LINEAR FS-LATTICES AND THEIR CHARACTERIZATION VIA FUNCTION SPACES

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ABSTRACT

Linear FS-lattices are special linked and bicontinuous lattices. Given a continuous lattice A, let $A{\multimap}A$ be the space of all maps $f\colon A \to A$ preserving suprema and $[A \to A]$ the space of maps preserving directed suprema where the order is defined pointwise. Then the inclusion $I_A\colon (A{\multimap}A) \to [A \to A]$ preserves suprema and has thus an upper adjoint $P_A\colon [A \to A] \to (A{\multimap}A)$. We show that A is a linear FS-lattice if and only if the map P_A preserves directed suprema. Furthermore, a complete lattice B is completely distributive if and only if it is a distributive linear FS-lattice; this is equivalent to the map P_B preserving suprema.

1 Linear FS-lattices

Continuous lattices [3] are complete lattices with some inherent form of approximation which makes them important objects of study in pure mathematics and theoretical computer science alike. We say that x is *way-below* y in a complete lattice L if and only if for all directed sets $D \subseteq L$ with $y \leq \vee D$ we have $x \leq d$ for some $d \in D$; we denote this by $x \ll y$. A complete lattices L is *continuous* if and only if every $y \in L$ is the supremum of elements $x \ll y$.

Such lattices provided the first mathematical model of the untyped lambda-calculus [17] and were the conceptual point of departure for the development of *domain theory* [1], a rich an subtle mathematical foundation for denotational semantics [13]. Let us just cite two results in pure mathematics where continuous lattices play a crucial role. First, endowed with their Scott-topology they are recognized as the injective T_0-spaces [3, Chapter II, Theorem 3.8]. Second, the distributive continuous lattices are the Stone duals of the locally compact sober spaces [6, 7]. But continuous lattices with their Scott-topology *are* such locally compact sober spaces and their Stone duals in turn are known to be the completely distributive lattices [4, 5]. Recall the a complete lattice L is completely distributive [14] if and only if for all families $(A_i)_{i\in I}$ of subsets of L we have

$$\bigwedge_{i\in I}\bigvee A_i = \bigvee_{f\in\prod_{i\in I}A_i}\ \bigwedge_{i\in I} f(i)$$

Their are two well-known characterizations of completely distributive lattices needed in this paper.

Theorem 1 *[3, Chapter I, Theorem 3.15] A complete lattice L is completely distributive if and only if L is distributive and L and L^{op} are continuous lattices.*

The other characterization of competely distributive lattices goes back to [15]; we define $x \lll y$ in the same way as $x \ll y$ only that we now allow to consider *all* subsets $D \subseteq L$ with $y \leq \vee D$: i.e., $x \lll y$ if and only if for all $D \subseteq L$ with $y \leq \vee D$ we have $x \leq d$ for some $d \in D$. In that case we say that x is way-way-below y. Call a complete lattice *prime-continuous* if and only if every element $y \in L$ is the supremum of $x \lll y$.

Theorem 2 *[1, 15, Theorem 7.1.3] A complete lattice is prime-continuous if and only if it is completely distributive.*

If we think of complete lattices as complete sup-semilattices then the homomorphisms are maps $f\colon L \to M$ which preserve suprema: $f(\vee X) = \vee f(X)$ for all $X \subseteq L$. Let $L\multimap M$ denote the function space of all such maps where the order is defined pointwise: $f \leq g$ in $L\multimap M$ if and only if $f(x) \leq g(x)$ for all $x \in L$. One readily notes that $L\multimap M$ is a complete lattice with

$$\bigvee \mathcal{F}(x) = \vee\{f(x)\colon f \in \mathcal{F}\}$$

for all $\mathcal{F} \subseteq L\multimap M$.

Alternatively, we may think of complete lattices as dcpos [1], partial orders with least element 0 such that all directed subsets have a supremum. Then the natural homomorphisms are maps $f\colon L \to M$ preserving directed suprema: $f(\vee D) = \vee f(D)$ for all directed sets $D \subseteq L$. Let $[L \to M]$ be the corresponding function space in the pointwise order. Again, this is a complete lattice where the supremum in $[L \to M]$ is evaluated pointwise. Evidently, $L\multimap M$ is a subset of $[L \to M]$ but the inclusion map preserves suprema since such suprema are calculated pointwise in either function space. The inclusion

$$I_{LM}\colon (L\multimap M) \to [L \to M]$$

therefore has an upper adjoint [3, Chapter 0, Corollary 3.5]

$$P_{LM}\colon [L \to M] \to (L\multimap M)$$

which renders for $f \in [L \to M]$ the greatest map preserving suprema below f. We write I_L and P_L if $L = M$.

A first step towards motivating and defining linear FS-lattices is to describe the continuity of a complete lattice L by appealing to higher-order notions: instead of realizing continuity via $\ll$ at the element level of L we only refer to the existence of certain functions in $[L \to L]$ which have the identity function as supremum.

For that we need to point out that $\ll$ satisfies the interpolation axiom [3] in a continuous lattice L: $x \ll y$ implies $x \ll z \ll y$ for some $z \in L$. This is *the* instrumental property in establishing a structure theory of continuous lattices.

Definition 1 *A self-map $f: L \to L$ of a complete lattice L is a* deflation *if and only if $f(x) \leq x$ for all $x \in L$ and the image of f is finite.*

Deflations $f \in [L \to L]$ give rise to elements of $\ll$:

Lemma 1 *Let $f \in [L \to L]$ be a deflation. Then $f(x) \ll x$ for all $x \in L$.*

Proof. Let $D \subseteq L$ be directed with $x \leq \vee D$. Then $f(x) \leq \vee f(D)$ as f preserves directed suprema. Since the image of f is finite and $f(D)$ directed there exists some $d^* \in D$ with $\vee f(D) = f(d^*)$ and thus $f(x) \leq f(d^*) \leq d^* \in D$. □

Proposition 1 *Let L be a complete lattice. Then L is continuous if and only if there exists a directed set of deflations $\mathcal{D} \subseteq [L \to L]$ with $\bigvee \mathcal{D} = \mathrm{id}_L$. In that case we have $x \ll y$ in L if and only if $x \leq f(y)$ for some $f \in \mathcal{D}$.*

Proof.

1. First, let $\mathcal{D}$ be such a set and $x \in L$. Then $f(x) \ll x$ for all $f \in \mathcal{F}$ by Lemma 1 and the supremum of $\{f(x): f \in \mathcal{F}\}$ equals x; thus L is continuous.
   Second, assume that L is continuous. For each finite subset $F \subseteq L$ define $d_F: L \to L$ as
   $$d_F(x) = \vee\{y \in F: y \ll x\}$$
   Clearly, the image of d_F is finite as it is contained in the sup-semilattice generated by F. One readily checks that $d_F \in [L \to L]$ using the interpolation property of $\ll$. Moreover, $d_F \leq \mathrm{id}_L$ shows that d_F is a deflation. Then $\{d_F: F \subseteq L \text{ finite}\}$ is a directed set of deflations in $[L \to L]$ and its supremum equals id_L, for L is continuous.
2. First, $x \ll y$ and $y = \vee\{f(y): f \in \mathcal{D}\}$ imply $x \leq f(y)$ for some $f \in \mathcal{D}$. Second, if $x \leq f(y)$ is the case for some $f \in \mathcal{D}$ then $f(y) \ll y$ implies $x \ll y$. □

Note that the pointwise supremum of two deflations in $[L \to L]$ is again a deflation. Thus we can also state that L is continuous if and only if id_L is the (directed) supremum of *all* deflations in $[L \to L]$. This results suggests to *define* linear FS-lattices L by demanding the existence of some directed family $\mathcal{D}$ in

$[L \to L]$ whose supremum equals id_L. The odd feature of the actual definition is that we *strengthen* it with respect to continuity by confining the family $\mathcal{D}$ to the smaller set $L \multimap L$ and by simultaneously *weakening* the notion of deflation. This weaker notion of finitely separated functions is due to A. Jung [12].

Definition 2 *Let $f, g\colon L \to M$ be two maps between complete lattices L and M. We say that f is* finitely separated *from g if and only if there exists some finite set $M \subseteq L$ such that for all $x \in L$ there is some $m_x \in M$ with $f(x) \leq m_x \leq g(x)$.*

We will primarily be interested in functions $f \in L \multimap L$ which are finitely separated from id_L. Clearly, any deflation on L is finitely separated from id_L by its image. The property of being finitely separated from id_L generalizes the notion of a deflation by weakening the finiteness of the image to some more general and abstract compactness condition.

Definition 3 *[11, 10] A complete lattice L is a linear FS-lattice if and only if there exists a directed set $\mathcal{D} \subseteq L \multimap L$ with $\vee\mathcal{D} = \mathrm{id}_L$ such that every $f \in \mathcal{D}$ is finitely separated from id_L.*

In the definition above, if L is just a dcpo and $\mathcal{D}$ a subset of $[L \to L]$ then this is the definition of FS-domains in [12]. One can show that linear FS-lattices are special linked, bicontinuous lattices [11, 10]. Viewed as complete sup-semilattices they form a $*$-autonomous subcategory of the $*$-autonomous category of complete sup-semilattices [2]; in fact, it is the greatest such $*$-autonomous category of linked continuous lattices with that given internal hom functor $L \multimap M$ [11, 10].

The algebraic linear FS-lattices are such that we can demand all $f \in \mathcal{D}$ to be *idempotent deflations*; further, their sup-retracts are linear FS-lattices where we can demand all $f \in \mathcal{D}$ to be *deflations*, not necessarily idempotent [11, 10]. Algebraic linear FS-lattices have already been studied as *profinite lattices* [**?**]. They have been interpreted at the element level via *interaction orders* [8] and they are the greatest class of algebraic lattices closed under the operation $L \mapsto (L \multimap L)$ [9].

In this paper we will show the following results:

- A continuous lattice L is a linear FS-lattice if and only if the map P_L preserves directed suprema;
- a complete lattice L is completely distributive if and only if it is a distributive linear FS-lattice;
- and a continuous lattice A is completely distributive if and only if the map P_A preserves suprema.

2 A characterization of linear FS-lattices

Let g be finitely separated from h by the finite set M. If $f \leq g$ then clearly f is finitely separated from h by M. This is all we need in showing one half of the characterization of linear FS-lattices.

Proposition 2 *Let L be a continuous lattice such that P_L preserves directed suprema. Then L is a linear FS-lattice.*

Proof. Since L is continuous we have a directed set $\mathcal{D} \subseteq [L \to L]$ of deflations such that id_L is the supremum of $\mathcal{D}$ (Proposition 1). Since P_L preserves directed suprema we have that $\mathrm{id}_L = P_L(\mathrm{id}_L)$ equals the supremum of the directed set $\{P_L(f): f \in \mathcal{D}\} \subseteq L \multimap L$. We are done if each $P_L(f)$ is finitely separated from id_L. But this is clear, for $P_L(f) \leq f$ and f, being a deflation, is finitely separated from id_L. □

The proof of the other implication is quite hard and involves an astonishingly subtle argument. But first we need to generalize Lemma 1 to the case of functions finitely separated from id_L.

Lemma 2 *Let L be a complete lattice such that $f \in [L \to L]$ is finitely separated from id_L. Then $f(y) \ll y$ for all $y \in L$; moreover, $x \leq f(y)$ implies $x \ll y$ in L.*

Proof. The second claim is immediate since $x \leq f(y) \ll y$ implies $x \ll y$. Let $\{a_i: i \in I\} \subseteq L$ be directed with $y \leq \vee\{a_i: i \in I\}$. Then $f(y) \leq \{f(a_i): i \in I\}$ as f preserves directed suprema. Let M be a finite set separating f from id_A. For $i \in I$ there exists some $m_i \in M$ with $f(a_i) \leq m_i \leq a_i$. The set $F = \{m_i: i \in I\}$ is finite; let J be a finite subset of I such that $F = \{m_j: j \in J\}$. As the family $\{a_i: i \in I\}$ is directed we have an upper bound a_k of $\{a_j: j \in J\}$. Then $f(y) \leq \vee_{j \in J} m_j \leq \vee_{j \in J} a_j \leq a_k$. □

Corollary 1 *Every linear FS-lattice is a continuous lattice.*

Proposition 3 *[11, 10] Let L and M be linear FS-lattice. Then $L \multimap M$ is a linear FS-lattices.*

Proof. Let $\mathcal{D} \subseteq L \multimap L$ and $\mathcal{E} \subseteq M \multimap M$ be directed sets with $\vee\mathcal{D} = \mathrm{id}_L$ and $\vee\mathcal{E} = \mathrm{id}_M$ such that all $f \in \mathcal{D}$ and $g \in \mathcal{E}$ are finitely separated from the respective identities. If M_f, respectively M_g, is a finite set separating $f \in \mathcal{D}$ from id_L, respectively $g \in \mathcal{E}$ from id_M, then we are done if $(f \multimap g)^2 \in (L \multimap M) \multimap (L \multimap M)$ is separated from $\mathrm{id}_{L \multimap M}$ by some finite set, where

$$f \multimap g(h) = g \circ h \circ f;$$

simply note that $\mathrm{id}_{L\multimap M}$ is the directed supremum of the set

$$\{(f\multimap g)^2 \colon f \in \mathcal{D},\ g \in \mathcal{E}\}$$

as composition preserves directed suprema.

We define an equivalence relation $\sim$ on $L\multimap M$ by

$$h_1 \sim h_2 \ \Leftrightarrow\ \forall m \in M_f\ \uparrow(g(h_1(m))) \cap M_g = \uparrow(g(h_2(m))) \cap M_g.$$

As M_f and M_g are finite, there are only finitely many equivalence classes on $L\multimap M$. Let M be an non-redundant and complete set of representatives of these classes. We claim that the finite set $f\multimap g(M)$ separates $(f\multimap g)^2$ from $\mathrm{id}_{L\multimap M}$. Given $h \in L\multimap M$, let $\bar{h}$ be the corresponding representative in M. For $a \in L$, we compute

$$\begin{array}{rcll} h(a) & \geq & h(m_f) & ;\text{ for some } m_f \in M_f \text{ with } f(a) \leq m_f \leq a \\ & \geq & m_g & ;\text{ for some } m_g \in M_g \text{ with } g(h(m_f)) \leq m_g \leq h(m_f) \\ & \geq & g(\bar{h}(m_f)) & ;\text{ as } g(h(m_f)) \leq m_g \text{ and } h \sim \bar{h} \\ & \geq & g(\bar{h}(f(a))) & ;\text{ as } f(x) \leq m_f. \end{array}$$

By symmetry, we obtain $\bar{h} \geq (f\multimap g)(h)$, so $h \geq f\multimap g(\bar{h}) \geq (f\multimap g)^2(h)$. □

This argument is due to A. Jung in [12] were it is used in the function space $[L \to M]$; we merely adapted it to cater for the space $L\multimap M$. This result implies that linear FS-lattices are a class of continuous lattices closed under the operation $L \mapsto (L\multimap L)$. There is, however, a more naive approach to obtaining such a class.

Lemma 3 *If L and M are continuous lattices then so is $[L \to M]$.*

2. *Let L be a complete lattice and M a continuous lattice equipped with maps $r \colon M \to L$ and $e \colon L \to M$ preserving directed suprema. If $r \circ e = \mathrm{id}_L$ then L is a continuous lattice.*

Proof. We provide succinct proofs by using Proposition 1.

1. We know that $\mathrm{id}_L = \bigvee \mathcal{D}$ and $\mathrm{id}_M = \bigvee \mathcal{E}$ for directed sets of deflations $\mathcal{D} \subseteq [L \to L]$ and $\mathcal{E} \subseteq [M \to M]$. Then

$$\mathcal{F} = \{[f \to g] \colon f \in \mathcal{D}, g \in \mathcal{E}\} \subseteq [[L \to M] \to [L \to M]]$$

is directed and its supremum equals $[\mathrm{id}_L \to \mathrm{id}_M] = \mathrm{id}_{[L\to M]}$ since composition preserves directed suprema. Thus $[L \to M]$ is continuous by Proposition 1.

2. Given $\mathcal{E}$ as in (1), the set $\{r \circ g \circ e \colon g \in \mathcal{E}\}$ is directed and has $r \circ \mathrm{id}_M \circ e = \mathrm{id}_L$ as supremum, for composition preserves directed suprema.

□

Given a continuous lattice L we therefore have that $[L \to L]$ is a continuous lattice as well. Since I_L is a lower adjoint of P_L it preserves suprema. Assuming that P_L preserves directed suprema we then infer that $L \multimap L$ is a continuous lattice by Lemma 3(2). In order to show that the class of continuous lattices L where P_L preserves directed suprema is closed under $L \mapsto (L \multimap L)$ we now need to verify that P_M preserves directed suprema for $M = L \multimap L$. This is far from obvious. The results presented in this paper show that it is indeed true and that this class of continuous lattices surprisingly coincides with the class of linear FS-lattices.

In showing the converse of Proposition 2 we show that P_L preserves directed suprema by proving that I_L preserves the way-below relation; we will see shortly that this is indeed a sound strategy.

Definition 4 *Let $f\colon L \to M$ be a function between two complete lattices. We say that f preserves the way-below relation if and only if $x \ll y$ in L implies $f(x) \ll f(y)$ in M.*

We cite the relevant parts of [1, Proposition 3.1.14] for the special case of continuous lattices.

Proposition 4 *Let L and M be complete lattices and $f\colon L \to M$ a lower adjoint of $g\colon M \to L$.*

1. *If the function g preserves directed suprema then f preserves the way-below relation.*
2. *If L is a continuous lattice then the converse of (1) is true as well.*

Finally, we have accumulated all the necessary concepts and facts for characterizing linear FS-lattices via the function spaces $\multimap$ and $[\ \to\ ]$.

Theorem 3 *Let L be a continuous lattice. Then L is a linear FS-lattice if and only if the map P_L preserves directed suprema.*

Proof. By Proposition 2 it suffices to show that P_L preserves directed suprema if L is a linear FS-lattice. In that case, Proposition 3 and Corollary 1 ensure that $L \multimap L$ is a continuous lattice. Using Proposition 4(2), it thus suffices to prove that I_L preserves the way-below relation. So let $f \ll g$ in $L \multimap L$. We need to show $f \ll g$ in $[\mathrm{L} \to L]$. By Lemma 2 we are done if we have $f \leq \Psi(g)$ for some $\Psi \in [[L \to L] \to [L \to L]]$ such that Ψ is finitely separated from id_L.

Since L is a linear FS-lattice we have a directed family of functions $\mathcal{D} \subseteq L \multimap L$ with supremum id_L such that each $h \in \mathcal{D}$ is separated from id_L by some finite set $M_h \subseteq L$. Let $h \multimap h$ be the element of $(L \multimap L) \multimap (L \multimap L)$ which sends each i to the map $h \circ i \circ h$. Clearly, $\mathrm{id}_{L \multimap L}$ is the supremum of $\{(h \multimap h)^2 \colon h \in \mathcal{D}\}$

as composition preserves directed suprema. Since $f \ll g$ in $L\multimap L$ and $g = \vee\{(h\multimap h)^2(g)\colon h \in \mathcal{D}\}$, we have $f \leq (h\multimap h)^2(g)$ for some $h \in \mathcal{D}$.

The subtle point is now this: since h is also in $[L \to L]$ we may define $[h \to h] \in [[L \to L] \to [L \to L]]$ whose action restricted to $L\multimap L$ *equals* the action of $h\multimap h$. Therefore

$$f \leq [h \to h]^2(g)$$

In [12] we then have a proof that $[h \to h]^2$ is finitely separated from $\mathrm{id}_{[L\to L]}$. In fact, this is the same proof as the one in Proposition 3 where we replace all $\multimap$ by $[\ \to\ ]$. □

3 A characterization of distributive linear FS-lattices

Before we strengthen the Theorem above to the case of distributive linear FS-lattices we want to uncover these latter lattices as being precisely the completely distributive ones.

Proposition 5 *A complete lattice L is completely distributive if and only if L is a distributive linear FS-lattice.*

Proof. First, let L be a distributive linear FS-lattice. Clearly $\mathbf{2} = \{0 < 1\}$ is a linear FS-lattice as it is finite. So $A\multimap\mathbf{2}$ is a linear FS-lattice by Proposition 3. But

$$A\multimap\mathbf{2} \cong A^{\mathrm{op}}$$

where the isomorphism is realized by

$$f \mapsto \vee f^{-1}(0_A)$$

Thus A^{op} is continuous by Corollary 1. Using Theorem 1 we infer that L is completely distributive.

Second, let L be completely distributive. Clearly, L is then distributive. For each finite subset $F \subseteq L$ define $e_F\colon L \to L$ by

$$e_F(x) = \vee\{y \in F\colon y \lll x\}$$

We reason as for the maps d_F to conclude that all maps e_F are deflations and that the family $\{e_F\colon F \subseteq L \text{ finite}\}$ is directed. The proof that $\ll$ satisfies the interpolation property can be successfully tranferred to $\lll$ whenever L is completely distributive [14]. Thus all maps e_F are in $L\multimap L$ and their supremum equals id_L by Theorem 2. Hence L is a linear FS-lattice. □

Proposition 6 *Let L be a completely distributive lattice. Then P_L preserves suprema and*

$$P_A(f)(a) = \vee\{f(w)\colon w \lll a\}$$

for all $f \in [L \to L]$ and all $a \in L$.

Proof. Let $f \in [L \to L]$ be given. Define $f^d(a) = \vee\{f(w): w \lll a\}$ for all $a \in L$. We claim that $f^d = P_L(f)$. Since L is completely distributive we known that $\lll$ satisfies the interpolation property and that every element in L is the supremum of elements way-way-below it. Clearly, f^d is monotone, so $\vee f^d(X) \leq f^d(\vee X)$ for $X \subseteq L$. Let $w' \lll f^d(\vee X)$. Then $w' \leq f(w)$ for some $w \lll \vee X$ by the definition of f^d. Let w'' be such that $w \lll w'' \lll \vee X$. Then $w'' \leq x$ for some $x \in X$ shows $w \lll x$. Thus $w' \leq f(w) \leq f^d(x) \leq f^d(\vee X)$ and $f^d \in L \multimap L$ has been shown. If $g \leq f$ with $g \in L \multimap L$ then $g \leq f^d$ readily follows as g preserves all suprema and as every element in L is the supremum of elements way-way-below it. This proves $P_L(f) = f^d$.

As every completely distributive lattice is a linear FS-lattice (Proposition 5) we have that P_L preserves directed suprema (Theorem 3). Thus it suffices to show that P_L preserves binary suprema. We compute

$$\begin{aligned}
(f \vee g)^d(a) &= \vee\{(f \vee g)(w): w \lll a\} \\
&= \vee\{f(w) \vee g(w): w \lll a\} \\
&= \vee\{f(w): w \lll a\} \vee (\vee\{g(w): w \lll a\}) \\
&= f^d(a) \vee g^d(a) \\
&= (f^d \vee g^d)(a)
\end{aligned}$$

for all $a \in L$. □

Recall that an element p in a complete lattice L is a $\vee$-prime if and only if for all $x, y \in L$ with $p \leq x \vee y$ we have $p \leq x$ or $p \leq y$.

Corollary 2 *Let L be a completely distributive lattice. Then the $\vee$-primes of $L \multimap L$ are exactly the $\vee$-primes of $[L \to L]$ which are elements of $L \multimap L$.*

Before we go on to prove the converse of Proposition 6, we need to establish a version of Proposition 4 for completely distributive lattice.

Lemma 4 *Let L and M be complete lattices and $f: L \to M$ a lower adjoint of $g: M \to L$.*

1. *If g preserves suprema then f preserves $\lll$.*
2. *If L is completely distributive then the converse of (1) is also true.*

Proof.

1. Let $a \lll b$ in L; we have to show $f(a) \lll f(b)$. Let $X \subseteq M$ such that $f(b) \leq \vee_M X$. We are done if $f(a) \leq x$ for some $x \in X$. Since g preserves all suprema we get $b \leq g(f(b)) \leq g(\vee_M X) = \vee_L g(X)$. Now, $a \lll b$ implies $a \leq g(x)$ for some $x \in X$ and $f(a) \leq f(g(x)) \leq x$ follows.
2. Let $X \subseteq M$ be arbitrary. As g is monotone we have $\vee_M g(X) \leq g(\vee_L X)$. We need to show the reverse inequality and by Theorem ? it suffices to show

$a \le \vee_M g(X)$ for all $a \lll g(\vee_L X)$. So let $a \lll g(\vee_L X)$ be given. Since f preserves $\lll$ we obtain $f(a) \lll f(g(\vee_L X)) \le \vee_L X$ which implies $f(a) \le x$ for some $x \in X$. Therefore, $a \le g(f(a)) \le g(x) \le \vee_M g(X)$.

□

Corollary 3 *Let L be a completely distributive lattice. Then I_L preserves $\lll$.*

Next we want to demonstrate that a continuous lattice L is completely distributive if P_L preserves suprema. For that we need to gain a better understanding of step functions, certain maps preserving directed suprema, and of their images under P_L.

Definition 5 *Let L be a complete lattice and $x, y, z \in L$. Define $x \searrow y \colon L \to L$ [1] to be the function with maps $\Uparrow(x)$ to y and $L \setminus \Uparrow(x)$ to 0_L, where*

$$\Uparrow(x) = \{y \in L \colon x \ll y\} \; [1,\; 3]$$

Further, let $z \nearrow y \colon L \to L$ be the function with maps $L \setminus \downarrow(z)$ to y and $\downarrow(z)$ to 0_L.

Lemma 5 *Let L be a continuous lattice and $x, y, z \in L$.*

1. *The map $x \searrow y \in L$ preserves directed suprema and is a deflation;*
2. *and $P_L(x \searrow y) = z \nearrow y$, where $z = \vee(L \setminus \Uparrow(x))$.*

Proof.

1. This is immediate as $\Uparrow(x)$ is an upper set inaccessible by directed suprema if L is continuous.
2. We first show $z \nearrow y \le x \searrow y$: If $a \in \Uparrow(x)$ then $(z \nearrow y)(a) \le y = (x \searrow y)(a)$. If $a \in L \setminus \Uparrow(x)$ then $a \le z$ implies $(z \nearrow y)(a) = 0_L \le (x \searrow y)(a)$. Now, let $g \in L \multimap L$ be such that $g \le x \searrow y$. We are done if $g \le z \nearrow y$. Since $g \le x \searrow y$ we have $L \setminus \Uparrow(x) = (x \searrow y)^{-1}(0_L) \subseteq g^{-1}(0_L)$. As g preserves suprema we conclude that $z = \vee(x \searrow y)^{-1}(0_L) \in g^{-1}(0_L)$. Therefore, $g(a) = 0_L \le (z \nearrow y)(a)$ for all $a \le z$. If $a \not\le z$ then $g(a) \le (x \searrow y)(a) \le y = (z \nearrow y)(a)$.

□

Theorem 4 *Let L be a continuous lattice. Then P_L preserves suprema if and only if L is completely distributive.*

Proof. By Proposition 6 it remains to show that L is completely distributive if P_L preserves suprema. Assuming the latter we utilize step functions: since L is continuous we know that

$$\mathrm{id}_L = \bigvee\{x \searrow y \colon y \ll x \text{ in } L\}. \; [3, \mathit{Chapter\ II}, \mathit{Exercise}\ 2.16(\mathit{iii})]$$

Since P_L is assumed to preserve suprema, we obtain

$$\begin{aligned}\mathrm{id}_L &= P_L(\mathrm{id}_L)\\ &= P_L(\bigvee\{x \searrow y : y \ll x \text{ in } L\})\\ &= \bigvee\{P_L(x \searrow y) : y \ll x \text{ in } L\}\\ &= \bigvee\{z \nearrow y : y \ll x,\ z = \vee(L \setminus \Uparrow(x))\}\end{aligned}$$

Each of the functions $z \nearrow y$ has at most two points in its image, so the image is certainly completely distributive. Therefore, each of these functions is a *tight* Galois Connection on L [16]. The supremum of tight Galois Connections is tight, for every Galois Connection has a least tight Galois Connection below it. Thus id_L is tight as well. By [16] this is the case if and only if L is completely distributive. □

Hence we can make $\bigvee\{P_L(x \searrow y) : y \ll x \text{ in } L\}$ a precise measure of complete distributivity.

Corollary 4 *Let L be a continuous lattice. Then the following are equivalent:*

1. *L is completely distributive.*
2. $\mathrm{id}_L = \bigvee\{P_L(x \searrow y) : y \ll x \text{ in } L\}$.

Furthermore, we always have the formula

$$\bigvee_{a \not\leq u} \bigwedge_{t \not\leq u} t = \bigvee\{P_L(x \searrow y) : y \ll x \text{ in } L\}$$

for all elements in a continuous lattice L.

Proof. The comments on the tightness of id_L above show that (2) implies (1). Let L be completely distributive. Then P_L preserves suprema and (2) is immediate. As L is completely distributive if and only if id_L is tight, and as $\bigvee\{P_L(x \searrow y) : y \ll x \text{ in } L\}$ is a tight Galois Connection below id_L satisfying the above equivalence, we infer that $\bigvee\{P_L(x \searrow y) : y \ll x \text{ in } L\}$ equals the greatest tight Galois Connection below id_L which is known to satisfy the above formula [16]. □

4 Open problems

There are essentially two open problems in the theory of linear FS-lattices. First, we know that a linear FS-lattice L is algebraic if and only if there exists some directed set $\mathcal{D} \subseteq L \multimap L$ of idempotent deflations whose supremum equals id_L. The retracts of these lattices are those complete lattices M which have some directed set $\mathcal{E} \subseteq M \multimap M$ of deflations whose supremum equals id_M; in particular, they are linear FS-lattices and linear FS-lattices are closed under sup-retracts (similar proof as in Lemma 3(2)). Of course, we would like to know whether every linear FS-lattice is the retract of some algebraic linear FS-lattice.

Question 1 *Is every linear FS-lattice the sup-retract of some algebraic linear FS-lattice?*

Note that such a statement holds for *continuous* lattices, for the ideal completion of a continuous lattice is an algebraic lattice. Moreover, this statement holds in the world of *distributive* linear FS-lattices, for we may realize such a lattice as the sup-retract of the completely distributive algebraic lattice of lower sets of its $\vee$-primes. However, it is not the case that the ideal completion of a linear FS-lattice is a linear FS-lattice and we have to come up with a generalization of the retract construction in the distributive case.

The second open problem is about characterizing linear FS-lattices as precisely those complete lattices L such that $L \multimap L$ is continuous. This has been shown for *algebraic* linear FS-lattices in [9]. We have seen that $L \multimap L$ is continuous for any linear FS-lattice. In [11, 10] we showed that L is bicontinuous if $L \multimap L$ is continuous. Moreover, if L is linked then the continuity of $L \multimap L$ forces L to be a linear FS-lattices. The problem is lies therefore in getting rid of the additional assumption of linkedness in proving this.

Question 2 *Let L be a (bicontinuous) lattice such that $L \multimap L$ is a continuous lattice. Is L linked?*

In that case L would indeed be a linear FS-lattice and these would be exactly those continuous lattices which are closed under $\multimap$.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3. Clarendon Press, 1995.

[2] M. Barr. *$*$-Autonomous Categories*, volume 752 of *Lecture Notes in Mathematics*. Springer Verlag, 1979.

[3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices.* Springer Verlag, 1980.

[4] R.-E. Hoffmann. Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactification. In B. Banaschewski and R.-E. Hoffmann, editors, *Continuous Lattices*, volume 871 of *Lecture Notes in Computer Science*, pages 159–208. Springer Verlag, 1981.

[5] R.-E. Hoffmann and K. H. Hofmann, editors. *Continuous Lattices and their Applications*, volume 101 of *Lecture Notes in Pure and Applied Mathematics.* Marcel Dekker Inc., 1987.

[6] K. H. Hofmann and J. Lawson. The spectral theory of distributive continuous lattices. *Transitions of the American Mathematical Society*, 246:285–310, 1978.

[7] K. H. Hofmann and M. Mislove. Local compactness and continuous lattices. In B. Banaschewski and R.-E. Hoffmann, editors, *Continuous Lattices, Proceedings Bremen 1979*, volume 871 of *Lecture Notes in Mathematics*, pages 209–248. Springer Verlag, 1981.

[8] M. Huth. Interaction Orders as Games. In C. Hankin and R. Nagarajan, editors, *Theory and Formal Methods 1994*, Workshops in Computer Science, pages pp.21–40. World Scientific Publishing, 1995. to appear.

[9] M. Huth. The Greatest Symmetric Monoidal Closed Category of Scott-domains. Technical Report 1734, Technische Hochschule Darmstadt, May 1995.

[10] M. Huth, A. Jung, and K. Keimel. Linear types and approximation. *submitted to the Journal of MSCS*, 1994.

[11] M. Huth, A. Jung, and K. Keimel. Linear types, approximation, and topology. In *Logic in Computer Science*, pages 110–114. IEEE Computer Society Press, 1994.

[12] A. Jung. The classification of continuous domains. In *Logic in Computer Science*, pages 35–40. IEEE Computer Society Press, 1990.

[13] P. D. Mosses. Denotational semantics. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, pages 577–632. North Holland, 1990.

[14] G. N. Raney. Completely distributive complete lattices. *Proc. AMS*, 3:667–680, 1952.

[15] G. N. Raney. A subdirect-union representation for completely distributive complete lattices. *Proc. AMS*, 4:518–522, 1953.

[16] G. N. Raney. Tight Galois connections and complete distributivity. *Trans. AMS*, 97:418–426, 1960.

[17] D. S. Scott. Continuous lattices. In E. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer Verlag, 1972.