leq \nu \iff \mu(U) \leq \nu(U) \quad (\forall U \in \sigma D)$
Probabilistic power domain	$\mathbb{V}_{\leq 1} D = \{\mu \mid \mu \text{ valuation}\}$
	– subdcpo of $[D, [0,1]]$
Probability measures	$\mathbb{V}_1 D = \{\mu \in \mathbb{V} D \mid \mu(D) = 1\}$
(for continuous DCPOs)	

Classical Approach Using Domains

- $\mathbb{V}_{\leq 1}, \mathbb{V}_1 : \mathit{DCPO} \to \mathit{DCPO}$ monads.

  Have an equational theory (Jones, 1989): $\forall r, s \in [0,1]. \forall a, b, c \in \mathbb{V}D$

Idempotence	$a +_r a = a$
Identity	$a +_1 b = a$
Skew commutativity	$a +_r b = b +_{1-r} a$
Skew associativity	$(a +_r b) +_s c = a +_{rs} (b +_{\frac{r-rs}{1-rs}} c) \quad rs \neq 1$

  Probabilistic algebra: dcpo D satisfying the above laws with $[0,1] \times D \times D \to D$ continuous in the usual $\times$ product Scott topologies

  Examples: $\mathbb{V}_{\leq 1} D, \mathbb{V}_1 D$ for D a DCPO.

Classical Approach Using Domains

- *Combining* $\mathbb{V}_{\leq 1}$ *with* $\mathcal{P}_*$

  Given monads $T, S \colon C \to C$, $T \circ S$ is a monad iff there is a distributive law $S \circ T \xrightarrow{\lambda} T \circ S$. (Beck)

  Theorem (Varacca) There is no distributive law for $\mathbb{V}_{\leq 1}$ and $\mathcal{P}_*$ for $* = L, U, C$.

  Additional problem: $\mathbb{V}_{\leq 1} \circ \mathcal{P}_*$ leads to bizarre laws among processes.

  Focus on $\mathcal{P}_* \circ \mathbb{V}_{\leq 1}$

Classical Approach Using Domains

- If D is a coherent probabilistic algebra, $X \subseteq D$ Lawson closed is *affine closed* if $X = \langle X \rangle \equiv \{x +_r y \mid x, y \in X, r \in [0,1]\}$ – also Lawson closed.

  On coherent probabilistic algebras, there are three *affine power domains:*

  $\mathcal{P}_{LA}$ the lower affine power domain, modeled as the *monad of non-empty affine-closed, Scott-closed sets*, under inclusion.

  $\mathcal{P}_{UA}$ the upper affine power domain, modeled as the *monad of non-empty Scott-compact, affine closed and saturated (= upper) sets*, under reverse inclusion.

  $\mathcal{P}_{CA}$ the (order-)convex affine power domain, modeled as the *monad of non-empty Lawson closed, order-convex, affine closed subsets*, under the Egli-Milner order.

Classical Approach Using Domains

- All three have equational theories, hence define monads of probabilistic algebras:

$\mathcal{P}_{LA}$: $p + q = \sup_r \{p +_r q \mid r \in [0,1]\}$

$\mathcal{P}_{UA}$: $p + q = \inf_r \{p +_r q \mid r \in [0,1]\}$

$\mathcal{P}_{CA}$: $p + q = \langle \{p +_r q \mid r \in [0,1]\} \rangle$

  Moreover, $\mathcal{P}_{LA}, \mathcal{P}_{UA} \colon COH_P \to BCD$, where COH_P is coherent probabilistic algebras.

- Each free algebra $\mathcal{P}_{*A} D$ is a retract of the corresponding power domain $\mathcal{P}_* D$.

Alternative Approach Using d-Cones

R. Tix (1999), Keimel, Plotkin and Tix (2004/2009) developed approach using techniques from functional analysis:

- A *cone* is a set C with $+\colon C \times C \to C$ and $\cdot\colon \mathbb{R}_+ \times C \to C$ satisfying:

  $$\begin{array}{ll} a + (b + c) = (a + b) + c & 1 \cdot a = a; \quad 0 \cdot a = 0 \\ a + b = b + a & (rs) \cdot a = r \cdot (s \cdot a) \\ a + 0 = a & r \cdot (a + b) = r \cdot a + s \cdot a; \quad (r + s) \cdot a = r \cdot a + s \cdot a \end{array}$$

  for $a, b, c \in C$ and $r, s \in \mathbb{R}_+$.

  C is an *ordered cone* if C has a partial order with $+$ and $\cdot$ monotone.

  C is a *d-cone* if C is a dcpo and $+$ and $\cdot$ are Scott continuous.

  Example: $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\}$ is a continuous d-cone.

  $f\colon C \to C'$ is *homogeneous* if $f(r \cdot a) = r \cdot f(a)$; f is *linear* if f also preserves $+$.

- Continuous d-cones are locally convex.

The Extended Probabilistic Domain

If D is a DCPO, $\mathbb{V}D = \{\mu \colon \sigma D \to \overline{\mathbb{R}_+} \mid \mu \text{ valuation}\}$.

Properties:

- $\mathbb{V}X$ is a d-cone for any topological space X
- $\mathbb{V}D$ is a continuous d-cone for D a domain. In addition:
  - $\mathbb{V}D$ has an additive way-below relation, and
  - $\mathbb{V}D$ is Lawson compact iff D is.

Hahn-Banach Theorem Let C be a continuous d-cone with additive way-below relation, and let $D \subseteq C$ be a d-subcone of C. Let $\Lambda \colon D \to \overline{\mathbb{R}_+}$ be linear and Scott-continuous, and let $p \colon C \to \overline{\mathbb{R}_+}$ be sublinear with

$$d \leq a + c,\ d, a \in D, c \in C \quad \Longrightarrow \quad \Lambda(d) \leq \Lambda(a) + p(c).$$

Then there is a Scott-continuous linear extension $\widehat{\Lambda} \colon C \to \overline{\mathbb{R}_+}$ with $\widehat{\Lambda} \leq p$.

The Extended Probabilistic Domain

- The Three Powercones

  Each affine power domain operator defines a related powercone on CONE, the category of d-cones and Scott-continuous linear maps:

  $\mathcal{H}$: The lower powercone of non-empty Scott-closed, affine subsets of C, ordered by inclusion.

  If $A, B \in \mathcal{H}(C)$, then $A +_{\mathcal{H}} B = \overline{A + B}$, $r \cdot_{\mathcal{H}} A = r \cdot A$, and $A \vee B = \overline{\langle A \cup B \rangle}$.

  $\mathcal{H}(C)$ is a continuous lattice if C is a domain.

  $\mathcal{S}$: The upper powercone of non-empty Scott-compact saturated affine subsets of C, ordered by reverse inclusion. $\mathcal{S}(C)$ is a continuous inf-semilattice if C is a domain.

  $\mathcal{B}$: The biconvex powercone of non-empty Lawson compact, order- and affine-convex subsets of C. $\mathcal{B}(C)$ is Lawson compact iff C is a coherent domain.

  Each of these is a left adjoint to an obvious inclusion functor.

Kegelspitzen: Combining Approaches

- *Goal: Integrate domain-theoretic models into d-cone approach*
- A *barycentric algebra* A is equipped with a family $+_r : A \times A \to A$, $r \in [0,1]$ satisfying

Idempotence	$a +_r a = a$
Identity	$a +_1 b = a$
Skew commutativity	$a +_r b = b +_{1-r} a$
Skew associativity	$(a +_r b) +_s c = a +_{rs} (b +_{\frac{r-rs}{1-rs}} c) \quad rs \neq 1$

  These are the same laws that characterize the equational theory for the valuations monad; they define *abstract convex sets*.

  A map $f : A \to B$ between barycentric algebras is *affine* if $f(a +_r a') = f(a) +_r f(a')$.

- If I is a set, then $\bigoplus_{i \in I} \mathbb{R}_+$ is the free cone over I, and

  $P_I = \{(x_i)_{i \in I} \mid \sum_{i \in I} x_i = 1\}$ is the free barycentric algebra over I.

Kegelspitzen: Combining Approaches

One adds:

A *point* 0 to ordered barycentric algebras to form *pointed barycentric algebras*

and a partial order to ordered barycentric algebras so that

$a \leq a' \implies a +_r b \leq a' +_r b$, for $a, a', b \in A$ and $r \in [0, 1]$.

Maps between such are 0-affine or *linear* if they are affine and preserve 0.

- If I is a set, the family

$$S_I = \{(x_i)_{i \in I} \in \bigoplus_{i \in I} \mathbb{R}_+ \mid \sum_{i \in I} x_i \leq 1\}$$

of finitely supported $(x_i)_{i \in I}$ is the free pointed barycentric algebra over I.

Kegelspitzen: Combining Approaches

An *b-cone* is an ordered cone C for which

$$+\colon C \times C \to C \text{ and } \cdot\colon [0,1] \times C \to C$$

are Scot-continuous, and in which every bounded directed set has a supremum.

A *d-cone* is a b-cone that is a dcpo.

Examples: $\mathbb{R}_+$ is a b-cone and $\overline{\mathbb{R}_+}$ is a d-cone.

A *Kegelspitze* is a pointed barycentric algebra K that is a dcpo in which

$$+_r\colon K \times K \to K \text{ and } \cdot\colon [0,1] \times K \to K$$

are Scott continuous.

Kegelspitzen: Combining Approaches

- *Next goal:* Embed a Kegelsptize in a b-cone, and the b-cone in a d-cone.

  Every Kegelspitze satisfies: $r \cdot x \leq r \cdot y \implies x \leq y$ for $0 < r < 1$.

  A Kegelspitze is *full* if $a \leq r \cdot b \implies (\exists a' \in K)\, a = r \cdot a'$ for $0 < r < 1$.

  Lemma A Kegelsptize can be embedded as a lower set in a cone iff it is full.

  Theorem Every full Kegelsptize can be order-embedded in a free b-cone $b\text{-}Cone(K)$, and $b\text{-}Cone(K)$ has a universal dcpo-completion $\overline{Cone(K)} = d\text{-}Cone(K)$ that is a d-cone.

  So we get: $K \hookrightarrow b\text{-}Cone(K) \hookrightarrow d\text{-}Cone(K)$.

  The free cone $b\text{-}Cone(K)$ is a quotient of the cone $\bigoplus_{i \in I} \mathbb{R}_+$ for a set I dependent on K.

  Example: $[0,1]^n \hookrightarrow \mathbb{R}_+^n \hookrightarrow \overline{\mathbb{R}_+}^n$.

Kegelspitzen: Combining Approaches

- *Continuity Results:* If K is a full Kegelspitze, then K is continuous iff $d\text{-}Cone(K)$ is continuous, in this case $\{r \cdot K \mid r \geq 0\}$ forms a basis for $d\text{-}Cone(K)$.
  Moreover, K is coherent iff $d\text{-}Cone(K)$ is coherent.
- *Valuations:* If P is a domain, then $\mathbb{V}_{\leq 1}$ is a Kegelspitze with $d\text{-}Cone(\mathbb{V}P) \simeq \mathbb{V}P$.

Kegelspitzen: Combining Approaches

- *Kraftkegelspitzen:* As before, there are three power Kegelspitzen:

$\mathcal{H}$: If K is a full Kegelspitze, then so is $(\mathcal{H}K, +_{rH}, \{0\})$, where $\mathcal{H}K$ is the family of non-empty Scott-closed convex subsets of K, and $X +_{rH} Y = \overline{X +_r Y}$. Moreover, $\mathcal{H}K$ is the universal join-semilattice Kegelsptize over K.

$\mathcal{S}$: If K is a full continuous Kegelspitze, then so is $(\mathcal{S}K, +_{rS}, K)$, where $\mathcal{S}K$ is the family of non-empty Scott-compact saturated and convex subsets of K, and $X \wedge Y) = \uparrow \langle X \cup Y \rangle)$. If convex combinations in K preserve way-below, then $\mathcal{S}K$ is the universal inf-semilattice continuous Kegelspitze over K.

$\mathcal{P}$: If K is a continuous coherent full Kegelspitze, then so is $(\mathcal{P}K, +_{rP}, \{0\})$, the family of non-empty Lawson-compact, order- and affine convex subsets of K in the Egli-Milner order: $X \sqsubseteq Y$ iff $X \subseteq \downarrow Y$ & $Y \subseteq \uparrow X$. If way-below on K is closed under convex combinations, then the same is true in $\mathcal{P}K$, and in this case, $\mathcal{P}K$ is the universal Kegelspitze semilattice over K.

Kegelspitzen and Linear / Nonlinear Modals

An *abstract model* of *linear logic* is induced by a Linear/Non-Linear (LNL) model[1]:

- A cartesian closed category $\mathbf{V}$.
- A symmetric monoidal closed category $\mathbf{C}$.
- A symmetric monoidal adjunction:

$$\mathbf{V} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbf{C}$$

together with some additional data which is irrelevant for this talk.

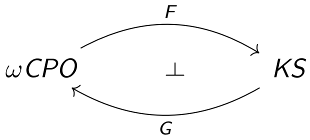
An LNL model is a model of Intuitionistic Linear Logic.

[1] Nick Benton. *A mixed linear and non-linear logic: Proofs, terms and models*. CSL'94

Kegelspitzen and Linear / Nonlinear Modals

A *concrete model* of linear logic is induced by a Linear/Non-Linear (LNL) model:

- The cartesian closed category ωCPO of posets having sups of countable chains and Scott-continuous maps.
- The symmetric monoidal closed category KS of Kegelspitzen and affine Scott-continuous maps.[1]
- A symmetric monoidal adjunction:[1]



[1] This all has to be validated.