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Intensional Double Glueing, Biextensional Collapse, and the Chu Construction

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Abstract

The superficial similarity between the Chu construction and the Hyland-Tan double glueing construction **G** has been observed widely. This paper establishes a more formal mathematical relationship between the two.

We show that double glueing on relations subsumes the Chu construction on sets: we present a full monoidal embedding of the category $\mathbf{chu}(\mathsf{Set}, K)$ of biextensional Chu spaces over K into $\mathbf{G}(\mathsf{Rel}^K)$, and a full monoidal embedding of the category $\mathbf{Chu}(\mathsf{Set}, K)$ of Chu spaces over K into $\mathbf{IG}(\mathsf{Rel}^K)$, where we define **IG**, the *intensional double glueing construction*, by substituting multisets for sets in **G**.

We define a biextensional collapse from **IG** to **G** which extends the familiar notion on Chu spaces. This yields a new interpretation of the monic specialisation implicit in **G** as a form of biextensionality.

1 Introduction

The **Chu** construction [2] and the Hyland-Tan double glueing construction **G** [21] have each produced models of multiplicative linear logic [8] which are fully complete in the sense of Abramsky and Jagadeesan [1]. The following superficial similarity between the two constructions has been observed widely. Each starts with a category $\mathbb{C}$ (with appropriate structure) and builds a star-autonomous category $\mathbb{C}'$ (hence a model of multiplicative linear logic [20]) in which:

- An object of $\mathbb{C}'$ possesses 'points' and 'copoints' in $\mathbb{C}$.
- Duality in $\mathbb{C}'$ interchanges points and copoints.
- A morphism in $\mathbb{C}'$ transforms points forwards and copoints backwards.

This paper establishes a more formal mathematical relationship between the two constructions.

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Relating the Chu construction and double glueing — ***Fig. 1***

$$\begin{tikzcd}
\mathbf{Chu}(\mathsf{Set}, K) \arrow[r, hook, "F^{+}"] \arrow[d, shift left, "\sim"] & \mathbf{IG}(\mathsf{Rel}^{K}) \arrow[d, shift left, "\sim"] \\
\mathbf{chu}(\mathsf{Set}, K) \arrow[r, hook, "F"] \arrow[u, hook, shift left] & \mathbf{G}(\mathsf{Rel}^{K}) \arrow[u, hook, shift left]
\end{tikzcd}$$

“ **Chu** $=$ **chu** $+$ multiplicity ”

“ **IG** $=$ **G** $+$ multiplicity ”

“ F^{+} $=$ F $+$ multiplicity ”

Both squares in the diagram commute, hooked arrows are full embeddings, and $(\sim)$ denotes biextensional collapse. The full embeddings F and F^{+} are monoidal. See main text for details.

Together with the commentary below, Figure 1 summarises the results presented in this paper. The first result corresponds to the lower edge in the commuting square of Figure 1:

(1) *The Hyland-Tan double glueing construction* $\mathbf{G}$ *on relations subsumes the biextensional Chu construction* **chu** *on sets.*

More precisely, we define (section 3) a full monoidal embedding F of $\mathbf{chu}(\mathsf{Set}, K)$ into $\mathbf{G}(\mathsf{Rel}^{K})$, where Set is the category of sets, K is any set, and Rel is the category of sets and binary relations. (In writing the functor category Rel^{K} we view K as a discrete category.) The biextensional Chu construction **chu** (lower case, following Pratt [17] and Barr [4]), is the **Chu** construction followed by restriction to biextensional (*i.e.*, separated and extensional) objects [4,5]. Following Pratt (*e.g.* [17]), the objects of $\mathbf{Chu}(\mathsf{Set}, K)$, and hence also of the biextensional full subcategory $\mathbf{chu}(\mathsf{Set}, K)$, are commonly known as *Chu spaces over* K. Figure 2 (overleaf) sketches the idea behind the embedding with a simple example.

The remaining results stem from the point of view captured by the slogan

“ **Chu** $=$ **chu** $+$ multiplicity ”

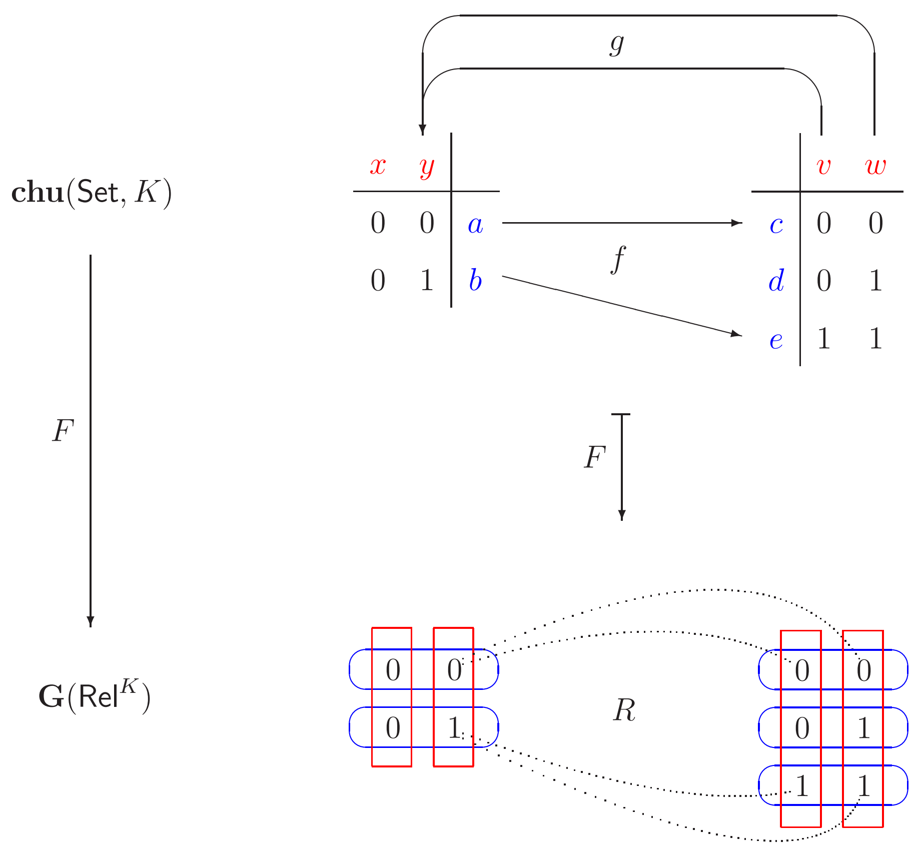
or “every Chu space can be viewed as an underlying biextensional Chu space together with multiplicity information”. To clarify our slogan, consider the Chu space $\mathcal{A}$ below-left (drawn according to the conventions of Figure 2):

	x	y	z
a	0	0	0
b	0	0	0
c	0	0	0
d	0	1	1
e	0	1	1

$\rightsquigarrow$

	$\{x\}$	$\{y, z\}$
$\{a, b, c\}$	0	0
$\{d, e\}$	0	1

Sketch of $F : \mathbf{chu}(\mathsf{Set}, K) \hookrightarrow \mathbf{G}(\mathsf{Rel}^K)$ — ***Fig. 2***



Top-right is a morphism (f, g) from a Chu space $\mathcal{A}$ (with two points a, b and two copoints (or states) x, y) to a Chu space $\mathcal{B}$ (with three points c, d, e and two copoints v, w). The matrix (or pairing) $\langle -, - \rangle$ of each is given by the tables (*e.g.* $\langle a, y \rangle = 0$, $\langle e, w \rangle = 1$; assume $0, 1 \in K$.) The graphs of f (forwards) and g (backwards) are shown (*e.g.* $f(b) = e$, $g(w) = y$).
Underneath is the image of $(f, g) : \mathcal{A} \to \mathcal{B}$ in $\mathbf{G}(\mathsf{Rel}^K)$ under F, a binary relation R between the four tokens of $F(\mathcal{A})$ (three 0-tokens and one 1-token) and the six tokens of $F(\mathcal{B})$ (three 0-tokens and three 1-tokens). The four edges of R are shown curved and dotted. By definition, an object of $\mathbf{G}(\mathbb{C})$ has 'values' and 'covalues', which in this case ($\mathbb{C} = \mathsf{Rel}^K$) are sets of tokens. Values are circumscribed by a rounded border, and covalues by an oblong border. (Thus $F(\mathcal{A})$ has two values and two covalues (each with two tokens), and $F(\mathcal{B})$ has three values (each with two tokens) and two covalues (each with three tokens)).
The definition of F on objects is as follows: matrix entries become tokens, matrix rows (resp. columns) become values (resp. covalues). On morphisms: R is a 'conjunction' of f and g (*e.g.* R relates the top-right token of $F(\mathcal{A})$ to the top-right token of $F(\mathcal{B})$ because $f(a) = c$ ("f(top) = top") and $g(w) = y$ ("g(right) = right")).

Since $\mathcal{A}$ has duplicate rows/columns (*e.g.* two copies of row 011), it is not biextensional. The slogan "**Chu** = **chu** + multiplicity" expresses the idea that $\mathcal{A}$ can be encoded as in the table to the right of $\mathcal{A}$: a biextensional Chu space $\widetilde{\mathcal{A}}$ (two distinct rows (00 and 01) and two distinct columns), with information regarding multiplicity (*i.e.*, regarding duplication of rows and columns in the original Chu space $\mathcal{A}$).

With this perspective in mind, we observe (section 3.4) that the full monoidal embedding $F : \mathbf{chu}(\mathsf{Set}, K) \hookrightarrow \mathbf{G}(\mathsf{Rel}^K)$ seems not to extend in any obvious way to a full embedding of the whole of $\mathbf{Chu}(\mathsf{Set}, K)$ into $\mathbf{G}(\mathsf{Rel}^K)$ because there is not enough 'room' in $\mathbf{G}$ to retain multiplicity information from **Chu**. To make room, we define (section 4) a variant **IG** of **G**, which we call the *intensional double glueing construction*, by analogy with our earlier Chu slogan:

$$\mathbf{IG} \quad = \quad \mathbf{G} \quad + \quad \text{multiplicity.}$$

We follow this prescription literally: where **G** attaches *sets* of points and copoints, **IG** attaches *multisets* of points and copoints. Our second result (section 5) corresponds to the upper edge in the commuting square of Figure 1:

(2) *The intensional double glueing construction* **IG** *on relations subsumes the* **Chu** *construction on sets.*

Specifically, we extend $F : \mathbf{chu}(\mathsf{Set}, K) \hookrightarrow \mathbf{G}(\mathsf{Rel}^K)$ to a full monoidal embedding $F^+ : \mathbf{Chu}(\mathsf{Set}, K) \hookrightarrow \mathbf{IG}(\mathsf{Rel}^K)$ in three steps, as illustrated by the dotted arrows in Figure 3, overleaf: given a Chu space $\mathcal{A}$, to define $F^+(\mathcal{A})$ first discard multiplicity by taking the biextensional collapse $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ (the downward dotted arrow in Figure 3), then use F to embed $\widetilde{\mathcal{A}}$ into $\mathbf{G}(\mathsf{Rel}^K)$ (the left-to-right dotted arrow), and finally restore any multiplicities that were present before the collapse (the upward dotted arrow), using the multisets available in $\mathbf{IG}(\mathsf{Rel}^K)$. (Note that the upwards "restore multiplicity" arrow is not a functor, since it uses multiplicity information about $\mathcal{A}$ back in $\mathbf{Chu}(\mathsf{Set}, K)$.) Thus the definition of F^+ adheres to the pattern of our previous two slogans:

$$F^+ \;=\; F \;+\; \text{multiplicity.}$$

In section 4.1 we define a functorial *biextensional collapse* ($\widetilde{\ }$) from **IG** to **G** by collapsing multisets to sets. Our third and final result (section 6) is:

(3) *Biextensional collapse on double glued categories subsumes the extant notion on Chu spaces.*

This corresponds to the square in Figure 1 commuting from top-left to bottom-right.

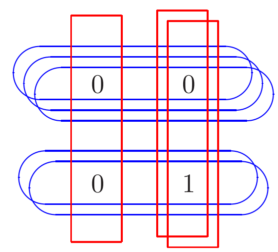
Biextensional versus intensional models. In the language of Hyland and Schalk's comprehensive study of glueing and orthogonality for models of linear logic [9], **IG** is the result of omitting the monic specialisation implicit in **G**. Hence our results provide a new interpretation of this monic specialisation as a

Defining F^+ by "$F^+ = F +$ multiplicity" — ***Fig. 3***

$$\begin{tikzcd}
\mathbf{Chu}(\mathsf{Set}, K) \arrow[r, hook, "F^+"] \arrow[d, dotted, "\text{discard multiplicity}"'] & \mathbf{IG}(\mathsf{Rel}^K) \\
\mathbf{chu}(\mathsf{Set}, K) \arrow[r, dotted, "F"] & \mathbf{G}(\mathsf{Rel}^K) \arrow[u, dotted, "\text{restore multiplicity}"']
\end{tikzcd}$$

	x	y	z
a	0	0	0
b	0	0	0
c	0	0	0
d	0	1	1
e	0	1	1

$\xmapsto{F^+}$

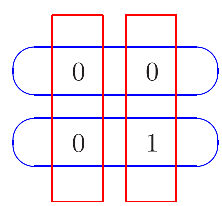


discard multiplicity

restore multiplicity

	$\{x\}$	$\{y, z\}$
$\{a, b, c\}$	0	0
$\{d, e\}$	0	1

$\overset{F}{\dashrightarrow}$



form of biextensionality. Correspondingly, one can view models of linear logic constructed using **G**, such as those of Hyland, Schalk, and Tan [21,9], and the work in progress of Blute, Hamano and Scott on double glued hypercoherences [7], as *biextensional*. In constrast, the Chu space model of [6] can be seen as *intensional* (non-biextensional).

This distinction between biextensional and intensional models of linear logic suggests further avenues for research, such as exploring the intensional counterparts of the aforementioned models based on **G**, and the general relationship between intensional and biextensional linear logical structure.

2 Background material

2.1 Chu spaces

A Chu space over a set K is an object of the category $\mathbf{Chu}(\mathsf{Set}, K)$, the result of applying Barr's construction **Chu** [2] to the category of sets with the set K as dualising object. Chu spaces have a remarkably rich structure, even in the simple case of K a two-element set $2 = \{0, 1\}$. Lafont and Streicher [12] made the elegant observation that the category of topological spaces embeds fully into $\mathbf{Chu}(\mathsf{Set}, 2)$, as does Girard's category of coherence spaces [8]. Via logical relations, $\mathbf{Chu}(\mathsf{Set}, 2)$ yields a fully complete model for multiplicative linear logic [6]. Pratt has uncovered connections between Chu spaces and a wide variety of fields [15,16,18,19].

The structure of the star-autonomous category $\mathbf{Chu}(\mathsf{Set}, K)$ is as follows. We adopt much of Pratt's terminology and notation [17].

- *Objects.* Triples (A, r, X), where A a set of ***points***, X a set of ***copoints*** [1], and r is a function $A \times X \to K$, the ***matrix*** or ***pairing***.
- *Morphisms.* A morphism $(A, r, X) \to (B, s, Y)$ is a pair (f, g) of functions $f : A \to B$ and $g : Y \to X$ satisfying the ***adjointness condition***:

$$r(a, g(y)) \;=\; s(f(a), y) \quad \text{for all points } a \in A \text{ and copoints } y \in Y.$$

- *Duality.* $(A, r, X)^{\perp} = (X, r^{\circ}, A)$, where $r^{\circ}(x, a) = r(a, x)$.
- *Internal hom and tensor.* Given $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$,

$$\begin{aligned}
\mathcal{A} \multimap \mathcal{B} &= \big(\hom(\mathcal{A}, \mathcal{B}), \mathsf{ev}, A \times Y\big) \\
\mathsf{ev}\big((f, g), \langle a, y\rangle\big) &= r(a, g(y)) \;\big(= s(f(a), y)\big) \in K \\
\mathcal{A} \otimes \mathcal{B} &= \big(A \times B, \mathsf{ev}^{\circ}, \hom(\mathcal{A}, \mathcal{B}^{\perp})\big) \\
\mathsf{ev}^{\circ}\big(\langle a, b\rangle, (h, k)\big) &= r(a, k(b)) \;\big(= s(b, h(a))\big) \in K
\end{aligned}$$

- *Tensor unit.* $(1, \pi_K, K)$, where $1 = \{0\}$ and $\pi_K(0, k) = k \in K$.

2.1.1 Biextensional collapse

Fix a Chu space $\mathcal{A} = (A, r, X)$ over K. Points $a, a' \in A$ are ***equivalent***, denoted $a \sim a'$, if they are indistinguishable in terms of their interaction with copoints: $r(a, x) = r(a', x)$ for all copoints $x \in X$. For example, points a and c of the 5-by-3 Chu space in Figure 3 (page 5) are equivalent. Dually, copoints $x, x' \in X$ are equivalent, also denoted $x \sim x'$, if they are indistinguishable in terms of their interaction with points: $r(a, x) = r(a, x')$ for all points $a \in A$. For example, copoints y and z of the 5-by-3 Chu space in Figure 3 are equivalent. We say that $\mathcal{A}$ is ***separated*** if it has no distinct equivalent points ($a \sim a'$

[1] We deviate from Pratt's terminology *states* in order to emphasise the duality.

only if $a = a'$), ***extensional*** if it has no distinct equivalent copoints ($x \sim x'$ only if $x = x'$), and ***biextensional*** if it is separated and extensional. (This terminology is in line with the view of $\mathbf{Chu}(\mathsf{Set}, 2)$ as a category of generalised topological spaces.) Following Pratt [17] and Barr [4], we write $\mathbf{chu}(\mathsf{Set}, K)$ for the biextensional full subcategory of $\mathbf{Chu}(\mathsf{Set}, K)$.

Write $\widetilde{q}$ for the $\sim$-equivalence class of a point or copoint q, and write $\widetilde{A}$ and $\widetilde{X}$ for the quotients of A and X by $\sim$, *i.e.*,

$$\begin{aligned} \widetilde{A} \quad &= \quad A/{\sim} \quad \equiv \quad \{\, \widetilde{a} \,:\, a \in A \,\} \\ \widetilde{X} \quad &= \quad X/{\sim} \quad \equiv \quad \{\, \widetilde{x} \,:\, x \in X \,\}. \end{aligned}$$

Define the quotient $\widetilde{r} : \widetilde{A} \times \widetilde{X} \to K$ of $r : A \times X \to K$ by $\widetilde{r}\,(\widetilde{a}, \widetilde{x}\,) = r(a, x)$. (Overloading the tilde notation $\widetilde{\alpha}$ streamlines our later presentation of the main results. Pronounce $\widetilde{\alpha}$ as "collapse α", whatever the type of α.) The ***biextensional collapse*** of a Chu space $\mathcal{A} = (A, r, X)$ is

$$\widetilde{\mathcal{A}} \quad = \quad (\widetilde{A}, \widetilde{r}, \widetilde{X}).$$

For example, the bottom-left (2-by-2) Chu space of Figure 3 (page 5) is the biextensional collapse of the 5-by-3 Chu space above it. Biextensional collapse is functorial from $\mathbf{Chu}(\mathsf{Set}, K)$ to $\mathbf{chu}(\mathsf{Set}, K)$: given $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$ in $\mathbf{Chu}(\mathsf{Set}, K)$ and a morphism $m = (f, g)$ from $\mathcal{A}$ to $\mathcal{B}$, the biextensional collapse of m is $\widetilde{m} = (\widetilde{f}, \widetilde{g}) : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$, where $\widetilde{f}\,(\widetilde{a}) = \widetilde{f(a)}$ and $\widetilde{g}(\widetilde{y}) = \widetilde{g(y)}$.

The biextensional full subcategory $\mathbf{chu}(\mathsf{Set}, K)$ of $\mathbf{Chu}(\mathsf{Set}, K)$ is star-autonomous with tensor $\widetilde{\otimes}$ and internal hom $\widetilde{\multimap}$ inherited from $\mathbf{Chu}(\mathsf{Set}, K)$ by biextensional collapse: $\mathcal{A} \mathbin{\widetilde{\otimes}} \mathcal{B} = \widetilde{\mathcal{A} \otimes \mathcal{B}}$ and $\mathcal{A} \mathbin{\widetilde{\multimap}} \mathcal{B} = \widetilde{\mathcal{A} \multimap \mathcal{B}}$.

2.2 The Hyland-Tan double glueing construction G

The double glueing construction $\mathbf{G}$, abstracting Loader's category LLP of linear logical predicates [11], was suggested by Hyland and developed in Tan's Ph.D. thesis [21]. When applied to a compact closed category $\mathbb{C}$ [10], the construction produces a star-autonomous category $\mathbf{G}(\mathbb{C})$ with a more refined structure: $\mathbf{G}(\mathbb{C})$ has a distinct tensor and par, and supports the mix rule iff (in $\mathbb{C}$) the identity is the only morphism $I \to I$. Tan shows how full completeness proofs for $\mathbf{G}(\mathbb{C})$ can be 'lifted' from the underlying compact closed category $\mathbb{C}$. In particular, she obtains a more abstract proof of Loader's full completeness result for $\mathsf{LLP} \cong \mathbf{G}(\mathsf{Rel})$.

Let $\mathbb{C}$ be a star-autonomous category with tensor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, tensor unit I, duality $(-)^{\perp} : \mathbb{C}^{\mathrm{op}} \to \mathbb{C}$, and internal hom $\multimap\, : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$. Let $\perp = I^{\perp}$, the dual of the tensor unit. For any object U of $\mathbb{C}$, define a U***-value*** to be a morphism $I \to U$ and a U***-covalue*** to be a morphism $U \to \perp$, and write

$$(-)_* \equiv \mathbb{C}(I,-) : \mathbb{C} \to \mathsf{Set} \qquad \text{(the } \textit{\textbf{values}} \text{ functor)}$$

$$(-)^* \equiv \mathbb{C}(-,\perp) : \mathbb{C} \to \mathsf{Set}^{\mathrm{op}} \qquad \text{(the } \textit{\textbf{covalues}} \text{ functor)}$$

Thus U_* is the set of U-values and U^* is the set of U-covalues. A morphism $R : U \to V$ maps values forwards (as R_*) and covalues backwards (as R^*):

$$U_* \xrightarrow{R_*} V_* \qquad\qquad R_*(a) = R \circ a$$

$$U^* \xleftarrow[R^*]{} V^* \qquad\qquad R^*(y) = y \circ R$$

$$\begin{tikzcd}
& I \arrow[dl, "a"'] \arrow[dr, "R_*(a)"] & \\
U \arrow[rr, "R"] \arrow[dr, "R^*(y)"'] & & V \arrow[dl, "y"] \\
& \perp &
\end{tikzcd}$$

The star-autonomous structure of $\mathbf{G}(\mathbb{C})$ is as follows.

- *Objects.* Triples (U, A, X) where
  - $\cdot$ U is an object of $\mathbb{C}$,
  - $\cdot$ A is a set of U-values (*i.e.*, $A \subseteq U_* = \mathbb{C}(I,U)$),
  - $\cdot$ X is a set of U-covalues (*i.e.*, $X \subseteq U^* = \mathbb{C}(U,\perp)$).
- *Morphisms.* A morphism $(U, A, X) \to (V, B, Y)$ is a morphism $R : U \to V$ in $\mathbb{C}$ such that $R_*(A) \subseteq B$ and $R^*(Y) \subseteq X$ (*i.e.*, such that $R_*(a) \in B$ whenever $a \in A$, and $R^*(y) \in X$ whenever $y \in Y$).
- *Duality.* $(U, A, X)^\perp = (U^\perp, X, A)$, modulo $(U^\perp)_* \cong U^*$ and $(U^\perp)^* \cong U_*$.
- *Internal hom and tensor.* Given $\mathcal{A} = (U, A, X)$ and $\mathcal{B} = (V, B, Y)$,

$$\mathcal{A} \multimap \mathcal{B} \;=\; \big(U \multimap V,\ \hom(\mathcal{A},\mathcal{B}),\ A \otimes Y^\perp\big)$$

$$\mathcal{A} \otimes \mathcal{B} \;=\; \big(U \otimes V,\ A \otimes B,\ \hom(\mathcal{A},\mathcal{B}^\perp)\big)$$

  where

$$\begin{aligned}
\hom(\mathcal{A},\mathcal{B}) &\subseteq \mathbb{C}(U,V) \;\cong\; (U \multimap V)_* \\
A \otimes Y^\perp &= \{\, a \otimes y^\perp \,:\, a \in A \text{ and } y \in Y \,\} \\
&\subseteq \mathbb{C}(I \otimes I, U \otimes V^\perp) \;\cong\; (U \multimap V)^* \\
A \otimes B &= \{\, a \otimes b \,:\, a \in A \text{ and } b \in B \,\} \\
&\subseteq \mathbb{C}(I \otimes I, U \otimes V) \;\cong\; (U \otimes V)_* \\
\hom(\mathcal{A},\mathcal{B}^\perp) &\subseteq \mathbb{C}(U, V^\perp) \;\cong\; (U \otimes V)^*.
\end{aligned}$$

- *Tensor unit.* $\big(I,\ \{\mathsf{id}_I\},\ I^* = \mathbb{C}(I,\perp)\,\big)$.

The original presentation of $\mathbf{G}$ in [21] applied to a compact closed category $\mathbb{C}$; the above generalisation to star-autonomous $\mathbb{C}$ is immediate.

Proposition 2.1 $\mathbf{G}(\mathbb{C})$ *is star-autonomous, with the above structure.*

Proof. See Hyland and Schalk [9], pages 28–9. $\square$

3 The biextensional full monodial embedding

In this section we present a full and faithful monoidal functor

$$F \;:\; \mathbf{chu}(\mathsf{Set}, K) \;\to\; \mathbf{G}(\mathsf{Rel}^K).$$

This functor runs from left to right in Figure 1 on page 2. We first review the (degenerate, compact closed) star-autonomous structure of Rel^K (section 3.1), then calculate $\mathbf{G}(\mathsf{Rel}^K)$ (section 3.2). The embedding is defined in section 3.3.

3.1 The category Rel^K of relations over K

In writing Rel^K we interpret the set K as a discrete category. To smooth the presentation of our main results, we identify an object $\langle U_k\rangle_{k\in K}$ of the functor category Rel^K, a K-indexed family of sets, with the pair (U, r), where U is the disjoint union of the U_k and $r : U \to K$ is $r(u) = k$ iff $u \in U_k(\hookrightarrow U)$. Given functions $U \xrightarrow{r} K \xleftarrow{s} V$, we use the following notation for K-fibred product (pullback):

$$\begin{aligned} U \times_K V \;&=\; \{\, \langle u, v\rangle \,:\, r(u) = s(v)\,\} \quad\subseteq\quad U \times V \\ r \times_K s \;\;(u, v) \;&=\; r(u) \;\big(= s(v)\big) \;\in\; K. \end{aligned}$$

The (degenerate, compact closed) star-autonomous structure of Rel^K is:

- *Objects.* Pairs (U, r) comprising a set U of ***tokens*** and a K***-colouring*** function $r : U \to K$.
- *Morphisms.* A morphism $(U, r) \to (V, s)$ is a binary relation $R \subseteq U \times V$ between tokens which respects colour: uRv only if $r(u) = s(v)$.
- *Duality.* Trivial: $(U, r)^\perp = (U, r)$.
- *Internal hom and tensor.* Both are given by K-fibred product (pullback): $(U, r) \multimap (V, s) \;=\; (U, r) \otimes (V, s) \;=\; (U \times_K V, r \times_K s)$.
- *Tensor unit.* (K, id_K).

3.2 The category $\mathbf{G}(\mathsf{Rel}^K)$ of double glued relations over K

Applying the Hyland-Tan double glueing construction to Rel^K yields the star-autonomous category $\mathbf{G}(\mathsf{Rel}^K)$. To reduce bracket clutter, we flatten the triplets $((U, r), A, X)$ coming out of the application of $\mathbf{G}$.

- *Objects.* Tuples (U, r, A, X) where
  - · U is a set of ***tokens***,
  - · $r : U \to K$ is a ***colouring*** function,
  - · A is a set of subsets of U, the ***values***,
  - · X is a set of subsets of U, the ***covalues***.
- *Morphisms.* A morphism $R : (U, r, A, X) \to (V, s, B, Y)$ is a binary relation $R \subseteq U \times V$ between tokens which
  - · respects colour, *i.e.*, uRv only if $r(u) = s(v)$;

· maps values to values by direct image, *i.e.*, for all $a \in A$,

$$R_*(a) \;\; \equiv \;\; \{\, v \in V \; : \; uRv \text{ for some } u \in a \,\}$$

is in B;

· maps covalues to covalues by inverse image, *i.e.*, for all $y \in Y$,

$$R^*(y) \;\; \equiv \;\; \{\, u \in U \; : \; uRv \text{ for some } v \in y \,\}$$

is in X.

- *Duality.* $(U, r, A, X)^\perp = (U, r, X, A)$, exchanging values and covalues.
- *Internal hom and tensor.* Given $\mathcal{A} = (U, r, A, X)$ and $\mathcal{B} = (V, s, B, Y)$,

$$\begin{aligned} \mathcal{A} \multimap \mathcal{B} \;&=\; \big(U \times_K V,\; r \times_K s,\; \mathrm{hom}(\mathcal{A}, \mathcal{B}),\; A \times_K Y \,\big) \\ \mathcal{A} \otimes \mathcal{B} \;&=\; \big(U \times_K V,\; r \times_K s,\; A \times_K B,\; \mathrm{hom}(\mathcal{A}, \mathcal{B}^\perp) \,\big) \end{aligned}$$

where

$$\begin{aligned} A \times_K Y \;&=\; \{\, a \times_K y \; : \; a \in A \text{ and } y \in Y \} \;\subseteq\; U \times_K V \\ A \times_K B \;&=\; \{\, a \times_K b \; : \; a \in A \text{ and } b \in B \,\} \;\subseteq\; U \times_K V. \end{aligned}$$

- *Tensor unit.* $\big(K, \mathsf{id}_K, \{K\}, \mathcal{P}(K)\big)$, where $\mathcal{P}$ denotes powerset.

3.3 The biextensional full monoidal embedding $F : \mathbf{chu}(\mathsf{Set}, K) \to \mathbf{G}(\mathsf{Rel}^K)$

Let $\mathcal{A} = (A, r, X)$ be a biextensional Chu space over K. Given a point $a \in A$ and a copoint $x \in X$, define[2]

$$\begin{aligned} \mathsf{row}(a) \;&=\; \{a\} \times X \;\subseteq\; A \times X \\ \mathsf{col}(x) \;&=\; A \times \{x\} \;\subseteq\; A \times X \end{aligned}$$

and define

$$\begin{aligned} \mathsf{rows}(r) \;&=\; \{\, \mathsf{row}(a) \; : \; a \in A \,\} \\ \mathsf{cols}(r) \;&=\; \{\, \mathsf{col}(x) \; : \; x \in X \,\} \end{aligned}$$

Note that $\mathsf{rows}(r) \cong A$ and $\mathsf{cols}(r) \cong X$ (since $\mathcal{A}$ is biextensional). On objects, define

$$F(A, r, X) \;\; = \;\; \big(A \times X,\; r,\; \mathsf{rows}(r),\; \mathsf{cols}(r) \big).$$

Given a morphism $(f, g) : (A, r, X) \to (B, s, Y)$ in $\mathbf{chu}(\mathsf{Set}, K)$, define the binary relation

$$F(f, g) \;\; \subseteq \;\; (A \times X) \times (B \times Y)$$

[2] Different to a common definition of "row" of (A, r, X) as any function $\rho : X \to K$ such that $\rho = r(a, -)$ for some $a \in A$.

by

$$\langle a, g(y) \rangle \;\; F(f,g) \;\; \langle f(a), y \rangle$$

for every point $a \in A$ and copoint $y \in Y$. See Figure 2 (page 3) for an example. To be a $\mathbf{G}(\mathsf{Rel}^K)$ morphism, the binary relation $F(f,g)$ must (1) respect colour, (2) map values forwards to values by direct image, and (3) map covalues back to covalues by inverse image. Property (1) follows immediately from the adjointness of f and g. Properties (2) and (3) will follow from the more general (non-biextensional) case in section 5. In the meantime, observe that (1)–(3) hold for the example in Figure 2 (page 3). That F is full, faithful, and monoidal will also follow from the more general case in section 5.

3.4 No obvious extension

Note that the definition of F above does not depend on biextensionality. The very same definition, verbatim, yields a functor from $\mathbf{Chu}(\mathsf{Set}, K)$ to $\mathbf{G}(\mathsf{Rel}^K)$. Although F is faithful on $\mathbf{chu}(\mathsf{Set}, K)$, this extension to $\mathbf{Chu}(\mathsf{Set}, K)$ is not: faithfulness fails on morphisms between Chu spaces with no points and more than one copoint (and vice versa). For example, let $\mathcal{A} \in \mathbf{Chu}(\mathsf{Set}, K)$ be a Chu space with two points and no copoints (hence non-biextensional). There are $2^2 = 4$ morphisms from $\mathcal{A}$ to $\mathcal{A}$, one for each function f from the two-element set to itself; however, there is only one morphism from $F(\mathcal{A})$ to $F(\mathcal{A})$ (the empty binary relation, since $F(\mathcal{A})$ has no tokens).

Thus the natural extension of F to a functor $\mathbf{Chu}(\mathsf{Set}, K) \to \mathbf{G}(\mathsf{Rel}^K)$ is a "near miss" for a nice relationship between the Chu construction and the Hyland-Tan double glueing construction. The intensional double glueing construction **IG** defined below was conceived as a "fix" for the failure of this extension to be faithful: in section 5, following the strategy outlined in Figure 3 (page 5), we extend $F : \mathbf{chu}(\mathsf{Set}, K) \hookrightarrow \mathbf{G}(\mathsf{Rel}, K)$ to a full monoidal embedding $F^+ : \mathbf{Chu}(\mathsf{Set}, K) \to \mathbf{IG}(\mathsf{Rel}^K)$.

4 The intensional double glueing construction IG

We define the ***intensional double glueing construction*** **IG** by substituting multisets for sets in Hyland and Tan's **G** [21]. The construction has similar properties to its progenitor **G**: when $\mathbb{C}$ is star-autonomous, $\mathbf{IG}(\mathbb{C})$ is star-autonomous, and when $\mathbb{C}$ is compact closed [10] (therefore with isomorphic tensor and par), $\mathbf{IG}(\mathbb{C})$ has distinct tensor and par. Thus **IG**, like **G**, is a potentially useful tool in the search for fully complete models of linear logic.

Define a ***multiset*** $A = (A, |-|_A)$ over a set V to be a set A equipped with a ***valuation*** $|-|_A : A \to V$. We shall typically omit subscripts from valuations. A morphism $(f, v) : A \to B$ between multisets A over V and B over W is a function $f : A \to B$ together with a function $v : V \to W$ which

tracks f in the sense that $|f(a)| = v(|a|)$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "|-|"'] & B \arrow[d, "|-|"] \\ V \arrow[r, "v"'] & W \end{tikzcd}$$

Let $\mathbb{C}$ be a star-autonomous category with tensor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, tensor unit I, duality $(-)^{\perp} : \mathbb{C}^{\mathrm{op}} \to \mathbb{C}$, and internal hom $\multimap\ : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$. Let $\perp = I^{\perp}$, the dual of the tensor unit. Recall the following shorthand from section 2.2:

$$(-)_* \equiv \mathbb{C}(I, -) : \mathbb{C} \to \mathsf{Set} \qquad \text{(the values functor)}$$

$$(-)^* \equiv \mathbb{C}(-, \perp) : \mathbb{C} \to \mathsf{Set}^{\mathrm{op}} \qquad \text{(the covalues functor)}$$

The star-autonomous structure of $\mathbf{IG}(\mathbb{C})$ is as follows. The reader may wish to make a line-by-line comparison with the structure of $\mathbf{G}(\mathbb{C})$ (page 7).

- *Objects.* Triples (U, A, X) where
  · U is an object of $\mathbb{C}$,
  · A is a multiset of U-values (*i.e.*, a multiset over $U_* = \mathbb{C}(I, U)$),
  · X is a multiset of U-covalues (*i.e.*, a multiset over $U^* = \mathbb{C}(U, \perp)$).
  We refer to the elements of A and X as ***points*** and ***copoints***, respectively. Each point $a \in A$ determines a U-value $|a| = |a|_A : I \to U$, and each copoint $x \in X$ determines a U-covalue $|x| = |x|_X : U \to \perp$.
- *Morphisms.* A morphism $(U, A, X) \to (V, B, Y)$ is a triple (R, f, g):
  · a morphism $R : U \to V$ in $\mathbb{C}$,
  · a function $f : A \to B$ on points, and
  · a function $g : Y \to X$ on copoints,
  such that R_* tracks f and R^* tracks g, *i.e.*, such that the squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "|-|"'] & B \arrow[d, "|-|"] \\ U_* \arrow[r, "R_*"'] & V_* \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "|-|"'] & Y \arrow[l, "g"'] \arrow[d, "|-|"] \\ U^* & V^* \arrow[l, "R^*"] \end{tikzcd}$$

  commute, *i.e.*, such that (f, R_*) and (g, R^*) are multiset morphisms.
- *Duality.* $(U, A, X)^{\perp} = (U^{\perp}, X, A)$, modulo $(U^{\perp})_* \cong U^*$ and $(U^{\perp})^* \cong U_*$.
- *Internal hom.* Given $\mathcal{A} = (U, A, X)$ and $\mathcal{B} = (V, B, Y)$,

$$\mathcal{A} \multimap \mathcal{B} \;=\; \big(U \multimap V,\ \hom(\mathcal{A}, \mathcal{B}),\ A \times Y\big)$$

with the following valuations on $\hom(\mathcal{A},\mathcal{B})$ and $A \times Y$:

$$
\begin{aligned}
|(R,f,g)| &= R \in \mathbb{C}(U,V) \cong (U \multimap V)_* \\
|\langle a,y\rangle| &= |a| \otimes |y| \in \mathbb{C}(I \otimes V, U \otimes \bot) \cong (U \multimap V)^*.
\end{aligned}
$$

- *Tensor.* Given $\mathcal{A} = (U,A,X)$ and $\mathcal{B} = (V,B,Y)$,

$$\mathcal{A} \otimes \mathcal{B} = \big(U \otimes V,\ A \times B,\ \hom(\mathcal{A},\mathcal{B}^\bot)\big)$$

with the following valuations on $A \times B$ and $\hom(\mathcal{A},\mathcal{B}^\bot)$:

$$
\begin{aligned}
|\langle a,b\rangle| &= |a| \otimes |b| \in \mathbb{C}(I \otimes I, U \otimes V) \cong (U \otimes V)_* \\
|(R,f,g)| &= R \in \mathbb{C}(U,V^\bot) \cong (U \otimes V)^*.
\end{aligned}
$$

- *Tensor unit.* $\big(\, I,\ \{\mathsf{id}_I\},\ I^* = \mathbb{C}(I,\bot)\,\big)$, with identity valuations.

Proposition 4.1 $\mathbf{IG}(\mathbb{C})$ *is star-autonomous.*

This proposition follows from abstract considerations in section 4.2.

4.1 Biextensional collapse

This section is motivated by, and parallels, the biextensional collapse of Chu spaces (section 2.1.1). Deine an object of $\mathbf{IG}(\mathbb{C})$ as ***biextensional*** if its valuations are injections. Up to equivalence (namely, taking objects whose valuations are inclusions, rather than injections), the biextensional full subcategory of $\mathbf{IG}(\mathbb{C})$ is Hyland and Tan's category $\mathbf{G}(\mathbb{C})$. Define the ***collapse*** of a multiset $M = (M, |{-}|_M)$ over V to be the image $|M| = \{\, |m|_M \,:\, m \in M \,\} \subseteq V$ of its valuation. Given an object $\mathcal{A} = (U,A,X) \in \mathbf{IG}(\mathbb{C})$, define its ***biextensional collapse*** $\widetilde{\mathcal{A}} \in \mathbf{G}(\mathbb{C})$ by collapsing its point and copoint multisets to sets:

$$\widetilde{\mathcal{A}} = (U, |A|, |X|).$$

Biextensional collapse is functorial from $\mathbf{IG}(\mathbb{C})$ to $\mathbf{G}(\mathbb{C})$: given $\mathcal{A} = (U,A,X)$ and $\mathcal{B} = (V,B,Y)$ in $\mathbf{IG}(\mathbb{C})$ and a morphism $m = (R,f,g) : \mathcal{A} \to \mathcal{B}$, its biextensional collapse $\widetilde{m}$ is simply $R : U \to V$. Since (f, R_*) and (g, R^*) are multiset morphisms, R maps values forwards and covalues backwards, as required of a morphism of $\mathbf{G}(\mathbb{C})$.

Analogous to the way in which the star-autonomous structure on biextensional $\mathbf{chu}(\mathsf{Set},K)$ is inherited from $\mathbf{Chu}(\mathsf{Set},K)$ by biextensional collapse (section 2.1.1), tensor $\widetilde{\otimes}$ and internal hom $\widetilde{\multimap}$ in $\mathbf{G}(\mathbb{C})$ can be viewed as being inherited from $\mathbf{IG}(\mathbb{C})$ by biextensional collapse: $\mathcal{A} \mathbin{\widetilde{\otimes}} \mathcal{B} = \widetilde{\mathcal{A} \otimes \mathcal{B}}$ and $\mathcal{A} \mathbin{\widetilde{\multimap}} \mathcal{B} = \widetilde{\mathcal{A} \multimap \mathcal{B}}$.

4.2 Abstract perspective

In this section we consider **IG** from a more abstract point of view, and prove that $\mathbf{IG}(\mathbb{C})$ is star-autonomous (Proposition 4.1).

The result of *glueing* along a functor $F : \mathbb{C} \to \mathbb{B}$ is the comma category $(\mathsf{id} \downarrow F)$ (see MacLane [13]). *Double glueing* along functors $\mathbb{B} \xleftarrow{F} \mathbb{C} \xrightarrow{G} \mathbb{D}$ appears under the notation $/F, G/$ as Pavlović's *category of interpolants* [14], and in Hyland and Schalk's comprehensive study of glueing and orthogonality for models of linear logic [9]. Objects of $/F, G/$ are triples (b, C, d) where C is an object of $\mathbb{C}$, b is a morphism in $\mathbb{B}$ into $F(C)$, and d is a morphism in $\mathbb{D}$ out of $G(C)$. A morphism $(b, C, d) \to (b', C', d')$ is a triple of morphisms $(f, g, h) \in \mathbb{B} \times \mathbb{C} \times \mathbb{D}$ where $g : C \to C'$ and the following squares commute:

$$
\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "f"'] & F(C) \arrow[d, "F(g)"] \\
\cdot \arrow[r, "b'"'] & F(C')
\end{tikzcd}
\qquad
\begin{tikzcd}
G(C) \arrow[r, "d"] \arrow[d, "G(g)"'] & \cdot \arrow[d, "h"] \\
G(C') \arrow[r, "d'"'] & \cdot
\end{tikzcd}
$$

Let $\mathbb{C}$ be a star-autonomous category with tensor unit I, and let $\perp = I^{\perp}$. The category $\mathbf{IG}(\mathbb{C})$ is the category of interpolants $/\mathbb{C}(I, -), \mathbb{C}(-, \perp)/$, *i.e.*, the result of double glueing along the functors

$$\mathsf{Set} \xleftarrow{\mathbb{C}(I,-)} \mathbb{C} \xrightarrow{\mathbb{C}(-,\perp)} \mathsf{Set}^{\mathrm{op}}$$

Proof that $\mathbf{IG}(\mathbb{C})$ is star-autonomous. We appeal to a more general result of Hyland and Schalk: in Proposition 4.14 on page 27 of [9], set $\mathbf{E} = \mathsf{Set}$ and $L = \mathbb{C}(I, -)$. This uses the fact that Set is symmetric monoidal closed with pullbacks.

5 Extending the full monoidal embedding

We saw in section 3.4 that the natural extension of $F : \mathbf{chu}(\mathsf{Set}, K) \hookrightarrow \mathbf{G}(\mathsf{Rel}^K)$ to the whole of $\mathbf{Chu}(\mathsf{Set}, K)$ fails to be faithful. In this section we "fix" the lack of faithfulness on $\mathbf{Chu}(\mathsf{Set}, K)$, extending F to a full monoidal embedding $F^+ : \mathbf{Chu}(\mathsf{Set}, K) \to \mathbf{IG}(\mathsf{Rel}^K)$. Figure 3 (page 5) outlines our strategy.

5.1 The category $\mathbf{IG}(\mathsf{Rel}^K)$ of intensionally double glued relations over K

To reduce bracket clutter, we flatten the triplets $((U, r), A, X)$ coming out of the application of **IG**, and present $\mathbf{IG}(\mathsf{Rel}^K)$ as follows. The reader may wish to make a line-by-line comparison with the structure of $\mathbf{G}(\mathsf{Rel}^K)$ (page 9).

- *Objects.* Tuples (U, r, A, X) where
  - · U is a set of ***tokens***,

- · $r : U \to K$ is a ***colouring*** function,
- · A is a multiset of subsets of U (*i.e.*, a multiset over $\mathcal{P}(U)$),
- · X is a multiset of subsets of U (*i.e.*, a multiset over $\mathcal{P}(U)$).

Each $a \in A$ is a ***point***, each $x \in X$ is a ***copoint***, $|a| = |a|_A \subseteq U$ is a ***value***, and $|x| = |x|_X \subseteq U$ is a ***covalue***.

- *Morphisms.* A morphism $(R, f, g) : (U, r, A, X) \to (V, s, B, Y)$ is:
  - · a binary relation $R \subseteq U \times V$ between tokens which respects colour, *i.e.*, uRv only if $r(u) = s(v)$,
  - · a function $f : A \to B$ on points, and
  - · a function $g : Y \to X$ on copoints,

  such that direct image R_* tracks f and inverse image R^* tracks g:

$$\begin{aligned} |f(a)| &= R_*\big(|a|\big) &\equiv& \;\{\, v \in V \,:\, uRv \text{ for some } u \in |a|\,\} \\ |g(y)| &= R^*\big(|y|\big) &\equiv& \;\{\, u \in U \,:\, uRv \text{ for some } v \in |y|\,\}. \end{aligned}$$

- *Duality.* $(U, r, A, X)^\perp = (U, r, X, A)$, interchanging points and copoints.
- *Internal hom and tensor.* Given $\mathcal{A} = (U, r, A, X)$ and $\mathcal{B} = (V, s, B, Y)$,

$$\begin{aligned} \mathcal{A} \multimap \mathcal{B} &= \big(U \times_K V,\ r \times_K s,\ \mathrm{hom}(\mathcal{A}, \mathcal{B}),\ A \times Y\,\big) \\ \mathcal{A} \otimes \mathcal{B} &= \big(U \times_K V,\ r \times_K s,\ A \times B,\ \mathrm{hom}(\mathcal{A}, \mathcal{B}^\perp)\big) \\ |(R, f, g)| &= R \qquad\qquad\qquad (\text{valuation in } \mathrm{hom}(\mathcal{A}, \mathcal{B}),\ \mathrm{hom}(\mathcal{A}, \mathcal{B}^\perp)) \\ |\langle a, q\rangle| &= |a| \times_K |q| \qquad\quad (\text{valuation in } A \times Y,\ A \times B) \end{aligned}$$

- *Tensor unit.* $\big(K, \mathsf{id}_K, \{K\}, \mathcal{P}(K)\big)$, with identity valuations.

The biextensional collapse $\widetilde{\mathcal{A}}$ of $\mathcal{A} = (U, r, A, X) \in \mathbf{IG}(\mathsf{Rel}^K)$ is $(U, r, |A|, |X|) \in \mathbf{G}(\mathsf{Rel}^K)$ and the biextensional collapse of a morphism (R, f, g) is R.

5.2 The full monoidal embedding $F^+ : \mathbf{Chu}(\mathsf{Set}, K) \to \mathbf{IG}(\mathsf{Rel}^K)$

Figure 3 (page 5) sketches the idea. Define F^+ on objects by

$$F^+(A, r, X) \;=\; \big(\,\widetilde{A} \times \widetilde{X},\ \widetilde{r},\ A,\ X\,\big)$$

where ($\widetilde{\ }$) is biextensional collapse on the components of a Chu space (section 2.1.1), and valuations on A and X (respectively) are:

$$\begin{aligned} |a| &= \{\,\widetilde{a}\,\} \times \widetilde{X} &\subseteq& \;\widetilde{A} \times \widetilde{X} \\ |x| &= \widetilde{A} \times \{\,\widetilde{x}\,\} &\subseteq& \;\widetilde{A} \times \widetilde{X} \end{aligned}$$

Thus the value of a point is its 'biextensional row', and the covalue of a copoint is its 'biextensional column'. See Figure 3.

Given a morphism $m = (f, g) : (A, r, X) \to (B, s, Y)$ in $\mathbf{Chu}(\mathsf{Set}, K)$, define $F^+(m) = (F(\widetilde{m}), f, g)$. Thus the binary relation

$$F(\widetilde{m}) \;\subseteq\; (\widetilde{A} \times \widetilde{X}) \times (\widetilde{B} \times \widetilde{Y})$$

relates $\langle \widetilde{a}, \widetilde{g(y)} \rangle$ and $\langle \widetilde{f(a)}, \widetilde{y} \rangle$ for all $a \in A$ and $y \in Y$.

The well-definedness, fullness and faithfulness of F^+ follow from the lemma below. Given non-empty sets A and X define a ***row*** (resp. ***column***) of $A \times X$ to be any subset of $A \times X$ of the form $\{a\} \times X$ (resp. $A \times \{x\}$).

Lemma 5.1 *Pairs of functions ($A \xrightarrow{f} B,\ X \xleftarrow{g} Y$) between non-empty sets are in bijection with binary relations $R \subseteq (A \times X) \times (B \times Y)$ whose direct image R_* maps rows of $A \times X$ to rows of $B \times Y$ and inverse image R^* maps columns of $B \times Y$ back to columns of $A \times X$.*

Proof. The correspondence is: $\big[f(a) = b \text{ and } g(y) = x\big]$ iff $\big[R_*(\{a\} \times X) = \{b\} \times Y \text{ and } R^*(B \times \{y\}) = A \times \{x\}\big]$ iff $\langle a, b\rangle R \langle x, y\rangle$. $\square$

F^+ is monoidal (but not strict or strong monoidal) with

$$\begin{aligned} F^+(\mathcal{A}) \otimes F^+(\mathcal{B}) &\longrightarrow F^+(\mathcal{A} \otimes \mathcal{B}) \\ (K, \mathsf{id}_K, \{K\}, \mathcal{P}(K)) &\longrightarrow F^+(1, \pi_K, K) \end{aligned}$$

defined as follows. Let $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$, so

$$F^+(\mathcal{A}) \otimes F^+(\mathcal{B}) \;=\; \big(\, (\widetilde{A} \times \widetilde{X}) \times_K (\widetilde{B} \times \widetilde{Y}),\ \widetilde{r} \times_K \widetilde{s},\ A \times B,\ \hom(\mathcal{A}, \mathcal{B}^\perp)\,\big)$$

$$F^+(\mathcal{A} \otimes \mathcal{B}) \;=\; \big(\, (\widetilde{A \times B}) \times \widetilde{\hom(\mathcal{A}, \mathcal{B}^\perp)},\ \widetilde{\mathsf{ev}},\ A \times B,\ \hom(\mathcal{A}, \mathcal{B}^\perp)\,\big)$$

(with the obvious valuations). Define the natural morphism from the former to the latter to be $(R, \mathsf{id}, \mathsf{id})$, where R is given by

$$\Big\langle\, \langle \widetilde{a}, \widetilde{g(b)} \rangle\,,\ \langle \widetilde{b}, \widetilde{f(a)} \rangle \,\Big\rangle \quad R \quad \Big\langle\, \widetilde{\langle a, b\rangle}\,,\ \widetilde{\langle f, g\rangle} \,\Big\rangle$$

for all $a \in A$, $b \in B$, and morphisms $(f, g) : \mathcal{A} \to \mathcal{B}^\perp$ in $\mathbf{Chu}(\mathsf{Set}, K)$. The map

$$(K, \mathsf{id}_K, \{K\}, \mathcal{P}(K)) \;\longrightarrow\; F^+(1, \pi_K, K) \;\cong\; (K, \mathsf{id}_K, \{K\}, K)$$

is $(\mathsf{id}, \mathsf{id}, \eta_K)$ where $\eta_K : K \to \mathcal{P}(K)$ takes $k \in K$ to $\{k\} \in \mathcal{P}(K)$.

Biextensional case. Since tensor $\widetilde{\otimes}$ and internal hom $\widetilde{\multimap}$ in the biextensional categories $\mathbf{chu}(\mathsf{Set}, K)$ and $\mathbf{G}(\mathsf{Rel}^K)$ are inherited from the corresponding larger categories by biextensional collapse ($\mathcal{A} \mathbin{\widetilde{\otimes}} \mathcal{B} = \widetilde{\mathcal{A} \otimes \mathcal{B}}$ and $\mathcal{A} \mathbin{\widetilde{\multimap}} \mathcal{B} = \widetilde{\mathcal{A} \multimap \mathcal{B}}$), the biextensional full embedding $F : \mathbf{chu}(\mathsf{Set}, 2) \hookrightarrow \mathbf{G}(\mathsf{Rel}^K)$ is also monoidal: take the biextensional collapse of $F^+(\mathcal{A}) \otimes F^+(\mathcal{B}) \to F^+(\mathcal{A} \otimes \mathcal{B})$.

6 Biextensional collapse

Biextensional collapse $(\tilde{\ })$: $\mathbf{IG} \to \mathbf{G}$ extends the familiar notion on Chu spaces, in the sense that the square in Figure 1 (page 3) commutes from top-left to bottom-right, *i.e.*, $F \circ (\tilde{\ }) = (\tilde{\ }) \circ F^+ : \mathbf{Chu}(\mathsf{Set}, K) \to \mathbf{G}(\mathsf{Rel}^K)$. (This fact is immediate from the definitions of F and F^+.)

In the terminology of Hyland and Schalk [9], $\mathbf{IG}$ is the result of omitting the specialisation to monic structure maps implicit in $\mathbf{G}$ [21]. Thus, since $\mathbf{G}(\mathbb{C})$ is (equivalent to) the biextensional full subcategory of $\mathbf{IG}(\mathbb{C})$, we obtain a new interpretation of the monic specialisation as a form of biextensionality. This leads to the distinction between *intensional* and *biextensional* models described in the introduction (page 4).

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