Semantic Models of Quantum Programming Languages: 

Recursion in Categorical Models

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Work Supported by US AFOSR

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Workshop on Higher Category Approach to Certifiably Correct Quantum Information Processing Systems
February 4, 2019
MURI Project
Semantics and Tools for
High Level Functional Quantum Programming Languages
A quantum computer is a classical computer with a quantum co-processor.

Circuit: sequence of unitary operators
Prototypical Quantum Computer

• We’ll elide measurements and focus on a classical functional language for *constructing circuits* and a linear language for *modeling them* as linear morphisms.

• A *quantum programming language* is a classical functional language together with a linear language of *quantum circuits*:

  ![Diagram](image)

  **Functional Language**  
  **Linear language**

• We study *circuit description languages* using Linear / Nonlinear Models
Proto-Quipper-M

• *Proto-Quipper-M* developed by Francisco Rios and Peter Selinger.

The types of the language:

Types \( A, B \) ::= \( \alpha \mid 0 \mid A + B \mid I \mid A \otimes B \mid A \to B \mid !A \mid \text{Circ}(T, U) \)

Intuitionistic types \( P, R \) ::= \( 0 \mid P + R \mid I \mid P \otimes R \mid !A \mid \text{Circ}(T, U) \)

M-types \( T, U \) ::= \( \alpha \mid I \mid T \otimes U \)

The term language:

Terms \( M, N \) ::= \( x \mid \ell \mid c \mid \text{let } x = M \text{ in } N \)

| \( \Box A M \) | \( \text{left}_{A, B} M \) | \( \text{right}_{A, B} M \) | \( \text{case } M \text{ of } \{ \text{left } x \to N \mid \text{right } y \to P \} \)
| \( \ast \mid M ; N \mid \langle M, N \rangle \mid \text{let } \langle x, y \rangle = M \text{ in } N \mid \lambda x^A . M \mid MN \)
| lift \( M \) | force \( M \) | box_T M | apply(M, N) | (\( \ell \), C, \( \ell' \))
There is only one form of type judgement.

Typing contexts \( \Phi, \Gamma, \ldots \) can be mixed.

Typing contexts \( Q \) are for circuit labels.

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**Combined Typing Judgement**

- \( \Phi, x : A; \emptyset \vdash x : A \) \( \text{(var)} \)
- \( \Phi; \ell : \alpha \vdash \ell : \alpha \) \( \text{(label)} \)
- \( \Phi; \emptyset \vdash c : A_c \) \( \text{(const)} \)

\[
\begin{align*}
\Gamma, x : A; Q \vdash M : B \\
\Gamma; Q \vdash \lambda x^A.M : A \rightarrow B & \quad \text{(abs)} \\
\Phi; \emptyset \vdash M : A \\
\Phi; \emptyset \vdash \text{lift}M : !A & \quad \text{(lift)} \\
\Gamma; Q \vdash M : !A & \quad \text{(force)} \\
\Phi; \Gamma_1; Q_1 \vdash M : \text{Circ}(T, U) \\
\Phi; \Gamma_2; Q_2 \vdash N : T & \quad \text{(apply)} \\
\Phi; \emptyset \vdash (\ell, C, \ell') : \text{Circ}(T, U) & \quad \text{(circ)}
\end{align*}
\]

Table 3: The typing rules of Proto-Quipper-M (excerpt)
Assume $H : Q \rightarrow Q$ is a constant representing the Hadamard gate.

Example

two-hadamard : Circ(Q, Q)
two-hadamard \equiv box_Q \lift \lambda q^Q.HHq

This program creates a completed circuit consisting of two $H$ gates. The term is intuitionistic (can be copied, deleted).
Example

Shor’s algorithm for integer factorization may be seen as an infinite family of quantum circuits – each circuit is a procedure for factoring an $n$-bit integer, for a fixed $n$.

Figure: Quantum Fourier Transform on $n$ qubits (subroutine in Shor’s algorithm).¹

¹Figure source: https://commons.wikimedia.org/w/index.php?curid=14545612
Proto-Quipper-M is used to describe families of morphisms in an arbitrary, but fixed, symmetric monoidal category, \( M \).

**Example**

If \( M = \text{FdCStar} \), then a program in our language is a family of quantum circuits.

**Example**

\( M \) also could be a category of string diagrams that is freely generated.

- Model Verilog, VHDL, similar hardware description languages, Petri Nets, etc.
Linear/Non-Linear models

A Linear/Non-Linear (LNL) model as described by Benton is given by the following data:

- A cartesian closed category $\mathcal{V}$.
- A symmetric monoidal closed category $\mathcal{C}$.
- A symmetric monoidal adjunction:

$$F \circ G = ! - \text{the lift comonad}$$

Remark

An LNL model is a model of Intuitionistic Linear Logic.

Nick Benton. A mixed linear and non-linear logic: Proofs, terms and models. CSL'94
Concrete models of Proto-Quipper-M

The original Proto-Quipper-M model is given by the LNL model:

$$\text{Set} \quad \perp \quad \text{Fam}[\overline{M}] \quad \perp \quad \text{Fam}[\overline{M}](I, -)$$

$\overline{M}$ – closed, product complete category containing given SMC $M$

- $\text{Fam}[\overline{M}] = \{(X, A) \mid X \text{ discrete category}, A : X \to \overline{M} \text{ functor}\}$.
- $(f, \phi) \in \text{Fam}[\overline{M}][(X, A), (Y, B)]$ if $f : X \to Y$ functor and $\phi : A \to B \circ f$ natural transformation.
- $(g, \psi) \circ (f, \phi) = (g \circ f, \psi f \circ \phi)$.

Theorem (Rios & Selinger)

The Families categorical model of Proto-Quipper-M is type-safe, sound, and computationally adequate.
Concrete models of Proto-Quipper-M

The original Proto-Quipper-M model is given by the LNL model:

\[
\text{Set} \xrightarrow{- \circ I} \text{Fam}[M] \xleftarrow{\perp} \text{Fam}[M](I, -)
\]

Sam Staton asked why the \text{Fam} construction is needed – it’s not:

A simpler model for Proto-Quipper-M satisfying the same properties is given by:

\[
\text{Set} \xrightarrow{- \circ I} \overline{M} \xleftarrow{\perp} \overline{M}(I, -)
\]

where in both cases \(\overline{M} = [M^{\text{op}}, \text{Set}]\).
Our Work: Adding Recursion

- Rename the language to *ECLNL*
  - Emphasizes *Enrichment, Combined typing judgement* and *LNL models*.
  - Doesn't tie the language to quantum programming *per se*.

- Describe an *abstract* categorical model for the same language.

- Extend language and abstract categorical model to support recursion.

- Prove soundness for abstract models, and computational adequacy for *concrete model*.

**Related work:** Rennela and Staton describe a different circuit description language, called EWire (based on QWire), for which they also use enriched category theory.
An abstract model for ECLNL

An *ECLNL model* is given by the following data:

1. A cartesian closed category $\mathcal{V}$ together with its self-enrichment $\mathcal{V}$ having finite $\mathcal{V}$-coproducts.

2. A $\mathcal{V}$-symmetric monoidal closed category $\mathcal{C}$ having finite $\mathcal{V}$-coproducts.

3. A $\mathcal{V}$-symmetric monoidal adjunction:

$$
\begin{array}{c}
\mathcal{V} \\
\downarrow \\
\mathcal{C},
\end{array}
\begin{array}{c}
\mathcal{V} \\
\uparrow \\
\mathcal{C}(I,-)
\end{array}
- \otimes I
$$

where $(- \otimes I)$ denotes the $\mathcal{V}$-copower of the tensor unit in $\mathcal{C}$.

4. A symmetric monoidal category $\mathcal{M}$ and a strong symmetric monoidal functor $E : \mathcal{M} \to \mathcal{C}$, the underlying category of $\mathcal{C}$.

**Theorem:** Absent condition 4, an LNL model canonically induces an ECLNL model.\(^2\)

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Soundness

Theorem (Soundness)
Every abstract model of ECLNL is computationally sound.
Concrete models of the base language

Fix an arbitrary symmetric monoidal category $\mathcal{M}$. Equipping $\mathcal{M}$ with the free $\text{DCPO}$-enrichment yields a concrete (order-enriched) ECLNL model:

$$\mathcal{M}(I, -)$$

where $\mathcal{M} = [\mathcal{M}^{\text{op}}, \text{DCPO}]$. 
Abstract models with recursion

Definition
An endofunctor $T : C \rightarrow C$ is \textit{parametrically algebraically compact}, if for every $A \in \text{Ob}(C)$, the endofunctor $A \otimes T(\_)$ has an initial algebra and a final coalgebra whose carriers coincide.

Theorem
A categorical model of a linear/non-linear lambda calculus extended with recursion is given by an LNL model:

\begin{center}
\begin{tikzpicture}
  \node (V) at (0,0) {$V$};
  \node (C) at (2,0) {$C$};
  \node (F) at (1,1) {$F$};
  \node (G) at (1,-1) {$G$};
  \draw[->] (V) to (F);
  \draw[->] (F) to (C);
  \draw[->] (C) to (G);
  \draw[->] (G) to (V);
\end{tikzpicture}
\end{center}

where $FG$ (or equivalently $GF$) is \textit{parametrically algebraically compact} \textsuperscript{3}.

\textsuperscript{3}Benton & Wadler. \textit{Linear logic, monads and the lambda calculus}. LiCS’96.
Definition
A categorical model of ECLNL extended with general recursion is given by a model of ECLNL, where in addition:

5. The comonad endofunctor:

\[ \mathcal{V} \xrightarrow{- \odot I} \perp \xleftarrow{\mathcal{C}(I,-)} \mathcal{C} \]

is parametrically algebraically compact.
Recursion

Extend the syntax:

\[
\frac{\Phi, x : !A; \emptyset \vdash m : A}{\Phi; \emptyset \vdash \text{rec } x^{A} m : A} \quad (\text{rec})
\]

Extend the operational semantics:

\[
(C, m[\text{lift } \text{rec } x^{A} m/x]) \Downarrow (C', v) \\
(C, \text{rec } x^{A} m) \Downarrow (C', v)
\]
Soundness

Theorem (Soundess)

Every model of ECLNL extended with recursion is computationally sound.
Concrete model of ECLNL extended with recursion

Let $M_*$ be the free $\text{DCPO}_{\perp!}$-enrichment of $M$ and $\overline{M_*} = [M_*^{\text{op}}, \text{DCPO}_{\perp!}]$ be the associated enriched functor category.

Remark

If $M = 1$, then the above model degenerates to the left vertical adjunction, which is a model of a LNL lambda calculus with general recursion.
Computational adequacy

Theorem

The following LNL model:

\[
\text{DCPO} \vdash_{\bot} \text{DCPO}_{\bot!} \quad U
\]

is computationally adequate at intuitionistic types for the circuit-free fragment of ECLNL.

- Use logical relations for proof.
- Problem with adding circuits is that structural induction over logical relations breaks down on tensors from \( \mathcal{M} \).
- Need more assumptions about \( \mathcal{M} \) for "traditional" approach to work.
1. Inductive / recursive types.
   - We can support inductive types, since both $\mathcal{C}$ and $\mathcal{V}$ are algebraically complete for endofunctors preserving $\omega$-colimits.
   - $\mathcal{C}$ is algebraically compact for endofunctors preserving $\omega$-colimits, but $\mathcal{V}$ is not.
   - Problem is identifying which parametrically algebraic compact bifunctors $T: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$ are intuitionistic. We believe we have solved this.
     Note: $e\text{-}p$ pairs arise here!

2. Dependent types (Fam/CFam constructions are well-behaved w.r.t. current models).

3. Dynamic lifting.
Conclusions

• One can construct a model of ECLNL by categorically enriching certain denotational models.

• We described a sound abstract model for ECLNL (with general recursion).

• Systematic construction for concrete models that works for any circuit (string diagram) model described by a symmetric monoidal category.

• The "domain theory" is at the most general level – DCPO, DCPO⊥,↑.
Thanks for your attention!

And

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Operational semantics

$(S, m)$ is a configuration if $S$ is a (partially completed) labeled circuit, and $m$ is a term.

\[
\begin{align*}
(S, m) \downarrow (S', v) & \quad (S', n) \downarrow (S'', v') \\
(S, \langle m, n \rangle) \downarrow (S'', \langle v, v' \rangle) & \\
(S, \text{lift } m) \downarrow (S, \text{lift } m) & \\
(S, m) \downarrow (S', \text{lift } m') & \quad (S', m') \downarrow (S'', v) & \quad (S, \text{let } \langle x, y \rangle = m \text{ in } n) \downarrow (S'', w) \\
(S, \text{force } m) \downarrow (S'', v) & \\
(S, m) \downarrow (S', \text{lift } n) & \quad \text{freshlabels}(T) = (Q, \vec{\ell}) & \quad (\text{id}_Q, n\vec{\ell}) \downarrow (D, \vec{\ell}') \\
(S, \text{box}_T m) \downarrow (S', (\vec{\ell}, D, \vec{\ell}')) & \\
(S, m) \downarrow (S', (\vec{\ell}, D, \vec{\ell}')) & \quad (S', n) \downarrow (S'', \vec{k}) & \quad \text{append}(S'', \vec{k}, \vec{\ell}, D, \vec{\ell}') = (S''', \vec{k}') \\
(S, \text{apply}(m, n)) \downarrow (S''', \vec{k}') & \\
(S, m) \downarrow (S', (\vec{\ell}, D, \vec{\ell}')) & \quad (S', n) \downarrow (S'', \vec{k}) & \quad \text{append}(S'', \vec{k}, \vec{\ell}, D, \vec{\ell}') \text{ undefined} \\
(S, \text{apply}(m, n)) \downarrow \text{Error} & \\
(S, (\vec{\ell}, D, \vec{\ell}')) \downarrow (S, (\vec{\ell}, D, \vec{\ell}'))
\end{align*}
\]
Recursion (contd.)

Extend the denotational semantics: \( [\Phi; \emptyset \vdash \text{rec } x^!A \ m : A] := \sigma[m] \circ \gamma[\Phi] \).

\[
\begin{array}{c}
[\Phi] \otimes ! [\Phi] \xleftarrow{\text{id} \otimes \text{lift}} [\Phi] \otimes [\Phi] \xrightarrow{\Delta} [\Phi] \\
\downarrow \text{id} \otimes ! \gamma[\Phi] \quad \quad \quad \quad \downarrow \gamma[\Phi] \\
[\Phi] \otimes ! \Omega[\Phi] \xleftarrow{\omega_{[\Phi]}^{-1}} \Omega[\Phi] \\
\downarrow \text{id} \\
[\Phi] \otimes ! \Omega[\Phi] \xrightarrow{\omega_{[\Phi]}} \Omega[\Phi] \\
\downarrow \text{id} \\
[\Phi] \otimes ! [A] \xrightarrow{\sigma[m]} [A] \\
\end{array}
\]