How Do Domains Model Topologies?

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Abstract

In this brief study we explicitly match the properties of spaces modelled by domains with the structure of their models. We claim that each property of the modelled topology is coupled with some construct in the model. Examples are pairs: (i) first-countability - strictly monotone map, (ii) developability - measurement, (iii) metrizability - partial metric, (iv) ultrametrizability - tree, (v) Choquet-completeness - dcpo, and more. By making this correspondence precise and explicit we reveal *how* domains model topologies.

1 Introduction

The idea that properties of certain topological spaces can be studied via an appropriate partially ordered set that "approximates" or "models" the space is present in early works such as Lacombe [29], Martin-Löf [38], Scott [42], and has been developed further in the work of Weihrauch and Schreiber [44] and Kamimura and Tang [24]. Since then, the connection between domain theory and "classical" mathematics has been exploited in a variety of applications including: real number computation [15], integration [17], [6], [10] and differential calculus [13], geometry [12], dynamical systems, fractals and measure theory [7], [8], and basic quantum mechanics [5]. [2]

There is a common pattern in all of the above research: one identifies a topology τ on the objects of interest X (usually it is a metric space), then defines partial approximants of the objects out of the resources available in the space (usually these are certain compact or closed sets) and a partial order

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[2] Most of the applications are surveyed in [9].

P between them. The construction makes the modelled space homeomorphic to the subset of maximal elements of P in the subspace Scott topology: $\langle X, \tau \rangle \cong \langle \max P, \sigma \mid_{\max P} \rangle$. Lastly, having prepared the setup, one studies the phenomena in the modelled space via the domain-theoretic tools available for the model. It comes as no surprise that the fundamental question of which topological spaces have maximal point models focused much attention in the past decade.

Lawson [31], [30] settled this question under certain assumptions (the model is second countable and the Scott and Lawson topologies restricted to the maximal elements of the model coincide[3]) by showing that a topological space has a model iff it is a Polish space. An explicit, elegant construction of such a domain was given by Edalat and Heckmann in [11] - later adapted to the special case of computable Banach spaces in [14] and used to introduce the notion of partial metric to domain theory [23]. Under the same assumptions Flagg and Kopperman [18] characterized complete separable ultrametric spaces as the ones which can be modelled by algebraic domains.

The domain-theoretic approach to Topology made it possible to gain new insight into the nature of *both* the spaces and their models. In a series of papers [34], [35], [32] following [36] Martin proposed a handful of useful techniques for studying topologies and their models, including the notion of measurement and ideal domain. Discarding Lawson's condition, he generalised Lawson's result by showing that spaces modelled by ω-continuous dcpos are regular iff they are Polish [33]. Moreover, outside the metrizable case, Reed and Martin characterised developable spaces as the ones modelled by continuous dcpos with measurement. Recently, Martin also observed a fundamental connection between order completeness of continuous domains and the Choquet-completeness of topological spaces [37].

Motivated by considerations from the area of injective spaces, people became concerned with finding *bounded complete models*. These domains have an especially pleasing property that every continuous mapping between modelled spaces extends to a Scott continuous function between the models; moreover, such an extension can be defined in a canonical way. The formal ball model for a metric space is not bounded complete in general; In [3] Ciesielski, Flagg and Kopperman gave a first (admirably involved) construction of a bounded complete model for Polish spaces and presented various topological and bitopological characterisations of second countable models. Finally, last year, Kopperman, Künzi and Waszkiewicz characterised all topologies that have bounded complete models of arbitrary cardinality [27]. Combining their results with a work of Künzi [28] resulted in a construction of such a model for arbitrary complete metric spaces.

The main objective of this paper is to explicitly match the properties of modelled spaces with the structure of their models. We thus inspect existing

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[3] the latter property will be referred to as: "Lawson's condition"

construction of models and reexamine considerations concerning the space of maximal elements of continuous domains. We claim that every structure of the modelled topology is mirrored by some construct in the model and *vice versa*. By making this correspondence precise and explicit we show how domains model topologies.

Our exposition is based on the author's doctoral dissertation [43].

2 Background

2.1 Domain theory

We review some basic notions from domain theory, mainly to fix the language and notation. See [1] for more information.

Posets
Let P be a poset. A pair of elements $x, y \in P$ is *consistent* (*bounded*), denoted $x \uparrow y$, if there exists an element $z \in P$ such that $z \sqsupseteq x, y$. The contrary case is written as $x \# y$. We say that a poset is *bounded complete* if each finite, bounded set of elements has a supremum. In particular, a non-empty bounded complete poset P has a least element, which arises as a supremum of the empty set. A subset A of P is *directed* if it is non-empty and any pair of elements of A has an upper bound in A. If a directed set A has a supremum, it is denoted $\bigsqcup^{\uparrow} A$. A poset P in which every directed set has a supremum is called a *dcpo*. A dcpo P is bounded complete iff every non-empty subset of P has an infimum.

Approximation
Let x and y be elements of a poset P. We say that x *approximates* (*is way-below*) y if for all directed subsets A of P, $y \sqsubseteq \bigsqcup^{\uparrow} A$ implies $x \sqsubseteq a$ for some $a \in A$. We denote it as $x \ll y$. If $x \ll x$ then x is called a *compact* element. The subset of compact elements of a poset P is denoted $K(P)$. Now, $\downarrow\!\!\!\downarrow x$ is the set of all approximants of x below it. $\uparrow\!\!\!\uparrow x$ is defined dually. We say that a subset B of a dcpo P is a (*domain-theoretic*) *basis* for P if for every element x of P, the set $\downarrow\!\!\!\downarrow x \cap B$ is directed with supremum x. A poset is called *continuous* if it has a basis. One can show that a poset P is continuous iff $\downarrow\!\!\!\downarrow x$ is directed with supremum x, for all $x \in P$. A poset is called a *domain* if it is a continuous dcpo. Note that $K(P) \subseteq B$ for any basis B of P. If $K(P)$ is itself a basis, the domain P is called *algebraic*. If a domain admits a countable basis, we say that it is ω-continuous (or ω-algebraic providing that $K(P)$ is a countable basis for P). A *Scott-domain* is a bounded complete ω-algebraic dcpo with a least element. A poset is *ideal* iff every element is either compact or maximal (or both). Obviously, ideal posets are algebraic.

Intrinsic topologies

A subset $U \subseteq P$ of a poset P is *upper* if $x \sqsupseteq y \in U$ implies $x \in U$. Upper sets inaccessible by directed suprema form a topology called the *Scott topology*; it is denoted $\sigma(P)$. A continuous poset P admits a countable domain-theoretic basis iff its Scott topology is second countable ([21], Theorem III-4.5). The Scott topology encodes the underlying order: $x \sqsubseteq y$ in P iff for all $U \in \sigma$ we have that $x \in U$ implies $y \in U$. This is the general definition of the so-called *specialisation order* for a topology. The collection $\{\mathord{\Uparrow} x \mid x \in P\}$ forms a basis for the Scott topology on a continuous poset P. The topology satisfies only weak separation axioms: It is always T_0 on a poset but T_1 only if the order is trivial. For an introduction to T_0 spaces, see [22]. An excellent general reference on Topology is [16].

Another two intrinsic topologies on a continuous poset P are: The *weak topology* $\omega(P)$ generated by the collection $\{P \setminus \mathord{\uparrow} x \mid x \in P\}$ and the *Lawson topology* defined as $\lambda(P) := \sigma(P) \vee \omega(P)$, the join of the Scott topology and the weak topology in the lattice of topologies on P. It has a basis of the form $\{\mathord{\Uparrow} x \setminus \mathord{\uparrow} F \mid x \in P, F \subseteq_{fin} P\}$ on any continuous poset P. The Lawson topology of any continuous poset is Hausdorff and for ω-continuous posets it is separable metrizable [21].

2.2 Partial metrics

We will briefly review basic definitions and facts about partial metric spaces from Heckmann [23], Matthews [39] and O'Neill [40], [41]. A partial metric on a set X is a map $p \colon X \times X \to [0, \infty)$ which satisfies for all $x, y, z \in X$,

(i) $p(x, y) = p(y, x)$ (symmetry).

(ii) $p(x, y) = p(x, x) = p(y, y)$ implies $x = y$ (T_0 separation axiom).

(iii) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ ($\Delta^\sharp$ – "the sharp triangle inequality").

(iv) $p(x, x) \leq p(x, y)$ (ssd – "small self-distances").

The *kernel* of p is the set $\ker p := \{x \in X \mid \exists y.\, p(x, y) = 0\}$.

Topology

The topology $\tau_p(X)$ induced by a partial metric p on a set X is given by the basis consisting of open balls of the form $B^p(x, \varepsilon) := \{y \in X \mid p(x, y) < p(x, x) + \varepsilon\}$ for an $x \in X$ and a radius $\varepsilon > 0$. It is not Hausdorff in general. Therefore, the specialisation order $\sqsubseteq_{\tau_p(X)}$ of $\tau_p(X)$ will be non-trivial in general. All of the $\tau_p(X)$-open sets, the open balls among them, are upper sets with respect to the order.

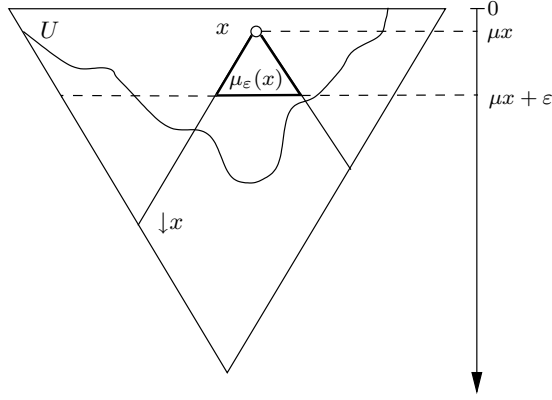## 2.3 Martin's theory

Our main reference is [36].

Fig. 1. The set of elements ε-close to x.

Quantitative approximation

Let P be a poset. For a monotone mapping $\mu\colon P \to [0, \infty)^{op}$ and any $x \in P$, $\varepsilon > 0$ we define

$$\mu(x, \varepsilon) := \{y \in P \mid y \sqsubseteq x \,\wedge\, \mu y < \mu x + \varepsilon\}.$$

We say that $\mu(x, \varepsilon)$ is the set of elements of P which are *ε-close to x*.

Measurement

We say that a monotone mapping $\mu\colon P \to [0, \infty)^{op}$ *induces the Scott topology* on a subset X of a poset P if

$$\forall U \in \sigma(P). \, \forall x \in U \cap X. \, \exists \varepsilon > 0. \, \mu(x, \varepsilon) \subseteq U.$$

We denote it as $\mu \to \sigma(X)$. If P is continuous, μ is Scott-continuous and $\mu \to \sigma(P)$, then we will say that μ *measures* P or that μ is a *measurement* on P. (Our definition of a measurement is a special case of the one given by Martin. In the language of [36] our maps are measurements which induce the Scott topology *everywhere*.)

Define the *kernel of* μ by $\ker\mu := \{x \in P \mid \mu x = 0\}$. The kernel is always a G_δ subset of maximal elements of P and as such is a topologically important object of study. We often seek a measurement on a domain with $\ker\mu = \max P$; this is called the *kernel condition* for measurements.

Let P be a continuous poset. A Scott-continuous map $\mu\colon P \to [0, \infty)^{op}$ is a *Lebesgue measurement* on P if for all Scott-compact subsets $K \subseteq \max P$ and for all Scott-open subsets $U \subseteq P$,

$$K \subseteq U \cap \max P \quad \Rightarrow \quad \exists \varepsilon > 0. \, \mu(K, \varepsilon) \subseteq U \cap \max P,$$

where $\mu(K, \varepsilon) := \bigcup\{\mu(x, \varepsilon) \mid x \in K\}$.

Definition 2.1 Let P be a continuous poset with a measurement $\mu\colon P \to [0, \infty)^{op}$. If for all consistent pairs $a, b \in P$, for all upper bounds r of a and b

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and for all $\varepsilon > 0$, there exists an $s \sqsubseteq a, b$ such that

$$\mu r + \mu s \leq \mu a + \mu b + \varepsilon,$$

then we say that μ is a *weakly modular* measurement on P.

It can be shown that every weakly modular measurement is a Lebesgue measurement [43].

2.4   Model of a space

Definition 2.2 A *model* of a topological space X is a continuous poset P together with a homeomorphism $\phi \colon X \to \max P$, where $\max P$ carries its subspace Scott topology inherited from P. A model P is *complete* if it is a dcpo; *bounded complete* if P is a bounded complete dcpo; *countably based* if P is ω-continuous; *ideal* if P is an ideal poset; a G_δ *model* if X is a G_δ subset in the Scott topology on P.

We write $\langle X, \tau \rangle \cong \langle \max P, \sigma \mid_{\max P} \rangle$ or simply $X \cong \max P$.

3   Useful techniques and facts

3.1   Ideal models of spaces

In [32], Martin observed that in the majority of cases, continuous models of spaces can be replaced with special algebraic models called ideal domains. In fact, he proved in [32] that any topological space X, which has a complete G_δ model P, has a complete G_δ ideal model. In [43] it has been noted that Martin's result remains valid in a more general setting:

Proposition 3.1 *If a topological space has a G_δ model P, then it has a G_δ ideal model E. In addition, if P is equipped with a measurement (kernel measurement, Lebesgue measurement, partial metric for the Scott topology), then E can be constructed in such a way that it admits a measurement (kernel measurement, Lebesgue measurement, partial metric for the Scott topology).*

3.2   Partial metrics versus measurements

A tight connection between partial metrics and measurements has been established in [43]. The shortest summary of the facts that are useful for this paper reads as follows:

Proposition 3.2 *Let P is a continuous poset.*

(i) *If P is equipped with a partial metric $p \colon P \times P \to [0, \infty)$ for the Scott topology, then the self-distance mapping of p is a Lebesgue measurement with the same kernel.*

(ii) *If P is equipped with a weakly modular measurement $\mu \colon P \to [0, \infty)^{op}$, then the map $p \colon P \times P \to [0, \infty)$ given by $p_\mu(x, y) := \inf\{\mu z \mid z \ll x, y\}$,*

$x, y \in P$ is a partial metric for the Scott topology with the same kernel as μ.

(iii) *If, in addition, P is algebraic, then it admits a partial metric for the Scott topology iff it admits a Lebesgue measurement with the same kernel.*

4 Basic relationship between spaces and their models

Consider a model $\langle P, \phi \rangle$ of a topological space $\langle X, \tau \rangle$. Since the specialisation preorder of the Scott topology on P agrees with the underlying order, the topology on P is always T_0. For the same reason the subspace Scott topology on $\max P$ is T_1. Therefore, any topology that can be modelled must be at least T_1. On the other hand, if the topology τ is not discrete, the Scott topology on P can not be T_1. At the moment we do not know if every T_1 space arise as a model of some continuous poset P. This question, however, seems to be far too general to be of any practical importance in computing.

In the case of interesting topologies it happens most often than they are G_δ subsets of their models. Martin [36] characterised this situation as follows:

Proposition 4.1 (Martin) *Let P be a continuous poset and $\langle X, \tau \rangle$ a topological space. P is a G_δ model of X iff $X \cong \ker \mu$ for some Scott-continuous mapping $\mu \colon P \to [0, \infty)^{op}$.*

It is worth to note that if X has a *complete* G_δ model, then it must be first-countable and Baire [34]; this does not hold in general if the model is not a dcpo.

The following result is a simultaneous generalization of the formal ball model proposed for metric spaces in [11] and its algebraic version described in [33]. It shows how to build domain models for first-countable spaces.

Proposition 4.2 *For a T_1 topological space $\langle X, \tau \rangle$ the following are equivalent:*

(i) *X is first-countable;*

(ii) *$X \cong \ker \mu$ for some Scott-continuous, strictly monotone mapping $\mu \colon P \to [0, \infty)^{op}$ on a continuous poset P.*

Proof. For (1)$\Rightarrow$(2), since X is first-countable, for every $a \in X$ we can pick a collection $N(a) := \{N(a, n) \mid n \in \omega\}$ of neighbourhoods of a with the property that $n \geq m$ implies $N(a, n) \subseteq N(a, m)$. Define $N(X, \omega) := \bigcup_{a \in X} N(a)$ and $X' := \{\{a\} \mid a \in X\}$. Let

$$P := \{(a, n) \in X \times \omega \mid N(a, n) \in N(X, \omega)\} \cup X'.$$

Consider a function $n \colon P \to \omega \cup \{\infty\}$ given by $n(x) := m$, when $x = (a, m) \in P \setminus X'$ and $n(x) := \infty$, whenever $x \in X'$. Consider a partial order $\sqsubseteq$ between

elements of P defined as the reflexive closure of

$$(a, r) \sqsubseteq (b, s) \text{ iff } (N(b, s) \subseteq N(a, r) \text{ and } s > r).$$

Clearly $X' = \max P$.

Observe that

(1) $\qquad\qquad \forall x, y \in P. \ n(x) = n(y) \text{ implies } (x = y \text{ or } x \# y),$

by the definition of the order on P (recall that $x \# y$ means that the subset $\{x, y\}$ of P has no upper bound).

Define a mapping $\mu \colon P \to [0, 1)^{op}$ by $\mu x = 2^{-n(x)}$ if $x \in P \setminus X'$ and $\mu x = 0$ otherwise. By definition and (1), $\ker \mu = X' = \max P$. It is also clear that the map is monotone and strictly monotone.

We will show that the function μ is Scott-continuous. Let D be a directed subset of P with supremum x.

Assume $x \in P \backslash X'$. Suppose that for any $d \in D$ we have $n(d) < n(x)$. Since D is nonempty, choose $d_1 \in D$ such that for any other $e \in D$, $n(e) \leq n(d_1)$. Now, if for arbitrary $d_2 \in D$ we have $d_2 \sqsubseteq d_1$, then $d_1 = x$, a contradiction as $n(d_1) < n(x)$. Otherwise, there is $d_2 \# d_1$ and by directness of D, there exists $d_3 \in D$ with $n(d_3) > n(d_1)$, a contradiction with our choice of d_1. We conclude that there exists an element $d \in D$ with $n(d) = n(x)$ and hence $x = d \in D$ by (1). We have proved that

(2) $\qquad\qquad \forall D \in P. \ (x = \bigsqcup^{\uparrow} D \text{ and } x \notin X') \text{ implies } x \in D.$

Therefore, $\bigsqcup^{\uparrow} \mu(D) = \mu x$.

Assume that $x \in X'$. Suppose that there exists $m \in \omega$ such that $n(d) \leq m$ for any $d \in D$. Without loss of generality we may choose the number m in such a way that $m = n(e)$ for some $e \in D$. If all elements of D are below e, then $x \sqsubseteq e$ and hence $x = e$, by maximality of x. This implies that $n(e) = n(x) = 0$, a contradiction. Otherwise, there exists $e_1 \in D$ with $e_1 \# e$. By directness of D, there is $e_2 \sqsupseteq e_1, e$ with $n(e_2) > n(e)$, which is again a contradiction. We have shown that

(3) $\quad \forall D \in P. \ (x = \bigsqcup^{\uparrow} D \text{ and } x \in X') \text{ implies } \{n(d) \mid d \in D\} \text{ is unbounded.}$

Hence, $\bigsqcup^{\uparrow} \mu(D) = 0 = \mu x$.

We conclude that the mapping μ is Scott-continuous.

We claim that every non-maximal element is compact. Let $z \in P \setminus X'$ and $z \sqsubseteq x = \bigsqcup^{\uparrow} D$ for some directed subset D of P. If $x \notin X'$, then $z \sqsubseteq x \in D$ by (2). Otherwise, say $x = \{a\}$ for some $a \in X$, and so there exists $k \in \omega$ such that $a \in U \subseteq z$ for some $U \in \mathcal{U}_k$. Without loss of generality, $k > n(z)$ and $n(e) = k$ for some $e \in D$ (the latter follows from (3)). Hence $a \in e \subseteq z$ and so $z \sqsubseteq e$. We have shown that $z \ll z$, whenever $z \in P \setminus X'$.

It is now easy to see that for any $x \notin X'$ we have $\downarrow x = \Downarrow x$ and so $x = \bigsqcup^{\uparrow} \Downarrow x$. Otherwise, if $x \in X'$ (say $x = \{a\}$), then by construction of P, $\Downarrow x$ is directed and $\{n(y) \mid y \ll x\}$ is unbounded. Clearly, if $\Downarrow x \sqsubseteq z$ for any other $z \in P$, then

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$n(z) = \infty$ and so $z \in X'$. Then $z = x$ by the T_1 axiom of the space X. We conclude that $x = \bigsqcup^\uparrow \downarrow x$. Therefore, P is an ideal poset and so it is continuous. Also, from the construction of P it is immediate that $\tau = \sigma(P) \mid_{X'}$.

For $(2) \Rightarrow (1)$, we will identify elements of X with $\ker\mu$, which is a subset of maximal elements of P. Let $x \in X$. Since $\downarrow x$ is directed, we can construct an increasing sequence (x_n) in $\downarrow x$ with $\mu x_n < 1/n$. It is now easy to see that $\Uparrow x_n \cap \max P$ is a basis at x in $\sigma \mid_{\max P} \cong \tau$. $\qquad \Box$

A simple modification of the above proof leads to a complete domain-theoretic characterisation of developable T_1 spaces. The following result was announced by Martin and Reed at the First Irish Conference on the Mathematical Foundations of Computer Science and Information Technology two years ago (but unpublished so far).

Proposition 4.3 (Reed & Martin) *For a topological space X, the following are equivalent:*

(i) *X is developable and T_1,*

(ii) *X is the kernel of a measurement on a continuous poset.*

Proof (Sketch) Let $\{\mathcal{U}_n\}_{n \in \omega} = N(X, \omega)$ be a development for X. Define functions n, μ and the poset P as in the proof of Proposition 4.2 (and use the same notation), prove that μ is Scott-continuous and strictly monotone. Show that P is ideal.

Finally, to conclude that the mapping μ measures P, take $x \in P$ and $x \in \Uparrow z \in P$. If $x \notin X'$, taking $\varepsilon := \mu x/2$ proves the claim. Otherwise, $x = \{a\}$ for some $a \in X$ and the claim follows from the fact that there exist $k \in \omega$ with $a \in \mathbf{St}(a, \mathcal{U}_k) \subseteq z$. $\qquad \Box$

Let us summarise the relationship between the structure of a modelled space and the structure of the model:

space X	model P
always T_1	always T_0; T_1 in degenerate cases
has G_δ model	$X \cong \ker\mu$ for μ continuous
first-countable	$X \cong \ker\mu$ for μ continuous, strictly monotone
developable	$X \cong \ker\mu$ for a measurement μ

5 Metrizable spaces and their models

We have shown that certain structural properties of topologies can be encoded in the existence of appropriate Scott-continuous mappings on the underlying model. This observation is valid also for the case of metrizable spaces and alike. Different equivalences presented in the following proposition have been

known and proved by Martin [36] and Heckmann [23]. The connection between partial metrics and measurements established in [43] allows to combine them in an elegant way.

Proposition 5.1 *The following are equivalent for a topological space X:*

(i) X *is metrizable;*

(ii) X *is the kernel of a Lebesgue measurement on a continuous poset;*

(iii) X *is the kernel of a partial metric that induces the Scott topology on a continuous poset;*

(iv) X *is the kernel of a partial metric that induces the Scott topology on an ideal poset.*

Proof. For (1)$\Rightarrow$(3) use the formal ball model $\mathbf{B}X$. (3)$\Rightarrow$(2) holds since the self-distance mapping of a partial metric is a Lebesgue measurement (cf. Proposition 3.2). For (2)$\Rightarrow$(4), use Proposition 3.1. (4)$\Rightarrow$(1) is trivial. □

It turns out that ultrametrizable spaces can be characterised in exactly the same way if we put some meaningful restriction on the order in the models:

Definition 5.2 A *tree* is a poset P such that

$$\forall x, y \in P. \ x{\uparrow}y \text{ implies } (x \sqsubseteq y \text{ or } y \sqsubseteq x).$$

A tree is *complete* if P is a dcpo. We will also assume that every tree has a bottom element $\bot$.

Proposition 5.3 *The following are equivalent for a topological space X:*

(i) X *is ultrametrizable;*

(ii) X *is the kernel of a Lebesgue measurement on an ideal tree;*

(iii) X *is the kernel of a partial metric that induces the Scott topology on an ideal tree;*

Proof. (1)$\Rightarrow$(3). Set $\mathcal{U}_n := \{B_d(x, 1/4^n) \mid x \in X\}$ with respect to some ultrametric $d\colon X \times X \to [0, \infty)$ compatible with the topology on X. The collection $\{\mathcal{U}_n\}$ is a development for X. We build an ideal model P of X as in Propositions 4.2, 4.3 (and use the notation from there). Since d is an ultrametric, for every x, y in $P \setminus X'$ such that $x{\uparrow}y$ we have either $x \sqsubseteq y$ or $y \sqsubseteq x$, and so P has a tree structure. Therefore the map μ (as defined in the proof of Proposition 4.3) is a measurement and is vacuously weakly modular. Hence, the induced partial semimetric p_μ is a partial metric with $\ker p_\mu = X$ by Proposition 3.2.(2). (3)$\Rightarrow$(2) is clear. For (2)$\Rightarrow$(1), observe that any partial metric p on a tree P satisfies

$$p(x, y) \leq \max\{p(x, z), p(z, y)\}$$

for any $x, y, z \in P$. Hence, the mapping p restricts to an ultrametric on its kernel. □

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6 Completeness of the spaces and their models

In his "Lectures on analysis" [2] Gustave Choquet proposed a notion of completeness for topological spaces.

Definition 6.1 Let $\langle X, \tau \rangle$ be a topological space and $\tau_* := \{(U, x) \mid x \in U, U \in \tau\}$. The space X is *Choquet-complete* if there exists a sequence of functions

$$f_n: \underbrace{\tau_* \times ... \times \tau_*}_{n \ times} \to \tau, \qquad n \in \omega$$

such that

(i) for each $((U_1, x_1), ..., (U_n, x_n))$ we have

$$x_n \in f_n((U_1, x_1), ..., (U_n, x_n)) \subseteq U_n$$

and

(ii) for any sequence (V_n, x_n) in τ_* with $V_{n+1} \subseteq f_n((V_1, x_1), ..., (V_n, x_n))$ for all $n \in \omega$ we have

$$\bigcap V_n \neq \emptyset.$$

Basic facts about Choquet completeness are:

Proposition 6.2 *The following hold:*

(i) *A Choquet complete space is Baire.*

(ii) *A metric space is Choquet complete iff it is completely metrizable.*

(iii) *G_δ subspaces of Choquet complete spaces are Choquet complete.*

(iv) *A locally compact sober space is Choquet complete.*

Proof. First two claims are demonstrated in [2]. For the proof of the third one, we refer to Exercise 8.16 of [25]. Finally, the last fact is proved in [35].□

As has been already remarked in [36], Choquet completeness does not assume any separation axioms to hold and, moreover, captures two fundamental aspects of computing: approximation and convergence. It seems therefore well-suited as a topological notion that characterise completeness of continuous domains. A single most important property of topological spaces that have complete models has been stated by Martin in a recent paper [37]:

Theorem 6.3 (Martin) *A topological space with a complete model is Choquet complete.*

Martin observes that this result implies that the space of maximal elements in a continuous dcpo is metrizable iff it is completely metrizable. We, however, do not need the full strength of Martin's theorem to prove next proposition:

Proposition 6.4 *The following are equivalent for a topological space X:*

(i) *X is completely metrizable;*

(ii) *X is the kernel of a Lebesgue measurement on a continuous dcpo;*

(iii) *X is the kernel of a partial metric that induces the Scott topology on a continuous dcpo;*

(iv) *X is the kernel of a partial metric that induces the Scott topology on an ideal dcpo.*

Proof. (4)$\Rightarrow$(1) it is clear that X is metrizable. Since X is a G_δ subset of a Choquet complete space (cf. Theorem 6.2(3)-(4)), it is completely metrizable. The rest of the proof obeys the same pattern as in Proposition 5.1 above and is therefore omitted. $\qquad\square$

Finally, we characterise Polish spaces in the spirit of the proposition above. Note that models of Polish spaces differ from models of complete metric spaces only by the assumption of second countability of the Scott topology (as one should expect). In the proposition below, we gather results on models of Polish spaces from [33] and [4], [3]. Again, the methods developed in [43] make the proof concise and transparent:

Proposition 6.5 *The following are equivalent for a topological space X:*

 (i) *X is Polish;*

 (ii) *X is the kernel of a Lebesgue measurement on an ω-continuous dcpo;*

(iii) *X is the kernel of a partial metric that induces the Scott topology on an ω-continuous dcpo;*

(iv) *X is modelled by an ω-continuous dcpo P such that $\max P$ is regular with respect to the subspace Scott topology.*

 (v) *X is modelled by an ω-continuous dcpo which satisfies the Lawson condition.*

(vi) *X is modelled by a countably based Lawson-compact dcpo.*

Proof. For (1)$\Rightarrow$(3) use the formal ball model $\mathbf{B}X$. (3)$\Rightarrow$(2) is immediate. (2)$\Rightarrow$(4). Let μ be a Lebesgue measurement on a countably based complete model P. Define $d\colon P \times P \to [0, \infty)$ by

$$d(x, y) := 2 \cdot \inf\{\mu z \mid z \ll x, y\} - \mu x - \mu y.$$

By Theorem 2.28, page 32 of [43], d can be extended to a metric ρ on P such that $\rho\mid_{\max P}$ induces the subspace Scott topology. That is, $\max P$ is metrizable and thus regular. (4)$\Rightarrow$(1) is proved by Martin in [33]. The argument is elegant and worth repeating: Since second-countability is hereditary, the subspace of maximal elements of P is second-countable and thus metrizable by Urysohn's Lemma (cf. [45], Theorem 23.1, page 166). The space $\max P$ is also a G_δ subset of a Choquet complete separable metric space, and hence Polish. (1)$\Rightarrow$(5) is proved in [3]. (5)$\Rightarrow$(6) is immediate. To complete the proof, we show (6)$\Rightarrow$(2). Let P be a countably based Lawson-compact model of X. By a result of Lawson [31], P admits a radially convex metric d for the Lawson topology.

Now, Martin proves (cf. Theorem 5.6.1 of [36], page 150) that P admits a kernel measurement μ such that for all $y \sqsubseteq x$ we have $d(x,y) \leq \mu y - \mu x$. Let K be a Scott-compact subset of a Scott-open set U in P. Hence K is Lawson-compact and U is Lawson-open. By the Lebesgue covering lemma (cf. [45], Theorem 22.5, page 163), there exists $\varepsilon > 0$ with $B_d(K, \varepsilon) \subseteq U$. Let $x \in K$ and $y \in \mu(x, \varepsilon)$. Then $d(x,y) \leq \mu y - \mu x \leq \mu y < \varepsilon$. Therefore $y \in B_d(K, \varepsilon) \subseteq U$ and we conclude that $\mu(K, \varepsilon) \subseteq U$, as required. $\qquad\square$

Bearing in mind that Choquet-completeness of the modelled space is reflected in the completeness of the model and *vice versa*, it takes no effort to restate Proposition 5.3 for complete (separable) ultrametric spaces. The proof, however, uses a new idea from [43] that the Choquet-completion of the space can be performed via the rounded ideal completion of the model.

Proposition 6.6 *The following are equivalent for a topological space X:*

(i) X *is a complete (separable) ultrametric space;*

(ii) X *is the kernel of a Lebesgue measurement on a complete (countably based) ideal tree;*

(iii) X *is the kernel of a partial metric that induces the Scott topology on a complete (countably based) ideal tree;*

Proof. For (1)$\Rightarrow$(3) use construction from Proposition 5.3 to build the model P of X. It is ideal and admits a Lebesgue measurement μ with $X \cong \ker\mu$. In [43] it is shown that the Choquet completion $\bar{X}$ of the maximal point space of P is given by the subset of maximal elements of the rounded ideal completion $\mathcal{I}(P)$ of P. Moreover, the measurement μ on P extends to a measurement $\bar{\mu}$ on $\mathcal{I}(P)$ with $\ker\bar{\mu} \cong \bar{X}$. But since X is already Choquet complete by Proposition 6.2.(4), this means that $\mathcal{I}(P)$ is a complete model for X equipped with a measurement $\bar{\mu}$. Observe that since P is a tree, its rounded ideals are chains and hence the tree structure is inherited by $\mathcal{I}(P)$. Moreover, $\bar{\mu}$ is vacuously weakly modular, and hence induces a partial metric $p_{\bar{\mu}}$ on $\mathcal{I}(P)$ by Proposition 3.2.(2).

The rest of the proof mimics Proposition 5.3 and the claims about second-countability present no difficulties. $\qquad\square$

To summarise, in this section we have shown that Choquet-completeness of modelled spaces corresponds precisely to completeness of their models. This correspondence is affirmed by Martin's theorem 6.3 and by the proof of the last proposition, where the rounded ideal completion of the model was used *as* the Choquet-completion of the space that "sits at the top of the model". Moreover, we observed that second-countability (or equivalently: separability in the metric case) of the space is reflected by the existence of a countable base in the model. It can be shown that this correspondence remains valid for just developable spaces as well.

space X	model P
developable + second countable	$X \cong \ker\mu$ for a measurement μ, P countably based
completely metrizable	$X \cong \ker\mu$ for a Lebesgue measurement μ (or: $X \cong \ker p$ for a partial metric p with $\tau_p = \sigma$) P directed-complete
completely metrizable + second-countable	as above + P countably based
completely ultrametrizable	$X \cong \ker\mu$ for a Lebesgue measurement μ (or: $X \cong \ker p$ for a partial metric p with $\tau_p = \sigma$) P directed-complete, tree
completely ultrametrizable + second-countable	as above + P countably based

7 Bounded complete models of spaces

In [27] a long awaited characterisation of all topological spaces with bounded complete models has been presented. The theorem extends in some sense the characterisation of Polish spaces given in [4] but goes far beyond the metrizable case. We will only state the result and sketch its basic consequences. For the introduction to bitopological and quasi-uniform spaces consult [26] and [20], respectively. A bitopological characterisation of posets is provided in [19].

Theorem 7.1 ([27]) *The following are equivalent for a T_1 topological space $\langle X, \tau \rangle$:*

(i) *There exists a compatible quasiproximity δ on X such that $\tau(\delta^{-1})$ is compact;*

(ii) *There exists a compatible quasiuniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}^{-1})$ is compact;*

(iii) *$\langle X, \tau \rangle$ is homeomorphic to $\langle \max P, \mathbf{p}\sigma \rangle$ where P is a pointed, coherent poset which has directed upper bounds and is equipped with an auxiliary, approximating, multiplicative binary relation $\prec$, and $\mathbf{p}\sigma$ is the pseudoScott topology on P;*

(iv) *X admits a bounded complete model;*

(v) *There is a compact topology $\tau^* \subseteq \tau$ on X such that $\langle X, \tau, \tau^* \rangle$ is pairwise completely regular.*

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A remarkable characterisation of complete metrizability has been given by Künzi in [28]:

Theorem 7.2 (Künzi) *A metrizable topological space X is completely metrizable iff there is a compatible quasiuniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}^{-1})$ is compact.*

The two theorems above yield an immediate corollary:

Corollary 7.3 ([27]) *Every complete metric space has a bounded complete model.*

It should be remarked (as it is noted in [4] for the second-countable case) that the fact that every locally compact Hausdorff space X can be modelled by its standard bounded complete model $U(X) := \{K \subseteq X \mid K \neq \emptyset \text{ compact}\}$ (ordered by the inverse inclusion) is a special case of Theorem 7.1. A similar remark applies to complete ultrametric spaces and the model proposed in Proposition 6.6.

We conclude that bounded completeness of models is reflected in certain compactness properties for bitopology characterising the modelled space.

8　Summary

We have shown explicitly how certain structural properties of topological spaces are modelled by mappings on domains that approximate the spaces. We hope that this concise study will be a good starting point for a *systematic* search for models of topological spaces of practical importance in computer science.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Clarendon Press, 1994.

[2] G. Choquet. *Lectures on Analysis*, volume 1. Benjamin, 1969.

[3] K. Ciesielski, R. Flagg, and R. Kopperman. *Polish spaces, computable approximations and bitopological spaces*. Topology and its Applications, to appear.

[4] K. Ciesielski, R.C. Flagg, and R. Kopperman. Characterizing Topologies With Bounded Complete Computational Models. *Electronic notes in Theoretical Computer Science*, 20:11, 1999.

[5] B. Coecke and K. Martin. A partial Order on Classical and Quantum States. Technical Report PRG-RR-02-07, Oxford University Computing Laboratory, 2002.

[6] A. Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.

[7] A. Edalat. Dynamical systems, measures and fractals via domain theory. *Information and Computation*, 120(1):32–48, 1995.

[8] A. Edalat. Power domains and iterated function systems. *Information and Computation*, 124:182–197, 1996.

[9] A. Edalat. Domains for Computation in Mathematics, Physics and Exact Real Arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, 1997.

[10] A. Edalat and M.H. Escardó. Integration in Real PCF (full version). *Information and Computation*, 160:128–166, 2000.

[11] A. Edalat and R. Heckmann. A computational model for metric spaces. *Theoretical Computer Science*, 193(1–2):53–73, 28 February 1998.

[12] A. Edalat and A. Lieutier. Foundation of a Computable Solid Modelling. In *Proceedings of ACM's Symposium on Solid Modeling'99*. ACM, 1999. Extended abstract. To appear in Theoretical Computer Science.

[13] A. Edalat and A. Lieutier. Domain Theory and Differential Calculus. In *Proceedings of IEEE Symposium on Logic in Computer Science*, 2002. Extended abstract.

[14] A. Edalat and P. Sünderhauf. Computable Banach Spaces via Domain Theory. *Theoretical Computer Science*, 219:169–184, 1999.

[15] A. Edalat and Philipp Sünderhauf. A Domain-theoretic Approach to Real Number Computation. *Theoretical Computer Science*, 210:73–98, 1998.

[16] R. Engelking. *General Topology*. Sigma Series in Pure Mathematics. Heldermann Verlag, 1989.

[17] M.H. Escardó. *PCF extended with real numbers: a domain theoretic approach to higher-order exact real number computation*. PhD thesis, Department of Computing, Imperial College of Science, Technology and Medicine, University of London, 1997.

[18] B. Flagg and R. Kopperman. Computational models for ultrametric spaces. In *Proceedings of MFPS XIII*. Electronic Notes in Theoretical Computer Science, 1997.

[19] R.C. Flagg and R.D. Kopperman. Tychonoff poset structures and auxiliary relations. *Ann. New York Acad. Sci.*, 767:45–61, 1995. Papers on General Topology and Applications (Andima et. al., eds.).

[20] P. Fletcher and W. F. Lindgren. *Quasi-Uniform Spaces*. Marcel-Dekker, 1982.

[21] G. K. Gierz, K. H Hoffmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott. *Continuous Lattices and Domains*. Encyclopedia of Mathematics and Its Applications, v. 93. Cambridge University Press, 2003.

[22] R. Heckmann. *Power Domain Constructions (Potenzbereich-Konstruktionen)*. PhD thesis, Universität des Saarlandes, December 1990.

[23] R. Heckmann. Approximation of metric spaces by partial metric spaces. *Applied Categorical Structures*, 7:71–83, 1999.

[24] T. Kamimura and A. Tang. Total objects of domains. *Theoretical Computer Science*, 34:275–288, 1984.

[25] A.S. Kechris. *Classical Descriptive Set Theory*. Springer Verlag, 1995.

[26] R. Kopperman. Asymmetry and duality in topology. *Topology and Appl.*, 66:1–39, 1995.

[27] R. Kopperman, H-P. A. Künzi, and P. Waszkiewicz. Bounded complete models of topological spaces. Draft paper, October 2002.

[28] H-P. A. Künzi. *Cocompactness and quasi-uniformizability of completely metrizable spaces*. Draft paper, November, 2001.

[29] D. Lacombe. Quelques procédés de définitions en topologie recursif. In *Constructivity in Mathematics*, pages 129–158. North-Holland, 1959.

[30] J. Lawson. Computation on metric spaces via domain theory. *Topology and its Applications*, 85:274–263, 1998.

[31] J. D. Lawson. Spaces of maximal points. *Mathematical Structures in Computer Science*, 7(5):543–555, October 1997.

[32] K. Martin. *Ideal Models of Computable Objects*. Theoretical Computer Science, to appear.

[33] K. Martin. *The Regular Spaces with Countably Based Models*. Theoretical Computer Science, to appear.

[34] K. Martin. Domain theoretic models of topological spaces. In *COMPROX III*, volume 13 of *Electronic Notes in Theoretical Computer Science*, 1998.

[35] K. Martin. Nonclassical techniques for models of computation. *Topology Proceedings*, 24, 1999.

[36] K. Martin. *A Foundation for Computation*. PhD thesis, Department of Mathematics, Tulane University, New Orleans, LA 70118, 2000.

[37] K. Martin. Topological games in domain theory. *Topology and Appl.*, 129(2), 2002.

[38] P. Martin-Löf. *Notes on Constructive Mathematics*. Almquist & Wiksell, 1970.

[39] S. G. Matthews. Partial metric topology. In *Proceedings of the 8th Summer Conference on Topology and Its Applications*, volume 728, pages 176–185, 1992.

[40] S.J. O'Neill. Partial metrics, valuations and domain theory. Research Report CS-RR-293, Department of Computer Science, University of Warwick, Coventry, UK, October 1995.

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[41] S.J. O'Neill. Two topologies are better than one. Research Report CS-RR-283, Department of Computer Science, University of Warwick, Coventry, UK, March 1995.

[42] D. Scott. Outline of the mathematical theory of computation. In *4th Princeton Conference on Informations Science*, 1970.

[43] P. Waszkiewicz. *Quantitative Continuous Domains*. PhD thesis, School of Computer Science, The University of Birmingham, Edgbaston, B15 2TT, May 2002.

[44] K. Weihrauch and U. Schreiber. Embedding metric spaces into cpo's. *Theoretical Computer Science*, 16:5–24, 1981.

[45] S. Willard. *General Topology*. Addison-Wesley Series in Mathematics. Addison-Wesley, 1970.