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Pseudo-distributive laws

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Abstract

We address the question of how elegantly to combine a number of different structures, such as finite product structure, monoidal structure, and colimiting structure, on a category. Extending work of Marmolejo and Lack, we develop the definition of a pseudo-distributive law between pseudo-monads, and we show how the definition and the main theorems about it may be used to model several such structures simultaneously. Specifically, we address the relationship between pseudo-distributive laws and the lifting of one pseudo-monad to the 2-category of algebras and to the Kleisli bicategory of another. This, for instance, sheds light on the preservation of some structures but not others along the Yoneda embedding. Our leading examples are given by the use of open maps to model bisimulation and by the logic of bunched implications.

1 Introduction

Categories with additional structure, such as symmetric monoidal structure, finite product structure, cartesian closed structure, both symmetric monoidal and finite product structure together [17,22], a monad [15], or a class of colimits [3,10], play a fundamental foundational role in theoretical computer science. Typically, one considers categories with several structures at once, with those structures interacting with each other in some way. For instance, Moggi's work on computational effects [15] involves both finite product structure and a monad, interacting with each other in the definition of strong monad. The logic of bunched implications involves a small symmetric monoidal category C with finite products and extends both the symmetric monoidal structure and the finite product structure along the Yoneda embedding $Y{:}C \longrightarrow [C^{op}, Set]$.

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In contrast, finite coproduct structure does not extend along the Yoneda embedding. The Yoneda embedding exhibits $[C^{op}, Set]$ as the free cocompletion of C, and, consequently, the monoidal and finite product structures are sent to monoidal closed and cartesian closed structures respectively. In the analysis of bisimulation using open maps [3,10], crucial use is explicitly made of the fact that the Yoneda embedding yields the free cocompletion of a small category, and structures such as finite product structure are analysed in that light.

Motivated by these examples, we seek a calculus of categories with structure: what does it mean to mean to be a "category with structure"? what elegant constructions, supported by theorems, can one make with categories with structure? how may one elegantly combine two or more structures on a category, again supported by theorems? There has been considerable abstract mathematical work on the first two of these questions over recent decades: a definitive definition of algebraic structure on a category appears in [2], with an account of some of the ideas directed towards theoretical computer scientists in [19]; the former paper also develops some constructions on categories with algebraic structure, with further development appearing in [6,7]. In this paper, we address the third question: how elegantly to combine two or more structures on a category? This question did not arise for us from abstract considerations, but was put to us by computer scientists working on bisimulation, and is also of immediate relevance to the work on bunched implications. The primary interest of the workers on bisimulation is where one of the structures is that of all small colimits, i.e., relating to the free cocompletion of a category.

The notion we develop here is that of a *pseudo-distributive law* between *pseudo-monads* [13,14]. The definition of a pseudo-distributive law generalises that of a distributive law between ordinary monads [1]. The generalisation is not routine because the pseudo-ness adds so much complication that, although it is fairly clear what is the right data for a pseudo-distributive law, it is less clear what are definitive axioms for the notion. The main result for pseudo-distributive laws is the generalisation of the equivalence between ordinary distributive laws and liftings [1]. Computational interest lies both in the lifting, given a pseudo-distributive law of one pseudo-monad S over another pseudo-monad T, of T to the 2-category of algebras of S and in the lifting of S to the Kleisli bicategory of T.

Except for a size concern that we analyse below, the leading examples for us of pseudo-monads on Cat are the 2-monads for finite products, symmetric monoidal structure, and all small colimits, and the pseudo-monads generated by their combinations. And the leading examples of pseudo-distributive laws for us are those between these structures. For a further wide class of examples, if T is a *pseudo-commutative* monad on Cat, as defined in [6,7], there is a canonical pseudo-distributive law of S over T, where S is the 2-monad on Cat whose algebras are small symmetric monoidal categories. This yields a substantial range of examples of relevance to computer science, many of them used to model contexts or parallelism. Pseudo-commutative monads have proved to be useful in the analysis of combining computational effects [8,9]. Further examples of pseudo-distributive laws arise involving *premonoidal* structure [20,21], as is increasingly used in modelling continuations [4].

The most relevant work to date in the direction of this paper, and work that is particularly helpful here, has been that of Marmolejo in [14] and Lack in [13], building on Kelly's work in [11]. Both papers rely on and are expressed in terms of definitions and results relevant to tricategories in [5,19], and neither paper is sullied by the presence of an example. Marmolejo's paper contains the central definition we need and goes a long way towards one result we regard as fundamental. But it is a long, dense paper, and it is not directed towards computer scientists, or, for that matter, towards any non-specialist in higher-dimensional category theory. Moreover, it does not address some of the issues of primary importance to us: for instance, it does not define the 2-category of algebras or the Kleisli bicategory of a pseudo-monad. Nor does the latter concept follow easily from the work in that paper: to give a definition in the spirit of that paper would require careful analysis of a three-dimensional colimit in a standard tricategory. Lack's paper is also dense and is also directed only towards experts in higher-dimensional category theory. It does contain a universal property that identifies the notion of the 2-category of algebras as we need it here, but, despite appearances, it does not actually describe that 2-category. Nor does it contain a definition of the Kleisli bicategory, even by identifying the appropriate universal property. It also assumes profound knowledge of coherence and of weighted enriched colimits. The gentle reader may be pleased to observe that we do not assume such knowledge in this paper.

We must now add a caveat: the example of all small colimits, as appears in our leading examples, is not explicitly covered by the above-mentioned definition. But the only reason for that is one of *size*: the free cocompletion of a small category is never small in non-trivial cases, so does not yield either a 2-monad, or, more generally, a pseudo-monad on Cat. The question of size can be addressed in various ways. For instance, in regard to bisimulation, one may restrict to a small class of small colimits as done in [3]. Alternatively, one may consider a larger universe, which is effectively equivalent to considering colimits of size less than κ for a strongly inaccessible cardinal κ. The work here is already complicated enough without explicit concern about size, and such techniques do exist to address the issue; so, for the purposes of this paper, we shall ignore the issue beyond our mentioning it here and making an occasional reference in the text as seems appropriate. We think a fundamental point of this paper that should be of considerable help to workers especially in bisimulation is that the bicategory $Prof$ is simply, except for this size issue, the Kleisli bicategory for a pseudo-monad on Cat.

The paper is organised as follows. In Section 2, we recall the definition of a pseudo-monad, and we provide examples that arise naturally in theoretical computer science. In Section 3, we define the 2-category of pseudo-algebras for a pseudo-monad, describe a universal property for it, and give computational examples. In Section 4, we define the Kleisli bicategory for a pseudo-monad, give a universal property for it, and give examples. And in Section 5, we recall the definition of a pseudo-distributive law, and we provide a theorem giving an equivalence between pseudo-distributive laws and liftings both to the 2-category of algebras and to the Kleisli bicategory.

2 Pseudo-monads

In this section, we introduce the notion of a pseudo-monad on a bicategory, a fortiori on a 2-category. Most of the examples of bicategories of primary interest to us are 2-categories. For instance, *Cat*, the 2-category of small categories, functors, and natural transformations, appears naturally as a 2-category rather than as a bicategory. So this paper is generally written in terms of pseudo-monads on 2-categories rather than on bicategories. But the usual expression of the definition of pseudo-monad nowadays is the same, as one suppresses the structural isomorphisms of the base bicategory in describing the axioms. So we give the bicategorical setting here.

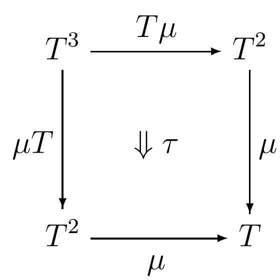
For some further examples of bicategories, *Rel* is the 2-category, indeed the locally ordered category, whose objects are sets, with a 1-cell from X to Y being a binary relation from X to Y, and with 2-cells given by inclusion of relations. A more sophisticated example, indeed one of our leading examples, of a bicategory is given by $Prof$, cf. [3,10].

Example 2.1 $Prof$ may be defined naturally in a number of different ways, some of them bicategorical and others 2-categorical. In all cases, its objects are small categories. One definition has an arrow from C to D defined to be a functor $C \longrightarrow [D^{op}, Set]$, with composition defined using a canonical lifting, given by left Kan extension, of any such functor to a functor with domain $[C^{op}, Set]$, then by using ordinary composition of functors. With this definition, $Prof$ naturally forms a bicategory, one that is evidently, in spirit, of the nature of a Kleisli bicategory. But an arrow in $Prof$ from C to D may alternatively be defined to be a colimit preserving functor from $[C^{op}, Set]$ to $[D^{op}, Set]$. This latter definition makes $Prof$ naturally into a 2-category, one that is equivalent to the previous definition. Yet another definition has a map from C to D defined to be a functor from $[C^{op}, Set]$ to $[D^{op}, Set]$ that has a right adjoint: that definition is isomorphic to the second definition, and it also naturally defines $Prof$ as a 2-category. $Prof$ is fundamental to the study of bisimulation using open maps [3,10].

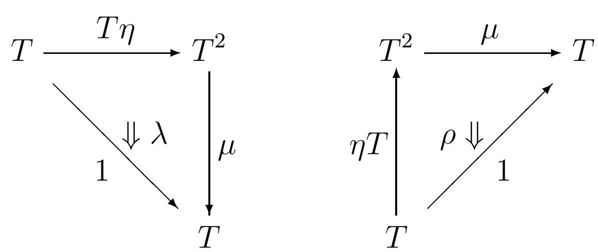
We give the definitions of *pseudo-functor*, *pseudo-natural transformation* and *modification* in Appendix A, following [23]. They are exactly the same as 2-functor, 2-natural transformation and modification except for the systematic replacement of equalities between arrows by invertible 2-cells subject to coherence axioms. In writing bicategorical diagrams, one typically suppresses the structural isomorphisms in the definition of bicategory, pseudo-functor and pseudo-natural transformation: the coherence conditions are sufficient to force there to be a unique choice in each case, and quite often, one's data is strict anyway. So we retain that convention in our diagrams here in order to avoid clutter.

Definition 2.2 A *pseudo-monad* on a bicategory C consists of

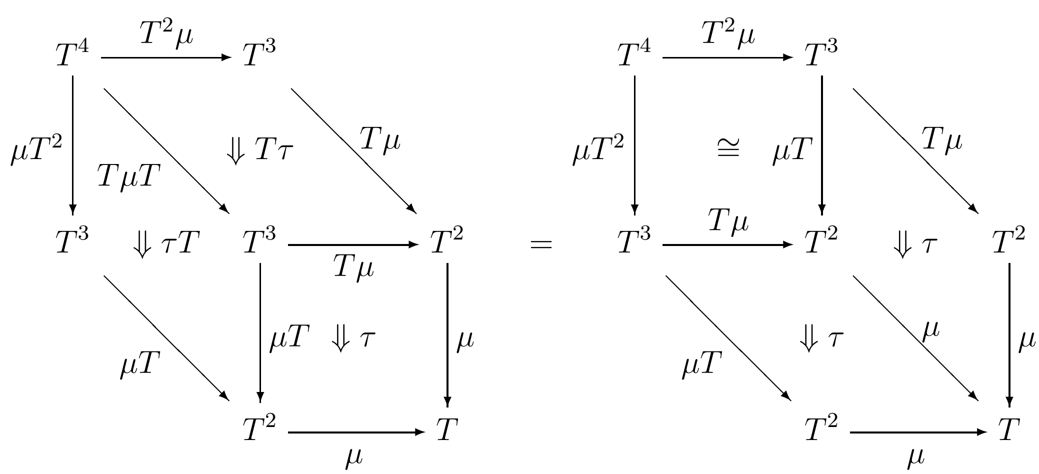
- a pseudo-functor $T : C \longrightarrow C$
- a pseudo-natural transformation $\mu : T^2 \to T$
- a pseudo-natural transformation $\eta : 1 \to T$
- an invertible modification



- invertible modifications



subject to two coherence axioms:



$$\begin{array}{ccccc} T^3 & \xrightarrow{1} & T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle T\eta T}\uparrow & \Downarrow \lambda T & \downarrow{\scriptstyle \mu T} \;\; \Downarrow \tau & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[1]{} & T^2 & \xrightarrow[\mu]{} & T \end{array} \quad = \quad \begin{array}{ccccc} T^3 & \xrightarrow{1} & T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle T\eta T}\uparrow & \Downarrow T\rho & & \nearrow{\scriptstyle 1} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[1]{} & T^2 & \xrightarrow{\mu} & T \end{array}$$

Example 2.3 Any 2-monad yields a pseudo-monad: given a 2-monad, regard the 2-functor trivially as a pseudo-functor and the two 2-natural transformations trivially as pseudo-natural transformations. And take the three invertible modifications to be identities.

Often, as was exploited heavily in [11] then in [2], one starts with a 2-monad on Cat rather than with the more general notion of pseudo-monad. That works far better than one might imagine for the purposes of studying categories with algebraic structure. But there are two ways in which pseudo-monads that are not 2-monads arise naturally, even in that study; and one of those ways is fundamental for us here.

Example 2.4 Consider the process of combining algebraic structures on Cat. One starts with 2-monads S and T, but one often only has a pseudo-distributive law of S over T, not a distributive law in the usual strict sense. For instance, taking S to be the 2-monad on Cat whose algebras are small symmetric monoidal categories, and taking T to be the 2-monad whose algebras are small categories with finite products, no natural choice of data for a distributive law of S over T satisfies the pentagon axiom:

$$\begin{array}{ccccc} S^2T & \xrightarrow{S\lambda} & STS & \xrightarrow{\lambda S} & TS^2 \\ {\scriptstyle \mu_S T}\downarrow & & & & \downarrow{\scriptstyle T\mu_S} \\ ST & & \xrightarrow[\lambda]{\qquad\qquad} & & TS \end{array}$$

But one may prove, by a combination of two of the main results in [6,11], that, making any natural choice of data for λ, this pentagon does commute up to coherent isomorphism. Consequently, although the 2-functor TS does not extend to a 2-monad, it does extend to a pseudo-monad as defined in [14]. This situation is typical, holding in general for pseudo-commutative T [6,7].

Example 2.5 One often has explicit descriptions of the various 2-monads on Cat, equivalently explicit descriptions of their free algebras. For instance, the free category with any class S of colimits on a small category C is given by the closure of C in $[C^{op}, Set]$ under that class of colimits, where C is considered as a subcategory of $[C^{op}, Set]$ via the Yoneda embedding $C \longrightarrow [C^{op}, Set]$ [12].

But that explicit description only agrees with the 2-monad for S-colimits up to equivalence. The explicit description always forms a pseudo-monad but rarely gives a 2-monad.

Example 2.6 The 2-monad on Cat for small categories with finite products extends to a pseudo-monad on $Prof$. Similarly for the 2-monad for small symmetric monoidal categories. These results will both follow, modulo size, from our work on pseudo-distributive laws and their liftings to Kleisli bicategories.

3 The 2-category of pseudo-algebras

In this section, given a pseudo-monad T on a 2-category C, we describe its 2-category Ps-T-Alg of pseudo-algebras. It is straightforward to define the notions of *pseudo-T-algebra*, *pseudo-map* of pseudo-T-algebras, and *algebra 2-cell*. But we shall need to consider delicate variants of the the definition as we proceed through the paper, so we shall give the definition of pseudo-algebra in detail. In the case of a 2-monad T, all the definitions are given compactly, in complete detail and in modern notation, but with one redundant axiom, at the start of [18]. For a pseudo-monad T, the definitions are almost given in [13,14]: there is an indexed version in [14] and there is a version for 2-functors in [13]; but neither paper formally has the definitions in the setting in which we use them here.

We mention, for cognoscenti, that if one adopts the spirit of [13,14], the distinction between 2-functors and pseudo-functors is more significant than it may appear: in order to make the generalisation to pseudo-functors, one uses the fact that the *Gray*-category 2-Cat_p of small 2-categories, pseudo-functors, pseudo-natural transformations, and modifications has a particular *Gray*-limit that might be called a relaxed three-dimensional limit, and such existence requires proof as 2-Cat_p is not complete.

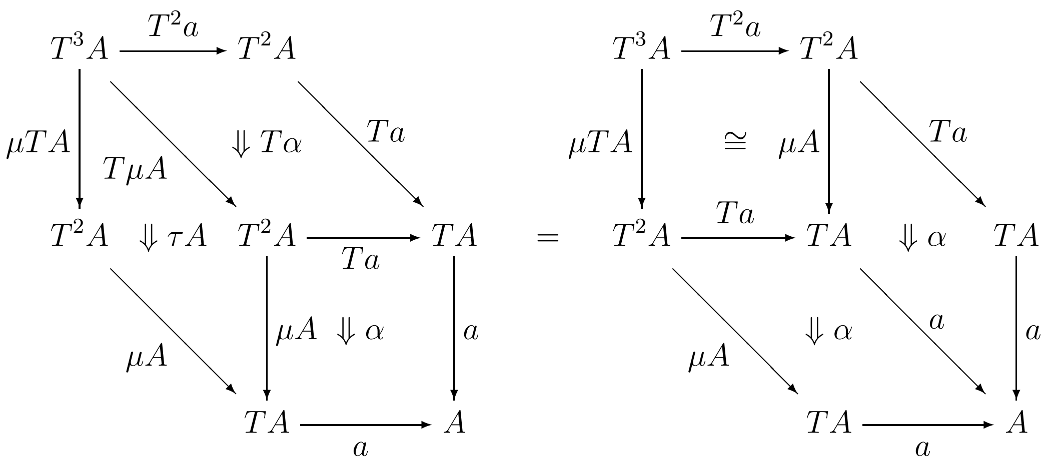
Recall that T has an underlying pseudo-functor that is not necessarily a 2-functor, so in the diagrams here, we are tacitly suppressing the coherence data.

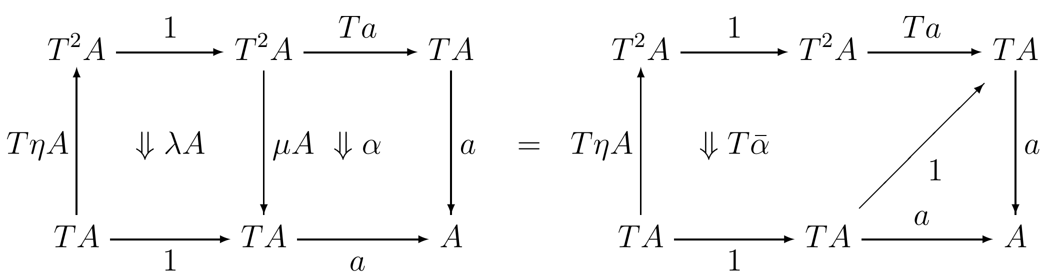
Definition 3.1 A *pseudo-T-algebra* consists of

- an object A of C
- an arrow $a : TA \longrightarrow A$
- invertible 2-cells

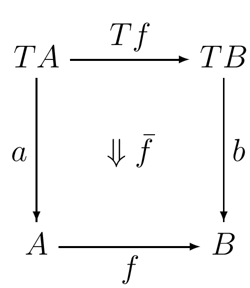
$$\begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu A}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\eta A} & TA \\ & {\scriptstyle 1}\searrow \ \Downarrow \bar{\alpha} & \downarrow{\scriptstyle a} \\ & & A \end{array}$$

subject to two coherence axioms:

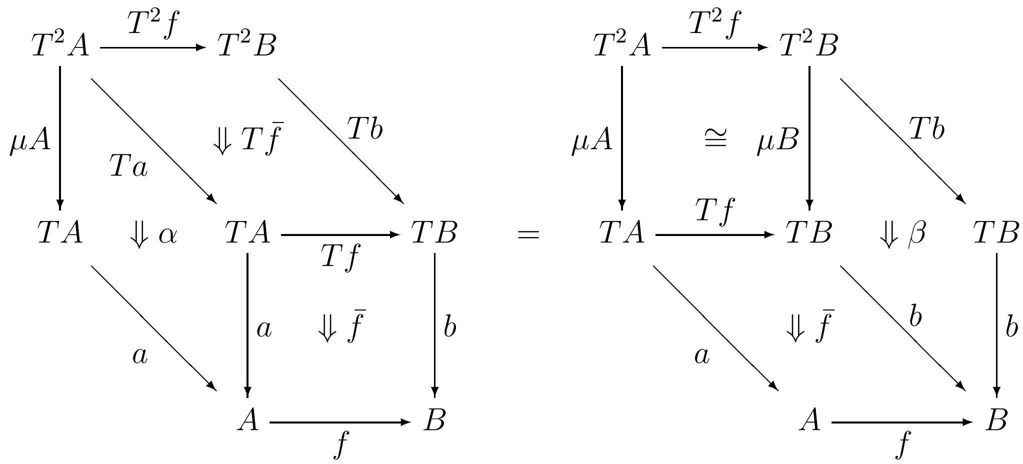


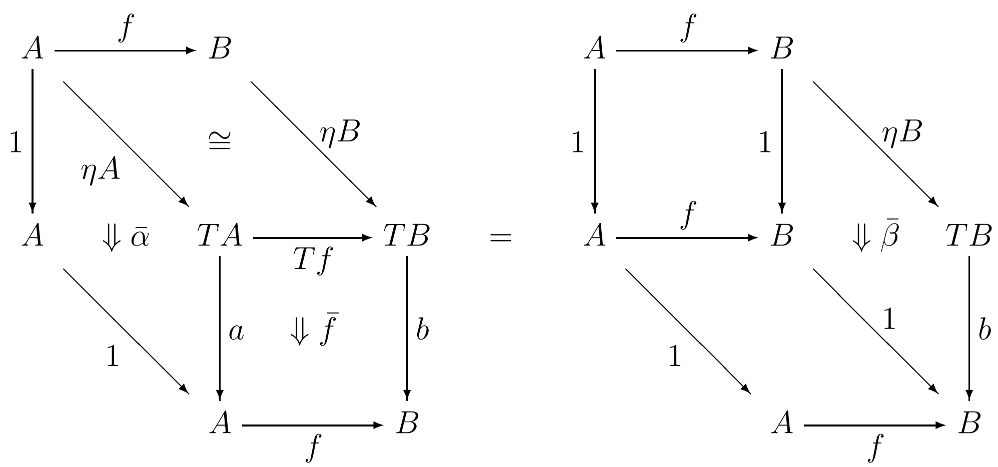


A second identity axiom, one for the composite of α with ηTA follows from these two axioms. A *pseudo-map* of pseudo-T-algebras from $(A, a, \alpha, \bar{\alpha})$ to $(B, b, \beta, \bar{\beta})$ consists of an arrow $f : A \longrightarrow B$ and an invertible 2-cell

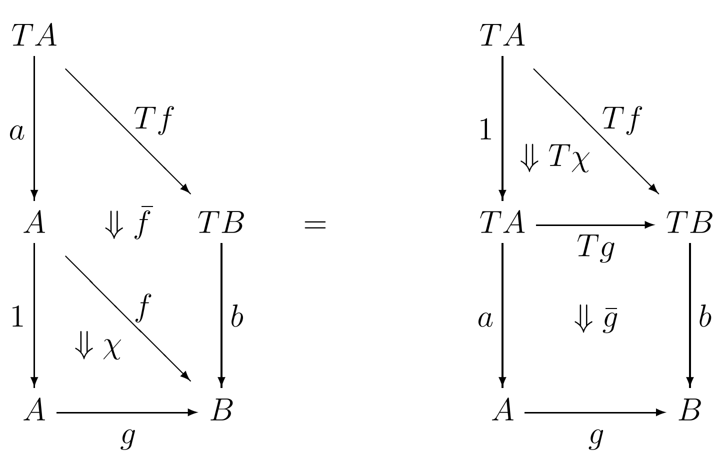


subject to two coherence axioms:





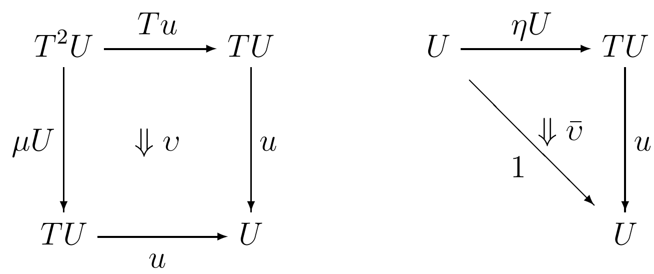
An algebra 2-cell from $(f, \bar{f})$ to $(g, \bar{g})$ is a 2-cell $\chi : f \Rightarrow g$ subject to one coherence axiom:



Observe that, for any 2-category C and any pseudo-monad T on C, there are

- a forgetful pseudo-functor $U : Ps\text{-}T\text{-}Alg \longrightarrow C$

- a canonical pseudo-natural transformation $u : TU \Rightarrow U$
- canonical invertible modifications



satisfying the axioms we demanded in the definition of a pseudo-T-algebra.

Proposition 3.2 *The data $(U, u, \upsilon, \bar{\upsilon})$ are universal, subject to the axioms, among all pseudo-functors with codomain C, pseudo-natural transformations, and invertible modifications, subject to the above-mentioned axioms.*

A proof follows from routine checking. Its significance is that, combined with [13] and the main result of [14], it will allow us to deduce an equivalence between pseudo-distributive laws and liftings of one pseudo-monad to the 2-category of algebras of the other. It also allows us to deduce other results of [13], including those one would reasonably expect of such a construction:

Corollary 3.3 *For any pseudo-monad T on any 2-category C*

- *the 2-category Ps-T-Alg yields a decomposition of T into a pseudo-adjunction.*
- *every pseudo-adjunction gives rise to a pseudo-monad and a comparison pseudo-functor into the induced 2-category of pseudo-algebras.*

Example 3.4 Let $C = Cat$. There are many examples of 2-monads on Cat of interest in theoretical computer science. In all the leading examples, including all those mentioned below, the 2-category Ps-T-Alg is biequivalent to the full sub-2-category determined by the strict T-algebras: see [18] for one of the two main general results to that effect. That full sub-2-category, often denoted T-Alg, was the focus of study of [2], which includes many specific examples. They include: the 2-category $FProd$ of small categories with finite products and functors that preserve finite products; the 2-category $SymMon$ of small symmetric monoidal categories and strong monoidal functors; for any small class S of colimits, the 2-category of small categories with S-colimits and functors that preserve S-colimits; the 2-category for which an object is a small category together with a monad on it; among many others.

Example 3.5 For a base 2-category other than Cat, let C be the 2-category $SymMon$. It will follow from our analysis of pseudo-distributivity that the 2-monad on Cat for small categories with finite products lifts to $SymMon$ and that an object of the 2-category of algebras of the lifting consists of a small symmetric monoidal category with finite products, for which the symmetric monoidal structure distributes over the finite product structure.

Example 3.6 Let $C = FCoprod$, the 2-category of small categories with finite coproducts. Put $TA = FProd(A^{op}, Set)$. For size reasons, TA is not a monad on $FCoprod$, but except for that caveat, $Ps\text{-}T\text{-}Alg$ would be the 2-category of categories with all small colimits. One can modify the description of TA in order to make a precise true statement here by making a size restriction along the lines we have explained above (see, for instance, [3,12]).

Example 3.7 Let $C = FProd$. It will follow from our analysis of pseudo-distributivity that, except for our usual size problem, $TA = [A^{op}, Set]$ would form a pseudo-monad on $FProd$ with pseudo-algebras given by categories with all small colimits and finite products, and with pseudo-maps given by functors that preserve such structure.

4 The Kleisli bicategory

In this section, we develop the notion of the Kleisli bicategory of a pseudo-monad. Except for a size issue addressed in [3], $Prof$ should be the Kleisli bicategory for a pseudo-monad on Cat given by $TA = [A^{op}, Set]$. An analysis of $Prof$ is fundamental to the study of bisimulation using open maps [10]. Winskel also needs a variety of Kleisli bicategory in order to analyse a variety of exponentials [16,24]. So we are keen to define the notion of Kleisli bicategory in a way that includes such variants.

However, it seems to be impossible to define the Kleisli bicategory of a pseudo-monad in a way that satisfies a result dual to Proposition 3.2, thus allowing us to adopt the theory of [13], while giving a construction that includes such leading examples and is easy to handle in practice. In fact, it is unclear whether a dual, in the sense required in [13], exists at all. Even if it does, it seems likely that it would be awkward to describe and it definitely would not have the same simple relationship with the usual Kleisli construction for ordinary categories as that we develop here: our construction is inherently bicategorical, while the general setting of [13,14] is inherently 2-categorical.

Nevertheless, with care, one can generalise the idea in [13], without becoming bogged down in the full generality of three-dimensional colimits in tricategories, to provide a construction that: agrees with the leading examples; is remarkably easy to describe; yet provides a dual up to biequivalence, which is sufficient for the main abstract proof of [13], albeit not the theorem as stated therein, to work. Indeed, the definition looks like an obvious idea for generalising the Kleisli construction: the hard part is to identify its universal property. The reason it is a bicategory rather than a 2-category is because the lifting requires the greater generality of the notion of bicategory in order to define the composition. The construction is surprisingly simple.

Definition 4.1 Given a pseudo-monad $(T, \mu, \eta, \tau, \lambda, \rho)$ on a 2-category C, the *Kleisli bicategory* of T, denoted $Kl(T)$, is defined by putting $Ob(Kl(T)) = ObC$ and $Kl(T)(A, B) = C(A, TB)$, with composition given by the composite:

$$C(B,TD)\times C(A,TB)\longrightarrow C(TB,T^2D)\times C(A,TB)\longrightarrow C(A,T^2D)\longrightarrow C(A,TD)$$

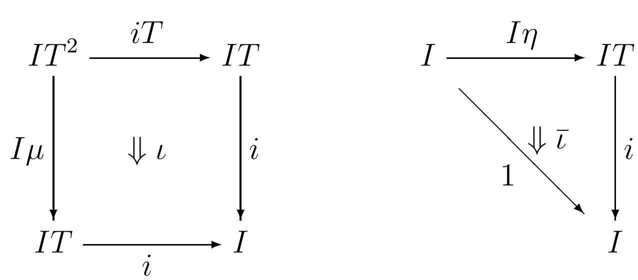
where the arrows in the composite are labelled using T, composition in C and μD respectively, with identities given by:

$$\eta A : A \longrightarrow TA$$

and with the coherence isomorphisms for the bicategorical structure of $Kl(T)$ given by τ, λ and ρ.

We should like to dualise Proposition 3.2 and use it both to deduce the results one would expect of a Kleisli construction and to give an equivalence between pseudo-distributive laws and liftings to Kleisli bicategories: but that duality does not hold of this construction. We can, however, rectify the situation with a little care for coherence. Mimicking the situation for algebras, observe that there are

- a canonical pseudo-functor $I : C \longrightarrow Kl(T)$
- a canonical pseudo-natural transformation $i : IT \Rightarrow I$
- canonical invertible modifications



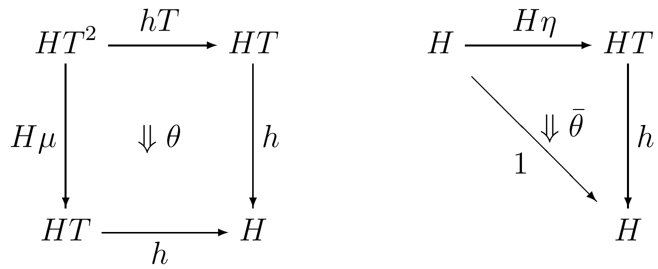
satisfying axioms corresponding to those we demanded in analysing Ps-T-Alg.

When T is a 2-monad, observe that $Kl(T)$ is a 2-category. But $Kl(T)$ is only a bicategory in general, hence the impossibility of dualising Proposition 3.2 directly.

Definition 4.2 For any pseudo-monad T on a 2-category C and for any bicategory B, define the bicategory $Cocone((C,T),B)$ such that an object consists of

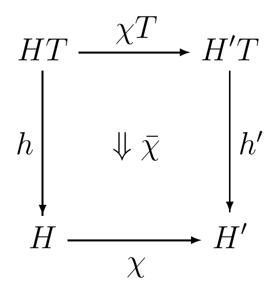
- a pseudo-functor $H : C \longrightarrow B$
- a pseudo-natural transformation $h : HT \longrightarrow H$

- invertible modifications



satisfying axioms as above.

An arrow in $Cocone((C,T),B)$ from $(H,h,\theta,\bar{\theta})$ to $(H',h',\theta',\bar{\theta}')$ consists of a pseudo-natural transformation $\chi : H \longrightarrow H'$ together with an invertible modification



subject to two axioms corresponding to the two axioms in the definition of a pseudo-map of algebras. A 2-cell from (χ,χ') to (ξ,ξ') is given by a modification $\zeta : \chi \Rightarrow \chi'$ subject to one coherence axiom, corresponding to that in the definition of an algebra 2-cell. The composition and identities for the bicategorical structure of $Cocone((C,T),B)$ are induced by those of B. The axioms required to prove that $Cocone((C,T),B)$ is indeed a bicategory follow routinely from the bicategorical axioms of B.

Theorem 4.3 *For any bicategory B, composition with $(I,i,\iota,\bar{\iota})$ induces a biequivalence of bicategories between $Cocone((C,T),B)$ and $Pseudo(Kl(T),B)$, the bicategory of pseudo-functors from $Kl(T)$ to B.*

A proof follows from routine but lengthy checking. The result provides a universal property for the Kleisli bicategory, allowing us to adopt the spirit of the development of [13], albeit with somewhat more subtle definitions or statements. For instance, adopting a natural tricategorical understanding of the notions of decomposition and comparison pseudo-functor, we may deduce the following results:

Corollary 4.4 *For any pseudo-monad T on any 2-category C*

- *the bicategory $Kl(T)$ yields a decomposition of T into a pseudo-adjunction.*
- *every pseudo-adjunction gives rise to a pseudo-monad and a comparison pseudo-functor from the induced Kleisli bicategory.*

Example 4.5 Let S be a small class of colimits. Then the pseudo-monad on Cat for small categories with S-colimits is given by sending a small category A to the free cocompletion of A in $[A^{op}, Set]$, with inclusion the Yoneda embedding $A \longrightarrow [A^{op}, Set]$, under S-colimits. Consequently, the Kleisli bicategory has objects being small categories and an arrow from A to B given by a functor from A to the closure of B in $[B^{op}, Set]$ under S-colimits. This fact is exploited in [3].

5 Pseudo-distributive laws

The central result about distributive laws for ordinary categories and ordinary monads is as follows [1]:

Theorem 5.1 *Given monads S and T on a category C, the following are equivalent:*

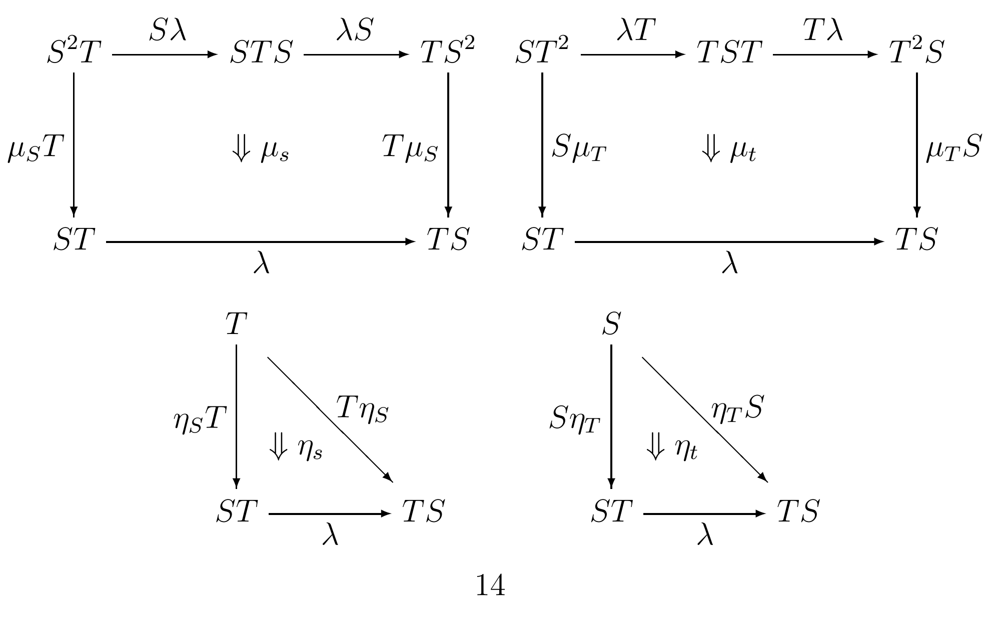
- *a distributive law $\lambda : ST \longrightarrow TS$ of S over T*
- *a lifting of T to S-Alg*
- *a lifting of S to $Kl(T)$*

It also follows from the definition of a distributive law of S over T that TS acquires a canonical monad structure, its category of algebras agrees with that for the lifting of T, and dually for the Kleisli construction.

As we have mentioned earlier, distributive laws rarely exist for 2-monads, even less for pseudo-monads. So one seeks an appropriate weakening of the notion, and Marmolejo's definition is definitive, given as follows:

Definition 5.2 A *pseudo-distributive law* of a pseudo-monad S over a pseudo-monad T consists of

- a pseudo-natural transformation $\lambda : ST \longrightarrow TS$
- invertible modifications



subject to nine coherence axioms: they are not complicated and they are spelt out clearly in [14], but we do not have space here to list them.

We have described examples, with forward references, through the course of the paper, so we do not give them in detail here. Leading ones are generated by the 2-monads on *Cat* for small categories with finite products and for small symmetric monoidal categories over the 2-monad for the free cocompletion of a small category under a small class of colimits. A non-example is given by replacing finite products by finite coproducts here, as the Yoneda embedding does not preserve finite coproducts. Another leading class of examples is given as follows:

Proposition 5.3 *Every pseudo-commutative monad T on Cat gives rise to a pseudo-distributive law of S, the 2-monad for small symmetric monoidal categories, over T.*

A proof follows from one of the main results from each of [6,7] and [11]. The main theorem about pseudo-monads mimics that for ordinary monads. To understand this result, one must adopt the bicategorical sense of equivalence as explained in the previous section.

Theorem 5.4 *Given pseudo-monads T and S on a 2-category C, the following are equivalent:*

- *a pseudo-distributive law of S over T*
- *a lifting T' of T to Ps-S-Alg*
- *a lifting S' of S to $Kl(T)$*

Proof. The equivalence of the first two items here can be deduced by combining [14] with [13] and Proposition 3.2. The equivalence of the first and last items follows from Theorem 4.3 together with a dual of the proof for algebras but with more delicacy systematically taken to account for the weaker notion of equivalence of Theorem 4.3. □

It is also shown in [14] that TS acquires the structure of a pseudo-monad.

Corollary 5.5 *For a pseudo-distributive law of S over T:*

- *Ps-T'-Alg is biequivalent to Ps-TS-Alg*
- *$Kl(S')$ is biequivalent to $Kl(TS)$*

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A Pseudo-functors, pseudo-natural transformations, and modifications

Definition A.1 Given bicategories C and D, a *pseudo-functor* from C to D consists of

- a function $H : ObC \longrightarrow ObD$
- for each pair of objects (A, B) of C, a functor $H : C(A, B) \longrightarrow D(HA, HB)$
- for each triple (A, B, E), an invertible natural transformation

$$\begin{array}{ccc}
C(B,E) \times C(A,B) & \xrightarrow{H \times H} & D(HB,HE) \times D(HA,HB) \\
\circ \big\downarrow & \Downarrow h & \big\downarrow \circ \\
C(A,E) & \xrightarrow[H]{} & D(HA,HE)
\end{array}$$

- for each object A, an invertible 2-cell $\bar{h}_A : 1_{HA} \Rightarrow H(1_A)$, equivalently an invertible natural transformation

$$\begin{array}{ccc}
1 & & \\
1_A \big\downarrow & \Downarrow \bar{h} & \searrow 1_{HA} \\
C(A,A) & \xrightarrow[H]{} & D(HA,HA)
\end{array}$$

subject to the coherence of three diagrams, in which we suppress the coherence data for the two bicategories:

$$\begin{array}{ccc}
C(E,F) \times C(B,E) \times C(A,B) & \xrightarrow{H \times H \times H} & D(HE,HF) \times D(HB,HE) \times D(HA,HB) \\
{\scriptstyle 1 \times \circ}\downarrow & \Downarrow H \times h & \downarrow{\scriptstyle 1 \times \circ} \\
C(E,F) \times C(A,E) & \xrightarrow[H \times H]{} & D(HE,HF) \times D(HA,HE) \\
{\scriptstyle \circ}\downarrow & \Downarrow h & \downarrow{\scriptstyle \circ} \\
C(A,F) & \xrightarrow[H]{} & D(HA,HF)
\end{array}$$

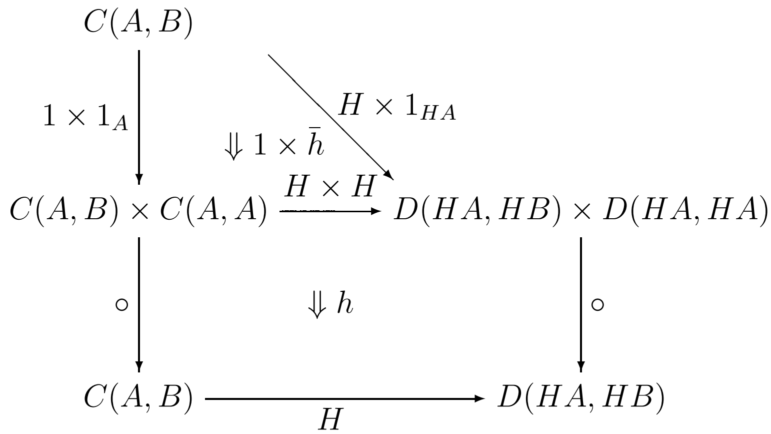
must equal

$$\begin{array}{ccc}
C(E,F) \times C(B,E) \times C(A,B) & \xrightarrow{H \times H \times H} & D(HE,HF) \times D(HB,HE) \times D(HA,HB) \\
{\scriptstyle \circ \times 1}\downarrow & \Downarrow h \times H & \downarrow{\scriptstyle \circ \times 1} \\
C(B,F) \times C(A,B) & \xrightarrow[H \times H]{} & D(HB,HF) \times D(HA,HB) \\
{\scriptstyle \circ}\downarrow & \Downarrow h & \downarrow{\scriptstyle \circ} \\
C(A,F) & \xrightarrow[H]{} & D(HA,HF)
\end{array}$$

and both

$$\begin{array}{ccc}
C(A,B) & & \\
{\scriptstyle 1_B \times 1}\downarrow & \Downarrow \bar{h} \times 1 \quad \searrow{\scriptstyle 1_{HB} \times H} & \\
C(B,B) \times C(A,B) & \xrightarrow{H \times H} & D(HB,HB) \times D(HA,HB) \\
{\scriptstyle \circ}\downarrow & \Downarrow h & \downarrow{\scriptstyle \circ} \\
C(A,B) & \xrightarrow[H]{} & D(HA,HB)
\end{array}$$

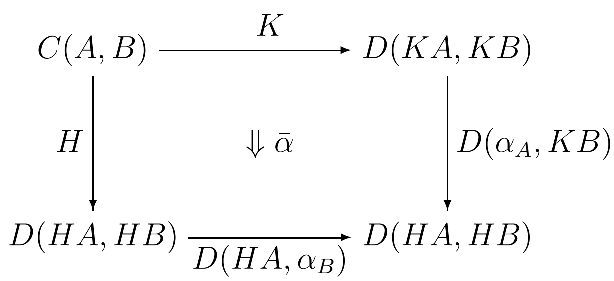
and



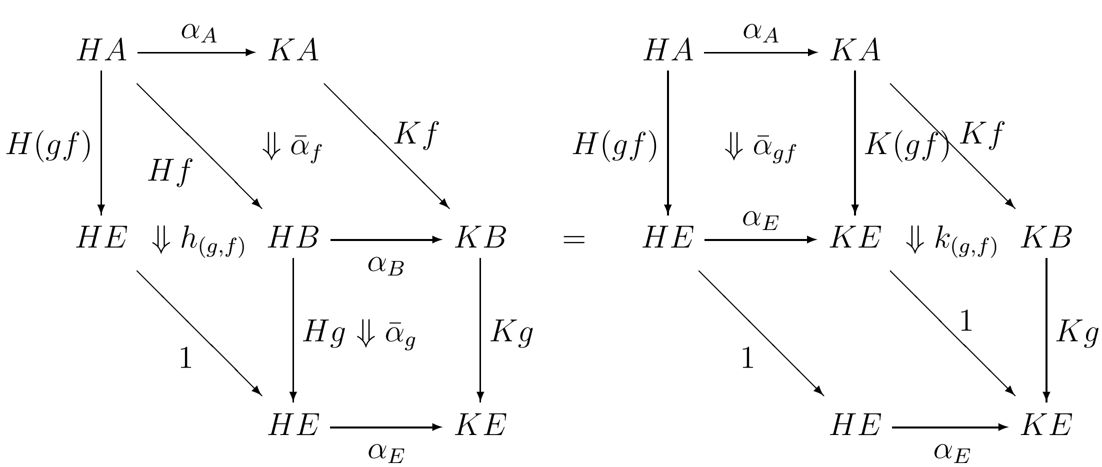
must be identities.

Definition A.2 A *pseudo-natural transformation* from $(H, h, \bar{h})$ to $(K, k, \bar{k})$ consists of

- for each object A, an arrow $\alpha_A : HA \longrightarrow KA$
- for each pair (A, B) a natural transformation



such that for every pair of composable arrows $(f : A \longrightarrow B, g : B \longrightarrow E)$, suppressing the coherence data in the definition of bicategory:



and

$$\begin{array}{ccc} HA & \xrightarrow{\alpha_A} & KA \\ {\scriptstyle 1}\downarrow & \Downarrow \bar{\alpha}_1 & \downarrow{\scriptstyle 1} \\ HA & \xrightarrow[\alpha_A]{} & KA \end{array}$$

is the identity.

Definition A.3 A *modification* from $(\alpha, \bar{\alpha})$ to $(\beta, \bar{\beta})$ consists of, for every object A, a 2-cell $\chi_A : \alpha_A \Rightarrow \beta_A$ such that for every arrow $f : A \longrightarrow B$, suppressing the bicategorical coherence data, we have:

$$\begin{array}{ccc} HA & & \\ {\scriptstyle Hf}\downarrow & \searrow^{\alpha_A} & \\ HB & \Downarrow \bar{\alpha}_f & KA \\ {\scriptstyle 1}\downarrow\ \Downarrow \chi_B & \searrow^{\alpha_B} & \downarrow{\scriptstyle Kf} \\ HB & \xrightarrow[\beta_B]{} & KB \end{array} \quad = \quad \begin{array}{ccc} HA & & \\ {\scriptstyle 1}\downarrow\ \Downarrow \chi_A & \searrow^{\alpha_A} & \\ HA & \xrightarrow[\beta_A]{} & KA \\ {\scriptstyle Hf}\downarrow & \Downarrow \bar{\beta}_f & \downarrow{\scriptstyle Kf} \\ HB & \xrightarrow[\beta_B]{} & KB \end{array}$$