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Compact Coverages Generate Spectral Frames

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Abstract

This note proves a useful characterization of spectral frames: a frame is spectral if and only if it can be generated from a *compact* coverage relation, where compactness is defined in the usual topological sense.

1 Introduction

The concept of frames (or locales) arises in the study of topology by taking the lattice of open sets as the starting point (this is the so-called pointless topology). Spectral (or coherent) frames form an important subclass of frames due to their property of being *spatial* [3]. Intuitively, they allow *points* to come back into the picture, in the sense that each element of the frame can be considered as a set of points, with the underlying partial order being recovered as set inclusion.

Technically, points are completely prime filters. As such, they have the logical standing of *models*. If one thinks of elements of a frame as propositional formulas, with the underlying partial order interpreted as logical implication, then spectral frames are *complete* when there exist enough models to capture implication: one formula entails another exactly when every model of the first formula is a model of the second.

Johnstone [3] introduces a way to generate a frame from a meet-semi-lattice using the so-called *coverage* relation. The generated frame consists of C-ideals under inclusion. He shows that spectral frames are exactly those which can be generated from a distributive lattice with the standard coverage relation associated with it. In this case C-ideals are precisely ideals and the generated frame corresponds to the ideal completion of the distributive lattice.

This note gives a characterization of spectral frames directly in terms of the coverage relation: *a frame is spectral if and only if it can be generated*

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from a compact coverage relation. Here we call a coverage relation compact if it has the property that when a set covers an element, a finite subset of the set already covers the element.

This result brings out the full advantage of the syntactic nature of the coverage relation: with minimal amount of data (a meet-semi-lattice and a coverage), we can not only generate a frame, but also know exactly when the generated frame is spectral. This is especially useful when compactness of the coverage relation is explicitly given, as in the case of sequent structures and hyperresolution [1].

2 Frames and coverage

This section gives a brief review of the coverage relation to fix notation and provide basic ideas.

A *frame* is a poset with finite meets and arbitrary joins which satisfies the infinite distributive law

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}.$$

For frames F and G, a frame morphism is a function $f : F \to G$ that preserves finite meets and arbitrary joins. Frames are also called *locales*.

Note that any function which preserves finite meets must be monotonic: if $a \leq b$ then

$$f(a) = f(a \wedge b) = f(a) \wedge f(b) \leq f(b).$$

Johnstone ([3], page 57) provides a way to construct a frame from a meet-semi-lattice based on the notion of *coverage relation*.

Definition 2.1 *Let $(S, \wedge, \leq)$ be a meet-semi-lattice. A* coverage *on S is a relation $\succ \subseteq 2^S \times S$ satisfying*

(i) *if $Y \succ a$ then a is an upper bound of Y with respect to $\leq$.*

(ii) *if $Y \succ a$ then for any $b \leq a$, $\{y \wedge b \mid y \in Y\} \succ b$.*

A coverage relation (or coverage) $\succ$ is called compact *if for every $X \subseteq S$ and every $a \in S$,*

$$X \succ a \text{ implies } Y \succ a \text{ for some finite } Y \subseteq^{\text{fin}} X.$$

A $\succ$-ideal determined by coverage $\succ$ is a subset I of S which is

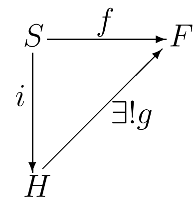
(i) *lower-closed: $a \in I \;\&\; b \leq a \Rightarrow b \in I$,*

(ii) *covered: $U \succ a \;\&\; U \subseteq I \Rightarrow a \in I$.*

A meet-semi-lattice S equipped with a coverage $\succ$ is called a site*. A frame H with $i : S \to H$ is said to be* generated *from a site $(S, \succ)$ if*

- *i preserves finite meets,*
- *i transforms covers to joins: $Y \succ a \Rightarrow i(a) = \bigvee i(Y)$, and*

- *H, i is universal, i.e., for any frame F and any meet-preserving and cover-to-join transforming function $f : S \to F$, there exists a unique frame morphism $g : H \to F$ such that the following diagram commutes:*



□

Note that such a universal property guarantees that the generated frame is unique up to isomorphism.

Note also that the requirement for U to be part of the lower set of a can be dropped in the formula $U \succ a$. This is because the other condition can be used to recover this property: if $U \succ a$, then $\{x \wedge a \mid x \in U\} \succ a \wedge a$.

The concept of coverage has a clear topological interpretation. If the elements of a frame are considered as open sets, then $U \succ a$ says that the collection of open sets U covers the open set a in the standard sense. When this happens, one can of course find a cover of a whose members are subsets of a. Under this interpretation the compactness property says that "basis" open sets in S are assumed to be topologically compact to start with.

Here is Johnstone's fundamental result for the coverage relation.

Theorem 2.2 (Coverage Theorem) *The collection of $\succ$-ideals under inclusion is the frame generated from a site $(S, \succ)$.*

Proofs for this basic theorem can be found in [3] and in [4]. We summarize some of the key ideas used in the proof below, some of which will be used in the next section.

Note that $\succ$-ideals are closed under arbitrary intersections, and S itself is the largest $\succ$-ideal. This property allows us to talk about the $\succ$-ideal $\mathsf{c}U$ generated by an arbitrary set $U \subseteq S$, which is the intersection of all $\succ$-ideals containing U. Also, since $\succ$-ideals are downwards closed, we clearly have $\mathsf{c}U = \mathsf{c}\downarrow U$, where $\downarrow U := \{x \in S \mid (\exists y \in U)\ x \leq y\}$. Sets with the property $U = \downarrow U$ are called *lower sets*.

To show that $\succ$-ideals form a frame, one needs to verify the infinite distributive law. However, this reduces to the fact that the mapping c preserves finite meets on lower sets, i.e.,

Lemma 2.3 *For any lower sets U, V, $\mathsf{c}(U \cap V) = \mathsf{c}U \cap \mathsf{c}V$.*

Once this is proven, then the intersection of two $\succ$-ideals is a $\succ$-ideal, and so

$$\begin{aligned} I \cap \bigvee_i J_i &= \mathsf{c}I \cap \mathsf{c}(\bigcup_i J_i) \\ &= \mathsf{c}(I \cap \bigcup_i J_i) \\ &= \mathsf{c}(\bigcup_i (I \cap J_i)) \\ &= \bigvee_i (I \cap J_i). \end{aligned}$$

where I and J_i are arbitrary $\succ$-ideals.

A crucial "quotient" construction is used to prove Lemma 2.3. For $K, U \subseteq S$, the quotient K/U is defined to be the set $\{x \in S \mid (\forall y \in U) x \wedge y \in K\}$. Quotients have several important properties:

(i) If K is a $\succ$-ideal then K/U is a $\succ$-ideal.

(ii) For any lowersets U, V, $U \cap V \subseteq K$ iff $U \subseteq K/V$ iff $V \subseteq K/U$.

(iii) For any lowersets U, V, we have $(U/V) \cap V \subseteq U$ and $V \subseteq U/(U/V)$.

To show $\mathsf{c}(U \cap V) \supseteq \mathsf{c}U \cap \mathsf{c}V$, let I be a $\succ$-ideal such that $U \cap V \subseteq I$. Since both I/U and $I/(I/U)$ are $\succ$-ideals, we have

$$\begin{aligned} &\mathsf{c}U \cap \mathsf{c}V \\ &\subseteq (I/(I/U)) \cap (I/U) \\ &\subseteq I. \end{aligned}$$

Now let $I = \mathsf{c}(U \cap V)$ to get the desired containment.

We end this section by indicating the morphisms used in the defining diagram for the generated frame (Definition 2.1). Let H be $\mathsf{Idl}_\succ(S)$, the collection of $\succ$-ideals of a site $(S, \succ)$, under inclusion. The mapping $i : S \to \mathsf{Idl}_\succ(S)$ is defined as $a \longmapsto \mathsf{c}\{a\}$ for all $a \in S$. For any other such map $f : S \to F$ with F a frame, define $g : \mathsf{Idl}_\succ(S) \to F$ by $I \longmapsto \bigvee f(I)$ for any $\succ$-ideal I.

One last property is used in showing that g is the unique mapping such that $f = g \circ i$: for any map $f : S \to F$ which preserves finite meets and transforms covers to joins, the set $f^{-1}(\downarrow x)$ is a $\succ$-ideal for any $x \in F$.

3 Spectral frames and compact coverages

In this section we prove the main result of this note, followed by a couple of examples.

Recall that a coverage relation $\succ$ over a meet-semi-lattice $(S, \wedge, \leq)$ is called *compact* if for every $X \subseteq S$ and every $a \in S$,

$$X \succ a \Rightarrow Y \succ a \text{ for some } Y \subseteq^{\text{fin}} X.$$

Theorem 3.1 *A frame is spectral if and only if it can be generated from a compact coverage relation.*

By a proposition of Johnstone ([3], page 64), any spectral frame is generated by a compact coverage relation, which is the standard one associated with a distributive lattice: $U \succ a$ if $U \subseteq \downarrow a$ and there exists a finite subset X of U such that $a = \vee X$.

We introduce a couple of lemmas first to prepare for the proof of the other direction.

Lemma 3.2 *If $(S, \succ)$ is a site for which the coverage relation $\succ$ is compact, then for any directed set F of $\succ$-ideals, we have*

$$\bigvee F = \bigcup F.$$

Proof. It suffices to show that $\bigcup F$ is a $\succ$-ideal. It is clearly lower-closed. It is also covered: suppose $X \subseteq \bigcup F$ and $X \succ a$. By the compactness of $\succ$, there is a finite subset Y of X such that $Y \succ a$. However, since $Y \subseteq \bigcup F$ with Y finite and F directed, we know that $Y \subseteq I$ for some $I \in F$. Therefore, $a \in I$ since I is covered. Hence $a \in \bigcup F$.

□

The next lemma characterizes compact $\succ$-ideals.

Lemma 3.3 *Suppose $(S, \succ)$ is a site and $\succ$ is compact. Then a $\succ$-ideal is finite (i.e. a compact element in the generated frame) if and only if it is generated by a finite subset of S.*

Proof. Consider $\mathsf{c}U$ for some finite subset U of S. If $\mathsf{c}U \subseteq \bigvee F$ for some directed set F, then $\mathsf{c}U \subseteq \bigcup F$, by Lemma 3.2. Since U is finite and $U \subseteq \bigcup F$, $U \subseteq I$ for some $I \in F$. Therefore, $\mathsf{c}U \subseteq I$. This shows that a $\succ$-ideal generated from a finite set is a compact element in the generated frame.

Suppose, on the other hand, that K is a compact element in $\mathsf{Idl}_{\succ}(S)$. We have

$$K \subseteq \bigvee \{\mathsf{c}X \mid X \subseteq^{\text{fin}} K\},$$

with the right hand side being a directed set. Therefore, $K = \mathsf{c}X$ for some finite subset X of K, by the compactness of K.

□

PROOF of Theorem 3.1. Let $(S, \succ)$ be a site with $\succ$ compact. We need to show that the generated frame $(\mathsf{Idl}_{\succ}(S), \subseteq)$ is spectral (or coherent).

According to Johnstone ([3], page 63), all we need to show is that

(i) every $\succ$-ideal is expressible as a join of finite elements, and

(ii) the finite elements form a sublattice of $\mathsf{Idl}_{\succ}(S)$, i.e., $\mathsf{c}\{\mathbf{1}\}$ is finite, where $\mathbf{1}$ is the top of S, and the meet of two finite elements is finite.

By Lemma 3.3, it is clear that every $\succ$-ideal is expressible as a (directed) join of finite elements, using the formula

$$I = \bigvee \{\mathsf{c}X \mid X \subseteq^{\text{fin}} I\}.$$

Also by Lemma 3.3, $\mathsf{c}\{\mathbf{1}\}$ is finite.

Now let I, J be finite elements. By Lemma 3.3, there exist finite sets X, Y such that $I = \mathsf{c}X$ and $J = \mathsf{c}Y$. It suffices to show that

$$I \cap J = \mathsf{c}\{x \wedge y \mid x \in X \;\&\; y \in Y\}$$

as $\{x \wedge y \mid x \in X \;\&\; y \in Y\}$ is a finite set. By Lemma 2.3, we have

$$I \cap J = \mathsf{c}X \cap \mathsf{c}Y = \mathsf{c}{\downarrow}X \cap \mathsf{c}{\downarrow}Y = \mathsf{c}({\downarrow}X \cap {\downarrow}Y).$$

Now the desired result follows because we have the equality

$${\downarrow}X \cap {\downarrow}Y = {\downarrow}\{x \wedge y \mid x \in X \;\&\; y \in Y\}.$$

□

For an example of a coverage relation which is not compact, consider the lattice $(\mathcal{P}(\omega), \subseteq)$, the powerset lattice consisting of all subsets of ω. For any $U \subseteq \mathcal{P}(\omega)$ and $x \in \mathcal{P}(\omega)$, write $U \succ x$ if $y \subseteq x$ for all $y \in U$ and $\bigcup U = x$. One can easily check that this is indeed a coverage relation. It is clearly not compact because x can be an infinite set. And indeed, the generated frame is not spectral, because $\mathsf{c}\{\omega\}$ is not finite: we have ${\downarrow}\{\omega\} = \bigvee\{\mathsf{c}X \mid X \subseteq^{\text{fin}} \omega\}$ and yet ${\downarrow}\{\omega\}$ is not a subset of any $\mathsf{c}X$ $({\downarrow}X)$ for finite X. Interestingly, the generated frame seems to be spatial, nevertheless.

Note that we cannot in general claim that the generated frame is not spectral if $\succ$ is not compact. The same frame may be isomorphic to a frame generated by a different, yet compact coverage. For a tighter correspondence, we must use the *weakly compact* property:

$$X \succ a \Rightarrow a \in \mathsf{c}Y \text{ for some } Y \subseteq^{\text{fin}} X.$$

This then will allow us to prove the following, whose proof is similar to the one for Theorem 3.1:

Theorem 3.4 *Let $(S, \succ)$ be a site. Then $(\mathsf{Idl}_{\succ}(S), \subseteq)$ is spectral if and only if $\succ$ satisfies the weakly compact property.*

We end the note with a positive example: the frames generated from the so-called entailment relations [1].

Example. An entailment relation is a set A together with a relation $\vdash$ on the set $\text{Fin}(A)$ of finite subsets of A, satisfying certain properties that should not concern us here. One can introduce a coverage relation over the meet-semi-lattice $(\text{Fin}(A), \cup, \supseteq)$ by the definition

$$\{\{a_1\} \cup X,\ \{a_2\} \cup X, \ldots,\ \{a_n\} \cup X\} \succ X \quad \text{iff} \quad X \vdash a_1, \ldots, a_n.$$

Since such a coverage relation is clearly compact, we know right away that the generated frame is spectral (and hence spatial), by Theorem 3.1. An interesting result reported in [1] is that the frame generated by this compact coverage relation is isomorphic to the collection of disjunctive states, derived independently from the hyper-resolution rule as used in disjunctive logic programs. Since the generated frame is spatial, we obtain a desirable notion of “models” (of disjunctive logic programs) for free, and we also obtain the completeness of hyper-resolution for free.

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