Semantic Models of Quantum Programming Languages:

Recursion in Categorical Models

Michael Mislove

Department of Computer Science
Tulane University

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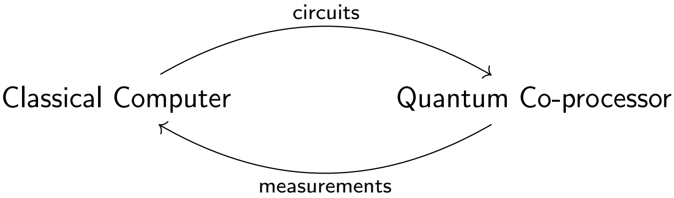
Semantics and Tools for High Level Functional Quantum Programming Languages

Prototypical Quantum Computer

- A *quantum computer* is a classical computer with a quantum co-processor



circuits

Classical Computer

Quantum Co-processor

measurements

- Circuit: sequence of unitary operators

Prototypical Quantum Computer

- We'll elide measurements and focus on a classical functional language for *constructing circuits* and a linear language for *modeling them* as linear morphisms.
- A *quantum programming language* is a classical functional language together with a linear language of *quantum circuits*:

$$\begin{tikzcd}\textit{Functional Language} \arrow[r, bend left] & \textit{Linear language} \arrow[l, bend left]\end{tikzcd}$$

- We study *circuit description languages* using Linear / Nonlinear Models

Proto-Quipper-M

- *Proto-Quipper-M* developed by Francisco Rios and Peter Selinger.

The types of the language:

$$\begin{array}{llcl}
\text{Types} & A, B & ::= & \alpha \mid 0 \mid A + B \mid I \mid A \otimes B \mid A \multimap B \mid {!A} \mid \mathbf{Circ(T, U)} \\
\text{Intuitionistic types} & P, R & ::= & 0 \mid P + R \mid I \mid P \otimes R \mid {!A} \mid \mathbf{Circ(T, U)} \\
\text{M-types} & T, U & ::= & \alpha \mid I \mid T \otimes U
\end{array}$$

The term language:

$$\begin{array}{llll}
\text{Terms} & M, N & ::= & x \mid \ell \mid c \mid \text{let } x = M \text{ in } N \\
& & & \mid \Box_A M \mid \text{left}_{A,B} M \mid \text{right}_{A,B} M \mid \text{case } M \text{ of } \{\text{left } x \to N \mid \text{right } y \to P\} \\
& & & \mid * \mid M; N \mid \langle M, N \rangle \mid \text{let } \langle x, y \rangle = M \text{ in } N \mid \lambda x^A.M \mid MN \\
& & & \mid \text{lift } M \mid \text{force } M \mid \mathbf{box_T M} \mid \mathbf{apply(M, N)} \mid \mathbf{(\vec{\ell}, C, \vec{\ell'})}
\end{array}$$

Combined Typing Judgement

- There is only one form of type judgement.
- Typing contexts $\Phi, \Gamma, \ldots$ can be mixed.
- Typing contexts Q are for circuit labels.

$$\frac{}{\Phi, x : A; \emptyset \vdash x : A}\ (\mathit{var}) \qquad \frac{}{\Phi; \ell : \alpha \vdash \ell : \alpha}\ (\mathit{label}) \qquad \frac{}{\Phi; \emptyset \vdash c : A_c}\ (\mathit{const})$$

$$\boxed{\frac{\Gamma, x : A; Q \vdash M : B}{\Gamma; Q \vdash \lambda x^A . M : A \multimap B}\ (\mathit{abs}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash M : A \multimap B \quad \Phi, \Gamma_2; Q_2 \vdash N : A}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash MN : B}\ (\mathit{app})}$$

$$\frac{\Phi; \emptyset \vdash M : A}{\Phi; \emptyset \vdash \mathrm{lift}\ M : !A}\ (\mathit{lift}) \qquad \frac{\Gamma; Q \vdash M : !A}{\Gamma; Q \vdash \mathrm{force}\ M : A}\ (\mathit{force})$$

$$\boxed{\frac{\Gamma; Q \vdash M : !(T \multimap U)}{\Gamma; Q \vdash \mathrm{box}_T\, M : \mathrm{Circ}(T, U)}\ (\mathit{box}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash M : \mathrm{Circ}(T, U) \quad \Phi, \Gamma_2; Q_2 \vdash N : T}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \mathrm{apply}(M, N) : U}\ (\mathit{apply})}$$

$$\frac{\emptyset; Q \vdash \vec{\ell} : T \quad \emptyset; Q' \vdash \vec{\ell}' : U \quad C \in \mathbf{M}_{\mathscr{L}}(Q, Q')}{\Phi; \emptyset \vdash (\vec{\ell}, C, \vec{\ell}') : \mathrm{Circ}(T, U)}\ (\mathit{circ})$$

Table 3: The typing rules of Proto-Quipper-M (excerpt)

Example

Assume $H : Q \multimap Q$ is a constant representing the Hadamard gate.

Example

two-hadamard : $\mathrm{Circ}(Q, Q)$
two-hadamard $\equiv \mathrm{box}_Q$ lift $\lambda q^Q.HHq$

This program creates a completed circuit consisting of two H gates. The term is intuitionistic (can be copied, deleted).

Circuit Model

Example

Shor's algorithm for integer factorization may be seen as an infinite family of quantum circuits – each circuit is a procedure for factoring an *n*-bit integer, for a fixed *n*.

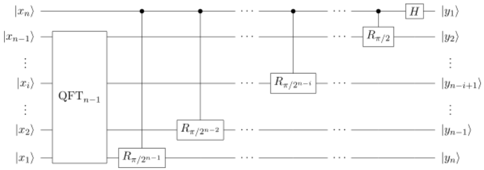


Figure: Quantum Fourier Transform on *n* qubits (subroutine in Shor's algorithm).[1]

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[1] Figure source: `https://commons.wikimedia.org/w/index.php?curid=14545612`

Circuit Model

Proto-Quipper-M is used to describe *families* of morphisms in an arbitrary, but fixed, symmetric monoidal category, **M**.

Example

If **M** = **FdCStar**, then a program in our language is a family of quantum circuits.

Example

M also could be a category of string diagrams that is freely generated.

- Model Verilog, VHDL, similar hardware description languages, Petri Nets, etc.

Linear/Non-Linear models

A Linear/Non-Linear (LNL) model as described by Benton is given by the following data:

- A cartesian closed category $\mathbf{V}$.
- A symmetric monoidal closed category $\mathbf{C}$.
- A symmetric monoidal adjunction:

$$\begin{tikzcd}
\mathbf{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, bend left, "G"]
\end{tikzcd}$$

$F \circ G = !$ – the lift comonad

Remark

An LNL model is a model of Intuitionistic Linear Logic.

Nick Benton. *A mixed linear and non-linear logic: Proofs, terms and models.* CSL'94

Concrete models of Proto-Quipper-M

The original Proto-Quipper-M model is given by the LNL model:

$$\begin{tikzcd}
\mathbf{Set} \arrow[rr, bend left, "- \odot I"] \arrow[rr, phantom, "\perp"] & & \mathbf{Fam}[\overline{\mathbf{M}}] \arrow[ll, bend left, "\mathbf{Fam}[\overline{\mathbf{M}}](I, -)"]
\end{tikzcd}$$

$\overline{\mathbf{M}}$ – closed, product complete category containing given SMC $\mathbf{M}$

- $\mathbf{Fam}[\overline{\mathbf{M}}] = \{(X, A) \mid X \text{ discrete category}, A \colon X \to \overline{\mathbf{M}} \text{ functor}\}$.
- $(f, \phi) \in \mathbf{Fam}[\overline{\mathbf{M}}]((X, A), (Y, B))$ if $f : X \to Y$ functor and $\phi \colon A \to B \circ f$ natural transformation.
- $(g, \psi) \circ (f, \phi) = (g \circ f, \psi f \circ \phi)$.

Theorem (Rios & Selinger)

The Families categorical model of Proto-Quipper-M is type-safe, sound, and computationally adequate

Concrete models of Proto-Quipper-M

The original Proto-Quipper-M model is given by the LNL model:

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] & \mathbf{Fam}[\overline{\mathbf{M}}] \arrow[l, bend left, "\mathbf{Fam}[\overline{\mathbf{M}}](I, -)"]
\end{tikzcd}$$

Sam Staton asked why the **Fam** construction is needed – it's not:

A simpler model for Proto-Quipper-M satisfying the same properties is given by:

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] & \overline{\mathbf{M}} \arrow[l, bend left, "\overline{\mathbf{M}}(I, -)"]
\end{tikzcd}$$

where in both cases $\overline{\mathbf{M}} = [\mathbf{M}^{\mathrm{op}}, \mathbf{Set}]$.

Our Work: Adding Recursion

- Rename the language to *ECLNL*
  - Emphasizes *Enrichment*, *Combined typing judgement* and *LNL models*.
  - Doesn't tie the language to quantum programming *per se*.
- Describe an *abstract* categorical model for the same language.
- Extend language and abstract categorical model to support recursion.
- Prove soundness for abstract models, and computational adequacy for *concrete model*.

Related work: Rennela and Staton describe a different circuit description language, called EWire (based on QWire), for which they also use enriched category theory.

An abstract model for ECLNL

An *ECLNL model* is given by the following data:

1. A cartesian closed category **V** together with its self-enrichment $\mathcal{V}$ having finite **V**-coproducts.
2. A **V**-symmetric monoidal closed category $\mathcal{C}$ having finite **V**-coproducts.
3. A **V**-symmetric monoidal adjunction:
$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] & \mathcal{C}, \arrow[l, bend left, "\mathcal{C}(I,-)"] \end{tikzcd}$$
where $(- \odot I)$ denotes the **V**-copower of the tensor unit in $\mathcal{C}$.
4. A symmetric monoidal category **M** and a strong symmetric monoidal functor $E : \mathbf{M} \to \mathbf{C}$, the underlying category of $\mathcal{C}$.

Theorem: Absent condition 4, an LNL model canonically induces an ECLNL model.[2]

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[2] Egger, Møgelberg, Simpson. *The enriched effect calculus: syntax and semantics*. Journal of Logic and Computation 2012

Soundness

Theorem (Soundness)

Every abstract model of ECLNL is computationally sound.

Concrete models of the base language

Fix an arbitrary symmetric monoidal category $\mathbf{M}$.
Equipping $\mathbf{M}$ with the free $\mathbf{DCPO}$-enrichment yields a concrete (order-enriched) ECLNL model:

$$\begin{tikzcd}
\mathbf{DCPO} \arrow[rr, bend left, "- \odot I"] \arrow[rr, phantom, "\perp"] & & \overline{\mathbf{M}} \arrow[ll, bend left, "\overline{\mathbf{M}}(I, -)"]
\end{tikzcd}$$

where $\overline{\mathbf{M}} = [\mathbf{M}^{\mathrm{op}}, \mathbf{DCPO}]$.

Abstract models with recursion

Definition

An endofunctor $T : \mathbf{C} \to \mathbf{C}$ is *parametrically algebraically compact*, if for every $A \in \mathrm{Ob}(\mathbf{C})$, the endofunctor $A \otimes T(-)$ has an initial algebra and a final coalgebra whose carriers coincide.

Theorem

A categorical model of a linear/non-linear lambda calculus extended with recursion is given by an LNL model:

$$\begin{tikzcd}
\mathbf{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, bend left, "G"]
\end{tikzcd}$$

where FG (or equivalently GF) is parametrically algebraically compact [3].

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[3] Benton & Wadler. *Linear logic, monads and the lambda calculus*. LiCS'96.

ECLNL extended with general recursion

Definition

A categorical model of ECLNL extended with general recursion is given by a model of ECLNL, where in addition:

5. The comonad endofunctor:

$$\begin{tikzcd}
\mathcal{V} \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\mathcal{C}(I,-)"]
\end{tikzcd}$$

   is parametrically algebraically compact.

Recursion

Extend the syntax:

$$\frac{\Phi, x :!A; \emptyset \vdash m : A}{\Phi; \emptyset \vdash \text{rec } x^{!A}\ m : A}\ (\text{rec})$$

Extend the operational semantics:

$$\frac{(C, m[\text{lift rec } x^{!A} m/x]) \Downarrow (C', v)}{(C, \text{rec } x^{!A} m) \Downarrow (C', v)}$$

Soundness

Theorem (Soundess)

Every model of ECLNL extended with recursion is computationally sound.

Concrete model of ECLNL extended with recursion

Let $\mathbf{M}_*$ be the free $\mathbf{DCPO}_{\perp !}$-enrichment of $\mathbf{M}$ and $\overline{\mathbf{M}_*} = [\mathbf{M}_*^{\mathrm{op}}, \mathbf{DCPO}_{\perp !}]$ be the associated enriched functor category.

$$\begin{tikzcd}
\mathbf{DCPO}_{\perp !} \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] & \overline{\mathbf{M}_*} \arrow[l, bend left, "\overline{\mathbf{M}_*}(I, -)"] \arrow[d, bend left, "\overline{U}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{DCPO} \arrow[u, bend left, "L"] \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] & \overline{\mathbf{M}} \arrow[l, bend left, "\overline{\mathbf{M}}(I, -)"] \arrow[u, bend left, "\overline{L}"]
\end{tikzcd}$$

Remark

If $\mathbf{M} = \mathbf{1}$*, then the above model degenerates to the left vertical adjunction, which is a model of a LNL lambda calculus with general recursion.*

Computational adequacy

Theorem

The following LNL model:

$$\begin{tikzcd} \mathbf{DCPO} \arrow[rr, bend left, "-_\perp"] \arrow[rr, phantom, "\perp"] & & \mathbf{DCPO}_{\perp !} \arrow[ll, bend left, "U"] \end{tikzcd}$$

is computationally adequate at intuitionistic types for the circuit-free fragment of ECLNL.

- Use logical relations for proof.
- Problem with adding circuits is that structural induction over logical relations breaks down on tensors from **M**
- Need more assumptions about **M** for "traditional" approach to work.

Ongoing / Future work

1. Inductive / recursive types.
   - We can support inductive types, since both $\mathcal{C}$ and $\mathcal{V}$ are algebraically complete for endofunctors preserving ω-colimits.
   - $\mathcal{C}$ is algebraically compact for endofunctors preserving ω-colimits, but $\mathcal{V}$ is not.
   - Problem is identifying which parametrically algebraic compact bifunctors $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ are *intuitionistic*. We believe we have solved this.
     Note: *e-p pairs arise here!*

2. Dependent types (Fam/CFam constructions are well-behaved w.r.t. current models).

3. Dynamic lifting.

Conclusions

- One can construct a model of ECLNL by categorically enriching certain denotational models.
- We described a sound abstract model for ECLNL (with general recursion).
- Systematic construction for concrete models that works for *any* circuit (string diagram) model described by a symmetric monoidal category.
- The "domain theory" is at the most general level – DCPO, $\mathrm{DCPO}_{\perp,!}$.

Thanks for your attention!

And

Happy Birthday, Achim!

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Operational semantics

(S, m) is a *configuration* if S is a (partially completed) labeled circuit, and m is a term.

$$\frac{(S,m) \Downarrow (S',v) \quad (S',n) \Downarrow (S'',v')}{(S,\langle m,n\rangle) \Downarrow (S'',\langle v,v'\rangle)} \qquad \frac{(S,m) \Downarrow (S',\langle v,v'\rangle) \quad (S',n[v \mathbin{/} x, v' \mathbin{/} y]) \Downarrow (S'',w)}{(S,\mathrm{let}\ \langle x,y\rangle = m\ \mathrm{in}\ n) \Downarrow (S'',w)}$$

$$\frac{}{(S,\mathrm{lift}\ m) \Downarrow (S,\mathrm{lift}\ m)} \qquad \frac{(S,m) \Downarrow (S',\mathrm{lift}\ m') \quad (S',m') \Downarrow (S'',v)}{(S,\mathrm{force}\ m) \Downarrow (S'',v)}$$

$$\frac{(S,m) \Downarrow (S',\mathrm{lift}\ n) \quad \mathrm{freshlabels}(T) = (Q,\vec{\ell}) \quad (\mathrm{id}_Q, n\vec{\ell}) \Downarrow (D,\vec{\ell}')}{(S,\mathrm{box}_T m) \Downarrow (S',(\vec{\ell},D,\vec{\ell}'))}$$

$$\frac{(S,m) \Downarrow (S',(\vec{\ell},D,\vec{\ell}')) \quad (S',n) \Downarrow (S'',\vec{k}) \quad \mathrm{append}(S'',\vec{k},\vec{\ell},D,\vec{\ell}') = (S''',\vec{k}')}{(S,\mathrm{apply}(m,n)) \Downarrow (S''',\vec{k}')}$$

$$\frac{(S,m) \Downarrow (S',(\vec{\ell},D,\vec{\ell}')) \quad (S',n) \Downarrow (S'',\vec{k}) \quad \mathrm{append}(S'',\vec{k},\vec{\ell},D,\vec{\ell}')\ \text{undefined}}{(S,\mathrm{apply}(m,n)) \Downarrow \mathrm{Error}} \qquad \frac{}{(S,(\vec{\ell},D,\vec{\ell}')) \Downarrow (S,(\vec{\ell},D,\vec{\ell}'))}$$

Recursion (contd.)

Extend the denotational semantics: $\llbracket \Phi; \emptyset \vdash \mathsf{rec}\ x^{!A}\ m : A \rrbracket := \sigma_{\llbracket m \rrbracket} \circ \gamma_{\llbracket \Phi \rrbracket}$.

$$\begin{tikzcd}
\llbracket \Phi \rrbracket \otimes ! \llbracket \Phi \rrbracket \arrow[d, dashed, "\mathrm{id} \otimes ! \gamma_{\llbracket \Phi \rrbracket}"'] & & \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \arrow[ll, "\mathrm{id} \otimes \mathbf{lift}"'] & \llbracket \Phi \rrbracket \arrow[l, "\Delta"'] \arrow[d, dashed, "\gamma_{\llbracket \Phi \rrbracket}"] \\
\llbracket \Phi \rrbracket \otimes ! \Omega_{\llbracket \Phi \rrbracket} \arrow[d, "\mathrm{id}"'] & & & \Omega_{\llbracket \Phi \rrbracket} \arrow[lll, "\omega^{-1}_{\llbracket \Phi \rrbracket}"'] \arrow[d, "\mathrm{id}"] \\
\llbracket \Phi \rrbracket \otimes ! \Omega_{\llbracket \Phi \rrbracket} \arrow[rrr, "\omega_{\llbracket \Phi \rrbracket}"] \arrow[d, dashed, "\mathrm{id} \otimes ! \sigma_{\llbracket m \rrbracket}"'] & & & \Omega_{\llbracket \Phi \rrbracket} \arrow[d, dashed, "\sigma_{\llbracket m \rrbracket}"] \\
\llbracket \Phi \rrbracket \otimes ! \llbracket A \rrbracket \arrow[rrr, "\llbracket m \rrbracket"'] & & & \llbracket A \rrbracket
\end{tikzcd}$$