Discrete Random Variables Over Domains, Revisited

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Outline

The main points:

- *Domains* are useful and popular models of computation, but...
- ... they don't handle probabilistic choice very well.
- An alternative is *random variables*, so
- ... I'll describe a *monad of discrete random variables* over domains, and
- ... indicate how the model can be extended to include *continuous* measures.

Domains and Probability

I. Domains are models of computation:

- DCPOs – directed complete partially ordered sets
  - $\emptyset \neq X \subseteq (D, \leq)$ *directed* if $\quad x, y \in X \;\Rightarrow\; (\exists z \in X)\, x, y \leq z$.
  - $(D, \leq)$ *directed complete* if $\sup X \in D$ ($\forall X \subseteq D$ directed)
- Domains are DCPOs with *approximation:*
  - $y \ll x$ if $x \leq \sup X$ directed implies $y \in \downarrow X$.
  - *D domain:* $\Downarrow x = \{y \mid y \ll x\}$ directed & $x = \sup \Downarrow x$ ($\forall x \in D$).
- Cartesian closed categories of domains and Scott-continuous maps
  - $f : D \longrightarrow E$ *Scott continuous* if f is monotone and $f(\sup X) = \sup f(X)$ ($\forall X \subseteq D$ directed).
  - Give models of untyped lambda calculus...
- For the purposes of this talk, think of the Failures or Failures/Divergences models of untimed *CSP*
  - Both are *bounded complete domains*.

Domains and Probability

II. $\mathbb{V}(D)$ – Subprobability measures *qua Scott continuous valuations:*

- $\mu\colon \mathcal{O}(D) \longrightarrow [0,1]$ with $\mu(\emptyset) = 0, \mu(D) \leq 1$,

  $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$.

  $\mu(\bigcup_i U_i) = \sup_i \mu(U_i)$ $(\forall \{U_i\} \subseteq \mathcal{O}(D)$ directed).
- $\mathbb{V}(D) \subseteq [\mathcal{O}(D) \longrightarrow [0,1]]$ is a subdcpo in pointwise order

What We Know About $(\mathbb{V}(D), \leq)$

Positive Results:

1980: Saheb-Djarhomi: $\mathbb{V}(D)$ is a dcpo & simple measures are sup-dense.

1989: Claire Jones: $\mathbb{V}$ is a monad on DCPO and on Dom;

Splitting Lemma: $\sum_i r_i \delta_{x_i} \leq \sum_j s_j \delta_{y_j}$ iff $(\exists \{t_{ij} \geq 0 \mid i, j\})$

with $r_i = \sum_j t_{ij}$, $\sum_i t_{ij} \leq s_j$; & $t_{ij} > 0 \Rightarrow x_i \leq y_j$

1998 Jung & Tix: $\mathbb{V}(D) \in$ Coh if D is;

$\mathbb{V}(T) \in$ BCD & $\mathbb{V}(T^{rev}) \in$ RB for each finite tree, T.

2016 M. (unpublished): $\mathbb{V}(C)$ is a continuous lattice if C is a complete chain.

Negative Results:

2003 Plotkin & Varacca: There is no distributive law for $\mathbb{V}$ over any of the power domains.

1980– No known CCC of domains invariant under $\mathbb{V}$.

What We Know About $(\mathbb{V}(D), \leq)$

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Purpose of this talk:

- Describe a *discrete random variable monad DCT* on BCD
  - Can be applied to models of untimed *CSP*
- Also describe an extension *CRV* that incorporates continuous measures, inspired by Hoare's normal termination, $\surd$

Discrete Random Variables over Domains

Random variable: $X\colon (S, \Sigma_S, \mu) \longrightarrow (T, \Sigma_T)$ measurable map.

E.g., S, T topological spaces, Σ_S, Σ_T Borel σ-algebras.

Example: Discrete coin tosses:

- $2^{n^\flat} = \{0,1\}^n_\perp$ – flat domain of outcomes of n coin tosses.
- $\mathcal{M} = \bigoplus_n 2^{n^\flat}$ – coalesced sum:

  $x \leq y$ iff $x = \perp$ or $x = y \in 2^n$.
- $\mathbb{V}(\mathcal{M}) = \{\sum_{n<N} r_n \delta_{x_n} \mid x_n \in \mathcal{M}, \sum_n r_n \leq 1 \;\&\; N \leq \omega\}$

  – sub-probabilities over $\mathcal{M}$.

  – Bounded complete domain b/c $\mathcal{M}$ is a tree.

  – All measures are discrete b/c $\mathcal{M}$ is countable.

Discrete Random Variables over Domains

Example: Discrete coin tosses:

- $DCT(D) = \{(\mu, X) \in \mathbb{V}(\mathcal{M}) \times [\mathsf{supp}_{\Sigma}\, \mu \longrightarrow D]\}$

  $(\mu, X) \leq (\nu, Y)$ iff $\mu \leq \nu$ & $X \circ \pi_{\mathsf{supp}_{\Sigma}\, \mu} \leq Y$.

  - $\mu \leq \nu \ \Rightarrow \exists \pi_{\mathsf{supp}_{\Sigma}\, \mu} \colon \mathsf{supp}_{\Sigma}\, \nu \longrightarrow \mathsf{supp}_{\Sigma}\, \mu$.
  - D bounded complete $\Rightarrow DCT(D)$ bounded complete
    - BCD is Cartesian closed
    - $DCT(D)$ is an $\mathbb{V}(\mathcal{M})$-indexed family $\{\mu\} \times [\mathsf{supp}_{\Sigma}\, \mu \longrightarrow D]$ of bounded complete domains.

- $f \colon D \longrightarrow E \ \Rightarrow \ DCT(f)(\mu, X) = (\mu, f \circ X)$.

Discrete Random Variables over Domains

Example: Discrete coin tosses:

- $DCT(D) = \{(\mu, X) \in \mathbb{V}(\mathcal{M}) \times [\mathsf{supp}_{\Sigma}\,\mu \longrightarrow D]\}$

  $(\mu, X) \leq (\nu, Y)$ iff $\mu \leq \nu$ & $X \circ \pi_{\mathsf{supp}_{\Sigma}\,\mu} \leq Y$.
- $f \colon D \longrightarrow E \;\Rightarrow\; DCT(f)(\mu, X) = (\mu, f \circ X)$.

An Example

A process flips a fair coin; if H occurs, then it executes $a \longrightarrow STOP$, if T occurs, it executes $b \longrightarrow SKIP$.
We wish to iterate this twice. Here's the result:

$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. $(0 = H; 1 = T)$ $\qquad X \colon \{0,1\}_{\perp} \longrightarrow CSP_{traces}$ by
$X(0) = a \longrightarrow STOP,\ X(1) = b \longrightarrow SKIP,\ X(\perp) = STOP$.

$(\mu, X); (\mu, X) = (\frac{1}{2}\delta_0 + \frac{1}{4}\delta_{10} + \frac{1}{4}\delta_{11}, X; (X; STOP))$, and

$(X; (X; STOP))(0) = a \longrightarrow STOP;$

$(X; (X; STOP))(10) = b \longrightarrow a \longrightarrow STOP;$

$(X; (X; STOP))(11) = b \longrightarrow b \longrightarrow STOP.$

Monadic Properties of $\mathbb{V}$ *qua* SProb

We claim $DCT : BCD \longrightarrow BCD$ is a monad. Here's why:

Let V be a real vector space.

$X \subseteq V$ is an *affine space* if $\exists\, [0,1] \times X \times X \longrightarrow X$ continuous.

If $[0,1] \cdot X \subseteq X$, then $0_V \in X$.

Comp – Compact Hausdorff spaces and continuous maps

CompAff – Compact Hausdorff affine spaces with zero and continuous affine maps preserving zero.

CompMon – Compact Hausdorff monoids and continuous monoid homomorphisms

CompAffMon – Compact Hausdorff affine monoids with zero and continuous affine monoid homomorphisms preserving zero.

Theorem

- SProb: Comp $\longrightarrow$ CompAff defines a monad.
- SProb: CompMon $\longrightarrow$ CompAffMon defines a monad.

Monadic Properties of $\mathbb{V}$ *qua* SProb

We claim $DCT: BCD \longrightarrow BCD$ is a monad. Here's why:

$(S, \cdot)$ compact monoid $\Rightarrow$ $*\colon \mathrm{SProb}\, S \times \mathrm{SProb}\, S \longrightarrow \mathrm{SProb}\, S$ is given by:

$\mu * \nu(A) = \mu \times \nu(\cdot^{-1}(A)) = \mu \times \nu(\{(x, y) \mid x \cdot y \in A\})$

$(\forall A \subseteq S$ measurable).

Moreover, $\mathrm{supp}\, \mu * \nu = \mathrm{supp}\, \mu \cdot \mathrm{supp}\, \nu$.

CompOrdMon – Compact ordered monoids and continuous monotone homomorphisms.

CompOrdAffMon – Compact ordered affine monoids with zero and continuous affine monotone homomorphisms preserving zero.

Theorem

- $\mathrm{SProb}\colon \mathrm{CompOrdMon} \longrightarrow \mathrm{CompOrdAffMon}$ defines a monad.

Monadic Properties of $\mathbb{V}$ *qua* SProb

We claim $DCT : BCD \longrightarrow BCD$ is a monad. Here's why:

Some facts about $\mathcal{M}$:

- $\mathcal{M}$ is a *coherent domain:* its Lawson topology is compact Hausdorff (b/c $\mathcal{M}$ is bounded complete)
- $\mathcal{M}$ is the free *ordered* monoid with zero over $\{0, 1\}$:

$$x \cdot y = \begin{cases} x & \text{if } y = \epsilon \\ y & \text{if } x = \epsilon \\ \perp & \text{if } x = \perp \text{ or } y = \perp \\ xy \in \{0,1\}^{|x|+|y|} & \text{otherwise} \end{cases}$$

- $(\mathcal{M}, \cdot)$ is a compact monoid.

Corollary $\mathcal{M} = \bigoplus_n 2^{n\flat}$ is a free compact ordered monoid, so

$(\mathbb{V}(\mathcal{M}), *)$ is a free compact ordered affine monoid.

Discrete Random Variables... (cont'd)

- *DCT* is a monad, with Kleisli lift:

$$
\begin{array}{ccc}
\{(\mu, X) \in \mathbb{V}(\mathcal{M}) \times [\mathsf{supp}_{\Sigma}\, \mu \longrightarrow D]\} & & \\
\big\uparrow{\scriptstyle \eta_D} & \searrow^{h^{\dagger}} & \\
D & \xrightarrow[h]{\qquad\qquad} & \{(\nu, Y) \in \mathbb{V}(\mathcal{M}) \times [\mathsf{supp}_{\Sigma}\, \nu \longrightarrow E]\})
\end{array}
$$

where $\eta_D(d) = (\delta_\epsilon, const_d)$, and

$$h^{\dagger}(\sum_m r_m \delta_{x_m}, X) = \left(\sum_m r_m(\delta_{x_m} * \pi_1 \circ h \circ X(x_m)), \overline{\pi_2 \circ h \circ X} \right)$$

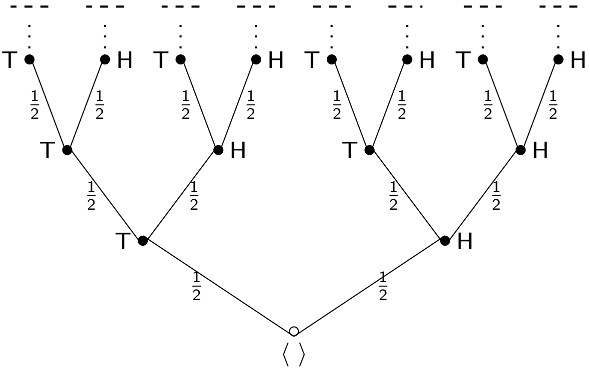
where $\overline{\pi_2 \circ h \circ X} : \overline{\bigcup_m \{x_m \cdot \mathsf{supp}_{\Sigma}(\pi_1 \circ h \circ X)(x_m))\}}^{\Sigma} \longrightarrow E$ is

$$\overline{\pi_2 \circ h \circ X}(x) = \bigwedge \{(\pi_2 \circ h \circ X)(x_m)(y) \mid x = x_m \cdot y \in (\{0,1\}^{|x_m|+|y|})_{\perp}\}.$$

Note: The many possible ways to factor x as $x_m y$ can lead to different possible outcomes.

The Cantor Tree as a Source of Randomness

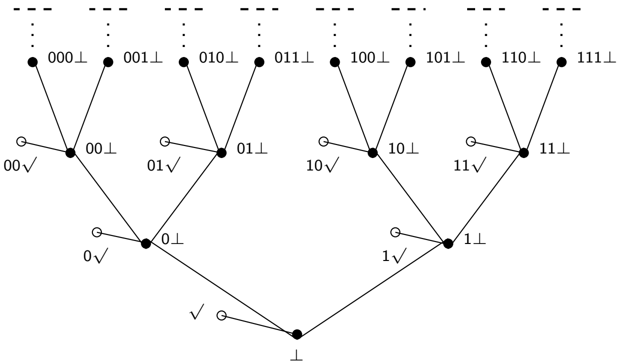
What about a model with continuous measures? $\mathcal{C} = \{0,1\}^\infty$



The Cantor Tree as a Source of Randomness

Instead of $\mathcal{C} = \{0,1\}^* \cup \{0,1\}^\omega$ in the prefix order, we use

$\mathbb{M}\{0,1\} = \{0,1\}^*\{\surd, \bot\} \cup \{0,1\}^\omega$, a sequential domain monoid.



$\mu = \frac{1}{4}\delta_{00\surd} + \frac{1}{4}\delta_{01\surd} + \frac{1}{4}\delta_{10\surd} + \frac{1}{4}\delta_{11\surd}$ is maximal, but

$\mu * \delta_\bot = \frac{1}{4}\delta_{00\bot} + \frac{1}{4}\delta_{01\bot} + \frac{1}{4}\delta_{10\bot} + \frac{1}{4}\delta_{11\bot} \leq \mu, \nu$ for all

uniform measures ν concentrated on $\{0,1\}^n\surd, \{0,1\}^n\bot$ for $n > 4$.

The Cantor Tree as a Source of Randomness

Then $\mathbb{V}\,\mathbb{M}\,\{0,1\}$ is an affine domain monoid.

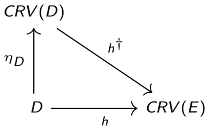
It's also in BCD because $\mathbb{M}\,\{0,1\}$ is a tree. So we define

$CRV(D) = \{(\mu, X) \in \mathbb{V}\,\mathbb{M}\,\{0,1\} \times [\mathrm{supp}_\Sigma\, \mu \longrightarrow D]\}$ with

$$(\mu, X) \leq (\nu, Y) \text{ iff } \mu \leq \nu \;\&\; X \circ \pi_{\mathrm{supp}_{\Sigma_\mu}} \leq Y, \text{ and}$$

$f\colon D \longrightarrow E \;\Rightarrow\; CRV(f)(\mu, X) = (\mu, f \circ X)$.

Theorem: *CRV* forms a monad on BCD.



where $\eta_D(d) = (\delta_{\surd}, const_d)$, and

$$h^\dagger(\sum_m r_m \delta_{x_m}, X) = \left(\sum_m r_m(\delta_{x_m} * \pi_1 \circ h \circ X(x_m)), \overline{\pi_2 \circ h \circ X}\right)$$

where $\overline{\pi_2 \circ h \circ X}\colon \overline{\bigcup_m \{x_m \cdot \mathrm{supp}_\Sigma(\pi_1 \circ h \circ X)(x_m))\}}^{\Sigma} \longrightarrow E$ is

$$\overline{\pi_2 \circ h \circ X}(x) = \bigwedge\{(\pi_2 \circ h \circ X)(x_m)(y) \mid x = x_m \cdot y \in (\{0,1\}^{|x_m|+|y|})_\perp\}.$$

And, finally...

Thanks, and Happy Birthday Bill!!