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Entropic Geometry from Logic

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Abstract

We establish the following equation:

Quantitative Probability = Logic + Partiality of Knowledge + Entropy

I.e.: 1. A finitary probability space Δ^n (= all probability measures on $\{1, \ldots, n\}$) can be fully and faithfully represented by the pair consisting of the abstraction D^n (= the object up to isomorphism) of the partially ordered set $(\Delta^n, \sqsubseteq)$ introduced in [3], and, Shannon entropy; 2. D^n itself can be obtained via a systematic purely order-theoretic procedure (which embodies introduction of partiality of knowledge) on an (algebraic) logic. This procedure applies to any poset A; $D_A \cong (\Delta^n, \sqsubseteq)$ when A is the n-element powerset and $D_A \cong (\Omega^n, \sqsubseteq)$, the domain of mixed quantum states also introduced in [3], when A is the lattice of subspaces of a Hilbert space.

1 Introduction

For a century the dominant formalization of uncertainty has been in terms of measures on a support. However, already in 1926 F. P. Ramsey proposed to conceive probability as *the logic of partial knowledge* [11]. D. S. Scott relied on a more general notion of partiality to propose the mathematical structure of a *domain* [12]. A deep connection between domains and measures of content was established by K. Martin in [9]. A domain $(\Delta^n, \sqsubseteq)$ of probability measures which has Shannon entropty as a measure of content and a domain $(\Omega^n, \sqsubseteq)$ of mixed quantum states which has von Neumann entropy as a measure of content were introduced in [3]. In this paper, we establish:

(i) Quantitative Probability = Qualitative Probability + Entropy

(ii) Qualitative Probability := Logic + Partiality of Knowledge

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[2] The phrase “Entropic Geometry” arose in exchanges with Keye Martin. I thank him for discussing the content and presentation of this paper. I thank Samson A. and Prakash P. for logistic, Dusko P. for recreational and Rhada J. for gastronomic support, and for their constructive feedback on [3]. All three referees provided constructive comments.

The first claim follows from the fact that D^n, the abstraction of $(\Delta^n, \sqsubseteq)$ as a partially ordered set (= *poset*), when equipped with Shannon entropy μ, fully and faithfully captures Δ^n: the identity is the only entropy-preserving order-isomorphism of $(\Delta^n, \sqsubseteq, \mu)$ — up to permutation of the names of its pure states (Corollary 5.2). Thus, uncertainty can be captured by combining qualitative (= domains) and quantitative (= entropy) notions of information.

A probability space does not only admit a notion of partiality (= domain structure); D^n can be purely order-theoretically constructed in terms of partial knowledge starting from an algebraic logic, namely the powerset of its maximal elements. Thus, no probability space is *a priori* required to produce D^n. This establishes the second claim. This result extends to the quantum case. It can be seen as the converse to [3] Theorems 4.8 and 4.11, were the the powerset $\mathrm{P}(\{1,\ldots,n\})$ and the lattice of subspaces $\mathbb{L}^n$ of a n-dimensional Hilbert space $\mathcal{H}^n$ are recoverd in order-theoretic manner respectively from $(\Delta^n, \sqsubseteq)$ and $(\Omega^n, \sqsubseteq)$. The fact that the *quantum logic* $\mathbb{L}^n$ constitutes the algebra of physical properties of a quantum system [1,5], as opposed to the classical logic $\mathrm{P}(\{1,\ldots,n\})$, justifies the utterance *probability from logic* (Section 6).

In fact, we produce a probability space with, *in addition*, a partial order relation on it (so the above equations are understatements). (Pre)orders have been in the study of probability [10], but never captured probability itself.

2 Preliminaries

In this section we recall results from [3]. Let Δ^n be all probability distributions on $\{1,\ldots,n\}$, that is, either a list $x = (x_1,\ldots,x_n) \in [0,1]^n$ or a map $x : \{1,\ldots,n\} \to [0,1] :: i \mapsto x_i$, with $\sum_{i=1}^{i=n} x_i = 1$. Decreasing monotone distributions in Δ^n, i.e., for all $i \in \{1,\ldots,n-1\}$ we have $x_i \geq x_{i+1}$, are denoted by Λ^n. The *spectrum* of x is the set $\mathrm{spec}(x) := \{x_i \mid 1 \leq i \leq n\}$. Denote the collection of all permutations $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ as $S(n)$. For a poset D, we set $\uparrow x = \{y \in D \mid x \sqsubseteq y\}$ and $\downarrow x = \{y \in D \mid y \sqsubseteq x\}$; we call $e \in D$ *maximal* iff $\uparrow e = \{e\}$; we denote the set of maximal elements of D by $\mathrm{Max}(D)$; the *bottom* $\perp$ (if it exists) of D is defined by $\uparrow \perp = D$. A poset D is a *chain* iff $x, y \in D$ either implies $x \sqsubseteq y$ or $y \sqsubseteq x$.

Definition 2.1 Let $n \geq 2$. For $x, y \in \Delta^n$, we have $x \sqsubseteq y$ iff there exists $\sigma \in S(n)$ such that $x \cdot \sigma, y \cdot \sigma \in \Lambda^n$ and if we have $\forall i \in \{1,\ldots,n-1\}$:

$$(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} \leq (x \cdot \sigma)_{i+1} (y \cdot \sigma)_i \,. \tag{1}$$

Theorem 2.2 *Let $n \geq 2$. Then, $(\Delta^n, \sqsubseteq)$ is a partially ordered set with*

$$\mathrm{Max}(\Delta^n) = \{e \in \Delta^n \mid \mathrm{spec}(x) = \{0,1\}\} \quad \& \quad \perp = (1/n, \ldots, 1/n) \,.$$

Moreover, it is a dcpo and admits the notions of partiality and approximation, *that is,* $(\Delta^n, \sqsubseteq)$ *is entitled to be called a domain.*[3] *Finally, Shannon entropy*

$$\mu : \Delta^n \to [0,1] :: x \mapsto -\sum_{i=1}^{n} x_i \log x_i$$

is a measure of content in the sense of [9].[4]

The intuition behind $x \sqsubseteq y$ is: "*State* y is more *informative* than *state* x". In epistemic terms this becomes: "*Observer* y has more *knowledge* about the system than *observer* x". Now we will formalize this intuition. Define the *Bayesian projections* $\{p_i\}_i$ such that for all $x \in \Delta^{n+1}$ with $x_i < 1$:

$$p_i(x) = \frac{1}{1-x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \in \Delta^n .$$

We then have for $x, y \in \Delta^{n+1}$ in terms of $(\Delta^n, \sqsubseteq)$:

$$x \sqsubseteq y \iff (\forall i)(x_i, y_i < 1 \Rightarrow p_i(x) \sqsubseteq p_i(y)) . \tag{2}$$

This interprets as follows. (For a detailed exposition see [3] §2.1 and §4.4.) The pure states $\{e_i\}_i$ are to be seen as the actual states the system can be in, while general mixed states x and y should be conceived as being epistemic. Equivalence (2) expresses: 1. Whenever a state x stands for less knowledge about the system than state y, then, after Bayesian update with respect to the new knowledge that the actual state of the system is not e_i, the state $p_i(x)$ still stands for less knowledge than $p_i(y)$ due to the initial advantage in knowledge of y as compared to x; 2. This behavior of $\sqsubseteq$ w.r.t. knowledge update exactly defines $\sqsubseteq$.[5] Indeed, the *inductive rule* (2) provides a definition equivalent to Definition 2.1 when a base case $n = 2$ is postulated as:

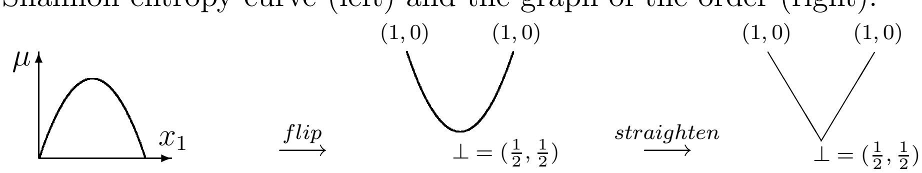
Definition 2.3 For $x, y \in \Delta^2$ we set

$$(x_1, x_2) \sqsubseteq (y_1, y_2) \iff (y_1 \le x_1 \le 1/2) \text{ or } (1/2 \le x_1 \le y_1) .$$

Theorem 2.4 *The order of Definition 2.3 is the only partial order on* Δ^2 *which has* $\perp = (1/2, 1/2)$ *and satisfies the mixing law:*

$$x \sqsubseteq y \text{ and } p \in [0,1] \implies x \sqsubseteq (1-p)x + py \sqsubseteq y .$$

The canonicity of this choice for the order on Δ^2 reflects in the shape of the Shannon entropy curve (left) and the graph of the order (right):



Conclusively, there exists an order on Δ^n which canonically arises from envisioning probability distributions as informative objects, and which is tightly intertwined with Shannon entropy.

[3] We refer to [3] for definitions and details on these domain-theoretic aspects. They are not essential for the developments in this paper.

[4] I.e., there is a tight connection between μ and the domain-theoretic properties of $(\Delta^n, \sqsubseteq)$.

[5] This gets extremely close to how order on physical properties is defined [7].

3 Symmetries and degeneration

For $x \in \Delta^n$, the map $x^\Lambda := x \cdot \sigma : \{1, \ldots, n\} \to [0,1]^n$ does not depend on the particular choice of σ when $\sigma \in S(n)$ is such that $x \cdot \sigma \in \Lambda^n$. It follows that $\sigma \in S(n)$ monotonizes $x \in \Delta^n$ iff σ makes the following diagram commute:

$$\begin{array}{ccc} \{1,\ldots,n\} & \xrightarrow{\ x\ } & [0,1] \\ \sigma\big\uparrow & \nearrow{\scriptstyle x^\Lambda} & \\ \{1,\ldots,n\} & & \end{array} \tag{3}$$

The inequalities (1) can now be restated without explicit reference to σ.

Proposition 3.1 *For $x, y \in \Delta^n$, we have $x \sqsubseteq y$ iff*

(i) *There exists at least one $\sigma \in S(n)$ such that $x \cdot \sigma$, $y \cdot \sigma \in \Lambda^n$;*

(ii) *For all $i \in \{1, \ldots, n-1\}$ we have $x^\Lambda_i \cdot y^\Lambda_{i+1} \leq x^\Lambda_{i+1} \cdot y^\Lambda_i$.*

Remark 3.2 When $x^\Lambda_{i+1} \neq 0 \neq y^\Lambda_{i+1}$ the inequalities express ratios:

$$x^\Lambda_i / x^\Lambda_{i+1} \ \leq \ y^\Lambda_i / y^\Lambda_{i+1} \, .$$

Let $x \in \Delta^n$. Let n^x be the cardinality of spec(x); let x^{spec} be the decreasingly ordered spectrum of x. Denote the multiplicity of value x^{spec}_j in the list x^Λ by n^x_j, or, n_j when it is clear from the context to which state this number applies. Then, set $K^{(x)}_1 := \{1, \ldots, n_1\}$ and set:

(i) $\forall j \in \{1, \ldots, n^x\} : \bar{n}_j := \sum_{i=1}^{i=j} n_i$

(ii) $\forall j \in \{2, \ldots, n^x\} : K^{(x)}_j := \{\bar{n}^{(x)}_{j-1} + 1, \ldots, \bar{n}^{(x)}_j\}$

that is $i \in K_j \ \Leftrightarrow \ x^\Lambda_i = x^{\mathrm{spec}}_j$. The diagram in eq.(3) then splits up in

$$\begin{array}{ccc} \{1,\ldots,n\} & \xrightarrow{\ x\ } & [0,1] \\ \sigma\big\uparrow & \nearrow{\scriptstyle x^{\mathrm{spec}}(1)} & \\ K_1 & & \end{array} \qquad \cdots \qquad \begin{array}{ccc} \{1,\ldots,n\} & \xrightarrow{\ x\ } & [0,1] \\ \sigma\big\uparrow & \nearrow{\scriptstyle x^{\mathrm{spec}}(n)} & \\ K_{n^x} & & \end{array}$$

where $x^{\mathrm{spec}}(1), \ldots, x^{\mathrm{spec}}(n)$ are constant maps. Requiring commutation then imposes an ordered partition $(\sigma[K_1], \ldots, \sigma[K_{n^x}])$ on $\{1, \ldots, n\}$.

For $i, j \in \{1, \ldots, n\}$ set $i \sim j$ whenever $x_i = x_j$. The corresponding equivalence classes then admit a total ordering $I^{(x)}_1 \succ \ldots \succ I^{(x)}_{n^x}$ which is such that $I_k \succ I_l$ whenever for $i \in I_k$ and $j \in I_l$ we have $x_i > x_j$. Thus

$$i \in I_j \ \Leftrightarrow \ x_i = x^{\mathrm{spec}}_j \, . \tag{4}$$

The cardinality of I_j is the same as that of K_j, namely n_j.

Lemma 3.3 *For $x \in \Delta^n$ and $\sigma \in S(n)$ we have $x \cdot \sigma \in \Lambda^n$ iff*

$$\forall j \in \{1, \ldots, n^x\} : \sigma[K_j] = I_j\,.$$

Proof. Since by diagram (3) we have $x \cdot \sigma \in \Lambda^n \Leftrightarrow \forall i \in \{1, \ldots, n\} : x_i^\Lambda = (x \cdot \sigma)(i)$ the equivalence follows from $\sigma(i) \in \sigma[K_j] \Leftrightarrow i \in K_j \Leftrightarrow x_j^{\rm spec} = x_i^\Lambda$ and $\sigma(i) \in I_j \Leftrightarrow x_j^{\rm spec} = x_{\sigma(i)} = (x \cdot \sigma)(i)$. □

Proposition 3.4 *Each $x \in \Delta^n$ is faithfully represented by the pair*

(i) *The ordered partition $\mathcal{I}^x := (I_1, \ldots, I_{n^x})$ on $\{1, \ldots, n\}$;*

(ii) *The $[0,1]$-valued n^x-element set ${\rm spec}(x)$.*

Conversely, each such pair defines a state $x \in \Delta^n$ iff $\sum_{j=1}^{j=n^x} n_j \cdot x_j^{\rm spec} = 1$.

Proof. Direction $\Rightarrow$ of eq.(4) fixes x given ${\rm spec}(x)$ and $(I_1, \ldots, I_{n^x})$. The converse follows by construction. □

The degeneration of the spectrum of $x \in \Delta^n$ which is now encoded in the ordered partition $\mathcal{I}^x$ is of crucial importance w.r.t. $\sqsubseteq$.

Lemma 3.5 (Degeneration) [3] *If $x \sqsubseteq y$ in Δ^n, then*

$$x_i = 0 \ \Rightarrow\ y_i = 0 \qquad \& \qquad y_i = y_j > 0 \ \Rightarrow\ x_i = x_j$$

Thus, degeneration admits a hierarchy in $(\Delta^n, \sqsubseteq)$:

zero-values/degeneration
non-degenerated non-zero values
degenerated non-zero values

Setting

$$\begin{cases} n_0^{(x)} := n^x & 0 \notin {\rm spec}(x) \\ n_0^{(x)} := n^x - 1\ ,\ \bar{n}_0 := \sum_{i=1}^{i=n_0} n_i\ ,\ I_0 := I_{n^x}\ ,\ K_0 := K_{n^x} & 0 \in {\rm spec}(x) \end{cases}$$

we can express the Degeneration Lemma in terms of $\mathcal{I}^x$.

Lemma 3.6 (Degeneration$^{\rm bis}$) *If $x \sqsubseteq y$ in Δ^n, then*

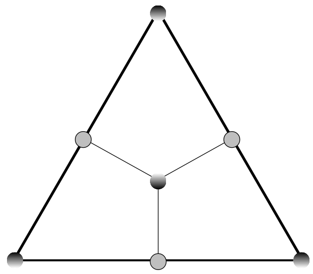
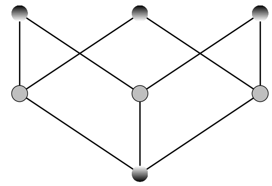
$$I_0^x \subseteq I_0^y \qquad \& \qquad \forall i \in \{1, \ldots, n_0^y\}, \exists j \in \{1, \ldots, n_0^x\} : I_i^y \subseteq I_j^x\,.$$

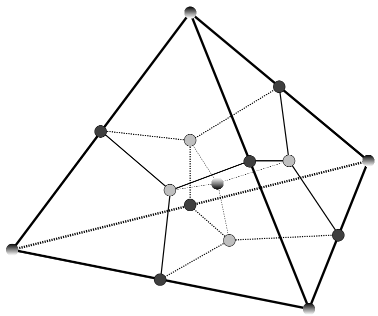
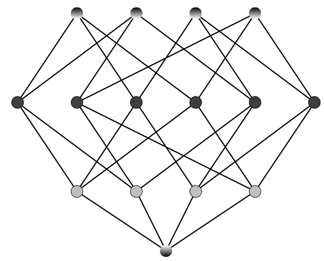
4 Coordinates

Definition 4.1 (Coordinates) Let ${\rm Coord}(\Delta^{\rm n})$ be all $x \in \Delta^n$ with an at most binary spectrum. Let the *degenerated coordinates* ${\rm Ir}_\perp(\Delta^{\rm n})$ be the set of all $x \in {\rm Coord}(\Delta^{\rm n})$ with $0 \in {\rm spec}(x)$. For $x \in {\rm Ir}_\perp(\Delta^{\rm n})$ let the *x-axis* be the set of all $y \in {\rm Coord}(\Delta^{\rm n})$ with $I_1^y = I_1^x$ (and thus also $I_2^y = I_2^x$).

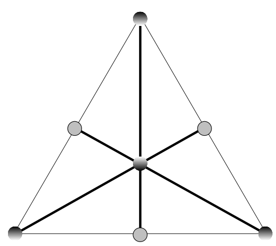
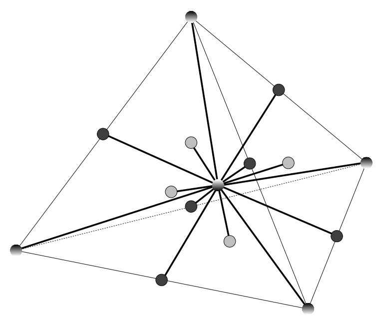
As shown in [3] §4.3, ${\rm Ir}_\perp(\Delta^{\rm n})$ constitutes a subposet of Δ^n which, when top and bottom are added to it, is isomorphic to the powerset $\mathcal{P}(\{1, \ldots, n\})$.

The illustrations below expose $\mathrm{Ir}_\perp(\Delta^{\mathrm{n}}) \cup \{\perp\}$ in the "triangle" Δ^3 and the "tetrahedron" Δ^4. The figures on the right are their Hasse diagrams.

The segments represent increase of the order and coincide on the left and the right, the increase being respectively radially and upwardly. The coordinate axes of Δ^3 and Δ^4 look as follows.

Proposition 4.2 *Coordinates and coodinate axes are order-theoretical:*

- $\mathrm{Ir}_\perp(\Delta^{\mathrm{n}}) \cup \{\perp\}$ *are the infima of sets in* $\mathcal{P}(\mathrm{Max}(\Delta^n)) \setminus \{\emptyset\}$.
- *If* $x \in \mathrm{Coord}_{\mathrm{Ir}}(\Delta^n) := \mathrm{Coord}(\Delta^{\mathrm{n}}) \setminus \mathrm{Ir}_\perp(\Delta^n)$ *then* $\downarrow x$ *is a chain. Conversely, if* $\downarrow x$ *is a chain then* $x \in \mathrm{Coord}(\Delta^{\mathrm{n}})$.
- *A coordinate axis is the completion of a maximal* $\mathrm{Coord}_{\mathrm{Ir}}(\Delta^n)$*-chain.*

Proof. Maximal elements and bottom are order-theoretical by definition and so are all $x \in \mathrm{Ir}_\perp(\Delta^n)$ since by [3] §4.3 we have $x = \bigwedge(\uparrow x \cap \mathrm{Max}(\Delta^n))$.

For $x \in \mathrm{Coord}(\Delta^n)\setminus\mathrm{Ir}_\perp(\Delta^n)$ we have $x = \perp$ or $\mathcal{I}^x = (I_1^x, I_2^x)$. Let $x \neq \perp$. If $y \sqsubseteq x$ by Lemma 3.6 we have $I_1^x \subseteq I^y$ and $I_2^x \subseteq J^y$ for some $I^y, J^y \in \mathcal{I}^y$. Thus, $\mathcal{I}^y = \mathcal{I}^x$ or $I_1^y = \{1, \ldots, n\}$. If $y, z \in \downarrow x$ with $y \neq \perp \neq z$ then $\mathcal{I}^y = \mathcal{I}^z = \mathcal{I}^x$ and either $y^+ \cdot z^- \leq z^+ \cdot y^-$ or $z^+ \cdot y^- \leq y^+ \cdot z^-$ so y and z compare. The cases $x = \perp$, $y = \perp$ and $z = \perp$ are trivial so $\downarrow x$ is a chain.

Let $x \notin \mathrm{Coord}(\Delta^n)$. Then $\{I_1^x, I_2^x, I_3^x\} \subseteq \mathcal{I}^x$. But $y, z \in \Delta^n$ defined by

$$\mathcal{I}^y = \{I_1^x\ , \{1,\ldots,n\} \setminus I_1^x\} \qquad \mathcal{I}^z = \{I_1^x \cup I_2^x\ , \{1,\ldots,n\} \setminus (I_1^x \cup I_2^x)\}$$

$$y_1^{\text{spec}} \cdot x_2^{\text{spec}} = x_1^{\text{spec}} \cdot y_2^{\text{spec}} \qquad\qquad z_1^{\text{spec}} \cdot x_3^{\text{spec}} = x_2^{\text{spec}} \cdot z_2^{\text{spec}}$$

(cfr. Proposition 3.4) don't compare although $y, z \sqsubseteq x$ so $\downarrow x$ is not a chain.

From the above we also know that for $x \in \text{Coord}_{\text{Ir}}(\Delta^n)$ and $y \sqsubseteq x$ we have $y \in \text{Coord}(\Delta^{\text{n}})$ and in particular that y belongs to the same axis as x. Thus for $y, z \in x$-axis with $z \neq x$ we have that $y \sqsubseteq w \sqsubseteq z$ forces $w \in x$-axis. Thus x-axis$\setminus\{x\}$ is a maximal chain in $\text{Coord}_{\text{Ir}}(\Delta^n)$. By [3] Proposition 2.16 we then have $x = \bigsqcup(x\text{-axis}\setminus\{x\})$. □

To $x \in \Delta^n \setminus \{\bot\}$ we attribute $\mathcal{C}^x = \{c(1), \ldots, c(n^x - 1)\} \subset \text{Coord}(\Delta^{\text{n}})$ as *its coordinates*, where, using Proposition 3.4, each $c(j)$ is defined by

$$\mathcal{I}^{c(j)} = \left\{ \bigcup_{i=1}^{i=j} I_i^x\ , \bigcup_{i=j+1}^{i=n^x} I_i^x \right\} \qquad\qquad c(j)_1^{\text{spec}} \cdot x_{j+1}^{\text{spec}} = x_j^{\text{spec}} \cdot c(j)_2^{\text{spec}}\,.$$

Further we set $\mathcal{C}^\bot = \emptyset$. If $0 \in \text{spec}(x)$ we set $c_0 := c(n^x - 1) \in \text{Ir}_\bot(\Delta^{\text{n}})$.

Theorem 4.3 (Decomposition in coordinates) *States $x \in \Delta^n$ and their coordinates $\mathcal{C}^x$ are in bijective order-theoretic correspondence:*

$$x = \bigsqcup \mathcal{C}^x \qquad \text{and} \qquad \mathcal{C}^x = \text{Max}(\text{Coord}(\Delta^{\text{n}}) \cap \downarrow x) \setminus \{\bot\}\,.$$

Proof. We exclude the trivial case $x = \bot$. Note that by counting we obtain

$$K_1^{c(j)} = \bigcup_{i=1}^{i=j} K_i^x \qquad K_2^{c(j)} = \bigcup_{i=j+1}^{i=n^x} K_i^x \qquad K_1^{c_0} = \bigcup_{i=1}^{i=n_0^x} K_i^x \qquad K_2^{c_0} = K_0^x\,.$$

Let $x \cdot \sigma_x \in \Lambda^n$. By Lemma 3.3 we have $\forall i \in \{1, \ldots, n^x\}$ that $\sigma[K_i] = I_i$ and as such we have for all $j \in \{1, \ldots, n_0^x - 1\}$

$$\sigma_x\left[K_1^{c(j)}\right] = \sigma_x\left[\bigcup_{i=1}^{i=j} K_i^x\right] = \bigcup_{i=1}^{i=j} \sigma_x\left[K_i^x\right] = \bigcup_{i=1}^{i=j} I_i^x = I_1^{c(j)}\,.$$

Analogously, $\sigma_x[K_2^{c(j)}] = I_2^{c(j)}$, $\sigma_x\left[K_1^{c_0}\right] = I_1^{c_0}$ and $\sigma_x\left[K_2^{c_0}\right] = I_2^{c_0}$. Thus, again by Lemma 3.3, for all $c(j) \in \mathcal{C}^x$ we have $c(j) \cdot \sigma_x \in \Lambda^n$ so x and $c(j)$ admit joint monotonization. Again, let $j \in \{1, \ldots, n_0^x - 1\}$. We have:

(i) $c(j)^\Lambda_{\bar{n}_j^x} / c(j)^\Lambda_{\bar{n}_j^x+1} = x^\Lambda_{\bar{n}_j^x} / x^\Lambda_{\bar{n}_j^x+1}$;

(ii) $c(j)^\Lambda_i / c(j)^\Lambda_{i+1} = 1 \leq x^\Lambda_i / x^\Lambda_{i+1}$ for $i \in \{1, \ldots, \bar{n}_0^x - 1\} \setminus \{\bar{n}_j^x\}$;

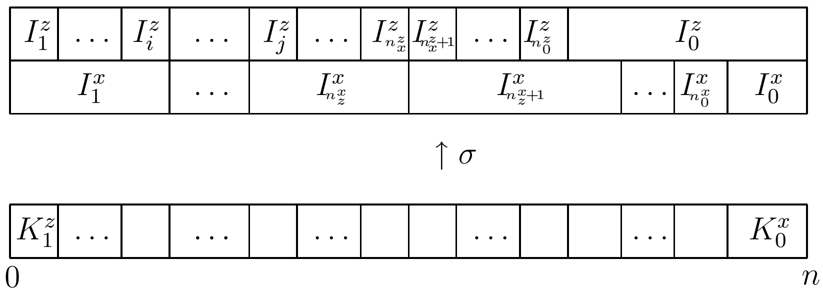
(iii) $c(j)^\Lambda_i \cdot x^\Lambda_{i+1} = c(j)^\Lambda_i \cdot 0 \leq c(j)^\Lambda_{i+1} \cdot x^\Lambda_i$ for $i \in \{\bar{n}_0^x, \ldots, n-1\}$.

Thus $c(j) \sqsubseteq x$ by Proposition 3.1. Analogously, in the case that $0 \in \text{spec}(x)$ we have $c_0 \sqsubseteq x$. Thus, x is an upper bound for $\mathcal{C}^x$.

Let $z \in \Delta^n$ be such that $\forall c \in \mathcal{C}^x : c \sqsubseteq z$ and $\sigma_x, \sigma_z \in S(n)$ such that $x \cdot \sigma_x \in \Lambda^n$ and $z \cdot \sigma_z \in \Lambda^n$. First we construct $\sigma \in S(n)$ that monotonizes both x and z. Set $n_z^x := \sup(\{0\} \cup \{j \in \{1, \ldots, n^x\} \mid K_j^x \cap K_0^z = \emptyset\})$.

Assume $n_z^x \neq 0$ (if not, skip this paragraph). We have for $i \in I_1^{c(1)} = I_1^x$

and for $k \in I_2^{c(1)} = \{1, \ldots, n\} \setminus I_1^x$ that $c(1)_i > c(1)_k \neq 0$. Since $c(1) \sqsubseteq z$ we have by Lemma 3.5 that $\forall i \in I_1^x, \forall k \in \{1, \ldots, n\} \setminus I_1^x : z_i > z_k$ so $\sigma_z[K_1^x] = I_1^x$.



By induction on $j \in \{1, \ldots, n_z^x\}$, since $c(j) \sqsubseteq z$ we have

$$\forall l \in \bigcup_{l=1}^{l=j-1} I_l^x, \forall i \in I_j^x, \forall k \in \{1, \ldots, n\} \setminus \bigcup_{l=1}^{l=j} I_l^x : z_l > z_i > z_k$$

so $\sigma_z[K_j^x] = I_j^x$. Let n_x^z be such that $\bigcup_{j=1}^{j=n_x^z} K_j^z = \bigcup_{j=1}^{j=n_z^x} K_j^x$. Setting

$$\forall j \in \{1, \ldots, n_x^z\} : \sigma[K_j^z] := \sigma_z[K_j^z] = I_j^z$$

we also obtain $\forall j \in \{1, \ldots, n_z^x\} : \sigma[K_j^x] = I_j^x$.

Next we set

- $\forall j \in \{n_x^z + 1, \ldots, n_0^z\} : \sigma[K_j^z] := \sigma_z[K_j^z] = I_j^z$
- $\sigma[K_{n_z^x+1}^x \cap K_0^z] := \sigma_x[K_{n_z^x+1}^x] \cap \sigma_z[K_0^z] = I_{n_z^x+1}^x \cap I_0^z$
- $\forall K_j^x \subseteq K_0^z : \sigma[K_j^x] := \sigma_x[K_j^x] = I_j^x$

Since $c(n_z^x + 1) \sqsubseteq z$ we obtain along the same lines as above that $\sigma[K_{n_z^x+1}^x] = I_{n_z^x+1}^x$ and $\sigma[K_0^z] = I_0^z$. Conclusively, σ monotonizes both x and z. We now verify the inequalities of Proposition 3.1 in order to prove that $x \sqsubseteq z$.

(i) $x_{\bar{n}_j^x}^\Lambda / x_{\bar{n}_j^x+1}^\Lambda = c(j)_{\bar{n}_j^x}^\Lambda / c(j)_{\bar{n}_j^x+1}^\Lambda \leq z_{\bar{n}_j^x}^\Lambda / z_{\bar{n}_j^x+1}^\Lambda$ for $j \in \{1, \ldots, n_0^x - 1\}$;

(ii) $x_i^\Lambda / x_{i+1}^\Lambda = 1 \leq z_i^\Lambda / z_{i+1}^\Lambda$ for $i \in \{1, \ldots, \bar{n}_0^z - 1\} \setminus \{\bar{n}_j^x | j \in \{1, \ldots, n_0^x - 1\}\}$;

(iii) $x_i^\Lambda \cdot z_{i+1}^\Lambda = x_i^\Lambda \cdot 0 \leq x_{i+1}^\Lambda \cdot z_i^\Lambda$ for $i \in \{\bar{n}_0^z, \ldots, n-1\}$.

Conversely, $\mathcal{C}^x = \mathrm{Max}(\mathrm{Coord}(\Delta^{\mathrm{n}}) \cap \downarrow x) \setminus \{\perp\}$ follows by Lemma 3.6 and the fact that $c(j)_1^{\mathrm{spec}} \cdot x_{j+1}^{\mathrm{spec}} = x_j^{\mathrm{spec}} \cdot c(j)_2^{\mathrm{spec}}$ maximizes those coordinates below x that are on the same axis. $\square$

One easily verifies that this decomposition is irreducible, that is, $\mathcal{C}^x$ is the infimum for inclusion of all finite $\mathcal{C} \subseteq \mathrm{Coord}(\Delta^n)$ with $x = \bigsqcup \mathcal{C}$. We proceed by characterizing the sets that arise as $\mathcal{C}^x$ for some x. It will follow that each $\mathcal{C}^x$ implicitly is an ordered list, the order being induced by the order on the irreducibles that label the axes to which each $c(j) \in \mathcal{C}^x$ belongs.

Proposition 4.4 $\{c(1), \ldots, c(m)\}$ *are the coordinates of some* $x \in \Delta^n$ *iff*

(i) $m \leq n - 1$

(ii) $x^1 \sqsupset \ldots \sqsupset x^m$ *where* $\forall j \in \{1, \ldots, m\} : c(j) \in x^j$*-axis*$\setminus\{\perp\}$

(iii) $c(j) = x^j \Rightarrow j = m$

Proof. For each $c(j)$ we obtain x^j such that $c(j) \in x^j$-axis by setting $\mathcal{I}^{x^j} = \mathcal{I}^{c(j)}$ and $0 \in \mathrm{spec}(x^j)$. (2.) is then easily verified. (1.) and (3.) are obvious. Conversely, defining $\mathcal{I}^x$ by intersecting the sets $\mathcal{I}^{c(j)}$ for all $j \in \{1, \ldots, m\}$ and imposing $c(j)_1^{\mathrm{spec}} \cdot x_{j+1}^{\mathrm{spec}} = x_j^{\mathrm{spec}} \cdot c(j)_2^{\mathrm{spec}}$ we construct $x \in \Delta^n$ which satisfies $\mathcal{C}^x = \{c(1), \ldots, c(m)\}$. □

5 Isomorphisms

Theorem 5.1 (Isomorphisms) *Order-isomorphisms of* $(\Delta^n, \sqsubseteq)$ *are in bijective correspondence with pairs consisting of*

- $\sigma \in S(n)$, ($\sim$ labeling the elements in $\mathrm{Max}(\Delta^n)$)
- $2^n - 2$ *order-isomorphisms of* $[0,1]$. ($\sim$ gauging each coordinate axis)

Proof. Let $h : \Delta^n \to \Delta^n$ be an order-isomorphism. We have $h(\bot) = \bot$. Since $h[\mathrm{Max}(\Delta^n)] = \mathrm{Max}(\Delta^n)$ this induces a permutation $\sigma \in S(n)$ via $\sigma(e_i) = h(e_i)$. This permutation σ extends to one on all $x \in \mathrm{Ir}_\bot(\Delta^\mathrm{n})$ since they are of the form $\bigwedge(\uparrow x \cap \max(\Delta^n))$ which on its turn extends by Proposition 4.2 to all coordinate axis (as a whole). For each coordinate axis set

$$f_x : x\text{-axis} \to x\text{-axis} :: y \mapsto h(y \cdot \sigma^{-1})$$

Since h is an order-isomorphism, so is f_x. The action on each $x \in \Delta^n$ is then implied by $x = \bigsqcup \mathcal{C}^x$. Conversely, let $\{f_x : x\text{-axis} \to x\text{-axis}\}$ be the $2^n - 2$ order-isomorphisms of $[0,1]$ and let $\sigma \in S(n)$. Define an order isomorphism

$$h : \Delta^n \to \Delta^n :: y \mapsto \bigsqcup \{f_x(c(j)) \cdot \sigma \mid c(j) \in \mathcal{C}^y, c(j) \in x\text{-axis}\}\,.$$

Existence of the suprema follows from Proposition 4.4, bijectivity from Theorem 4.3 and monotonicity from $\mathcal{C}^x = \mathrm{Max}(\mathrm{Coord}(\Delta^\mathrm{n}) \cap \downarrow x) \setminus \{\bot\}$. Indeed, when $x \sqsubseteq y$ then this forces each $c(j) \in \mathcal{C}^x$ to have an upper bound in $\mathcal{C}^y$ since then $\downarrow x \subseteq \downarrow y$. Applying this argument to h^{-1} yields strictness. □

Corollary 5.2 *The identity is the only order-isomorphism of* $(\Delta^n, \sqsubseteq)$ *which preserves both* $\mathrm{Max}(\Delta^n)$ *and Shannon entropy (or any other map that is strictly increasing on coordinate axis).*

Proof. By Theorem 5.1 it suffices to verify that Shannon entropy is strictly increasing on each coordinate axis. Then its preservation forces all maps $\{f_x : x\text{-axis} \to x\text{-axis} \mid x \in \mathrm{Ir}_\bot(\Delta^n)\}$ to be identities. □

By definition of D^n there exists an order-isomorphism $h : D^n \to (\Delta^n, \sqsubseteq)$. A map $\mu : D^n \to [0,1]$ is induced by commutation of

$$\begin{array}{ccc} [0,1] & \xrightarrow{\ id\ } & [0,1] \\ \uparrow \mu & & \uparrow \mu \\ D^n & \xrightarrow{\ h\ } & (\Delta^n, \sqsubseteq) \end{array}$$

Corollary 5.2 implies that if $\mu : D^n \to [0,1]$ is fixed, no other order-isomorphism $h' : D^n \to (\Delta^n, \sqsubseteq)$ satisfying $\forall i : h(e_i) = h'(e_i)$ makes the diagram commute. Thus, the pair $(D^n, \mu : D^n \to [0,1])$ defines a unique gauge $h : D^n \to \Delta^n$ which assigns to each $x \in D^n$ a unique list of numbers $h(x) \in \Delta^n$.

6 Probability from logic

We will reconstruct D^n from $A := \mathrm{P}(\{1,\ldots,n\})$ in order-theoretic manner.

Formal procedure. Let A be a bounded poset. Let Γ be a bounded chain.[6]

(i) Denote by $A^*_{0,1}$ the poset obtained by removing the top and bottom from A and by reversing the order.

(ii) Let $\mathrm{MChain}(A^*_{0,1})$ be all maximal chains $\vec{a} = \{a_1 \sqsupset \ldots \sqsupset a_{n-1}\}$ in $A^*_{0,1}$. In benefit of lucidity we assume that all these chains have length $n-1$.[7]

(iii) Denote by $\mathrm{Cl}_\top(\Gamma^{n-1})$ the set of all Γ-valued tuples $\vec{\gamma} = (\gamma_1, \ldots, \gamma_{n-1})$ subjected to the closure[8]

$$\forall i < j \in \{1,\ldots,n-1\} : \gamma_i = \top \ \Rightarrow\ \gamma_j = \top \,.$$

(iv) Set $[A^*_{0,1}, \Gamma] := \{\vec{a}\cdot\vec{\gamma} \mid \vec{a} \in \mathrm{MChain}(A^*_{0,1})\,, \vec{\gamma} \in \mathrm{Cl}_\top(\Gamma^{n-1})\}\,.$

(v) Introduce the pointwisely induced relation

$$\vec{a}\cdot\vec{\gamma} \sqsubseteq \vec{b}\cdot\vec{\varphi} \iff \vec{a} = \vec{b} \text{ and } \forall i \in \{1,\ldots,n-1\} : \gamma_i \sqsubseteq \varphi_i \,.$$

(vi) Define the indices:

$$I(\vec{\gamma}) := \{i \in \{1,\ldots,n-1\} \mid \gamma_i \notin \{\bot, \top\}\}\,;$$
$$\iota(\vec{\gamma}) := \inf\{i \in \{1,\ldots,n-1\} \mid \gamma_i = \top\}\,.$$

Let $\overline{[A^*_{0,1}, \Gamma]}$ be the set of equivalence classes in $[A^*_{0,1}, \Gamma]$ obtained for

$$\vec{a}\cdot\vec{\gamma} = \vec{b}\cdot\vec{\varphi} \iff \vec{\gamma} = \vec{\varphi} \text{ and } (i \in I(\vec{\gamma}) \cup \{\iota(\vec{\gamma})\} \Rightarrow a_i = b_i)\,.$$

(vii) Finally, $\overline{[A^*_{0,1}, \Gamma]}$ inherits the relation $\sqsubseteq$ on $[A^*_{0,1}, \Gamma]$, explicitly,

$$\vec{a}\cdot\vec{\gamma} \sqsubseteq \vec{a}\cdot\vec{\varphi} \implies [\vec{a}\cdot\vec{\gamma}] \sqsubseteq [\vec{a}\cdot\vec{\varphi}]\,.$$

Proposition 6.1 $\left(\overline{[A^*_{0,1}, \Gamma]}\,, \sqsubseteq\right)$ *is a poset with a bottom.*

Proof. We have to prove anti-symmetry and transitivity of $\sqsubseteq$ on $\overline{[A^*_{0,1}, \Gamma]}$.

Anti-symmetry. Let $\vec{a}\cdot\vec{\gamma} \sqsubseteq \vec{a}\cdot\vec{\varphi}$ and $\vec{b}\cdot\vec{\gamma} \sqsupseteq \vec{b}\cdot\vec{\varphi}$ with $[\vec{a}\cdot\vec{\gamma}] = [\vec{b}\cdot\vec{\gamma}]$ and $[\vec{a}\cdot\vec{\varphi}] = [\vec{b}\cdot\vec{\varphi}]$. We must then for all $i \in \{1,\ldots,n-1\}$ both have $\gamma_i \sqsubseteq \varphi_i$ and $\varphi_i \sqsubseteq \gamma_i$ from which $\vec{a}\cdot\vec{\gamma} = \vec{a}\cdot\vec{\varphi}$ and thus $[\vec{a}\cdot\vec{\gamma}] = [\vec{a}\cdot\vec{\varphi}]$ follows.

[6] The construction and Proposition 6.1 still hold for Γ any bounded poset.

[7] The construction and Proposition 6.1 still hold without this assumption.

[8] $\mathrm{Cl}_\top$ indeed acts as a closure operator on the pointwisely ordered complete lattice Γ^{n-1}, and thus, $\mathrm{Cl}_\top(\Gamma^{n-1})$ is itself a complete lattice. For all $n \geq 2$ monotone states constitute complete lattices since $(\Lambda^n, \sqsubseteq) \cong \mathrm{Cl}_\top([0,1]^{n-1})$. Moreover, $(\Delta^n, \sqsubseteq)$ admits arbitrary non-empty infima and any subset of Δ^n with an upper bound has a supremum w.r.t. $\sqsubseteq$ [2].

Transitivity. Let $\vec{a}\cdot\vec{\gamma}^- \sqsubseteq \vec{a}\cdot\vec{\gamma}$ and $\vec{b}\cdot\vec{\gamma} \sqsubseteq \vec{b}\cdot\vec{\gamma}^+$ with $[\vec{a}\cdot\vec{\gamma}] = [\vec{b}\cdot\vec{\gamma}]$. We have to prove that $[\vec{a}\cdot\vec{\gamma}^-] \sqsubseteq [\vec{b}\cdot\vec{\gamma}^+]$. We define $\vec{c} \in \mathrm{MChain}(A^*_{0,1})$ as follows. For $i \in I(\vec{\gamma}) : c_i := a_i = b_i$, for $i \in \{\iota(\vec{\gamma}), \ldots, n-1\} : c_i := a_i$ and in all other cases, that is $\gamma_i = \bot$, we set $c_i := b_i$. Since $\gamma_i^- \sqsubseteq \gamma_i$ implies $\gamma_i = \bot \Rightarrow \gamma_i^- = \bot$ and $\gamma_i \sqsubseteq \gamma_i^+$ implies $\gamma_i = \top \Rightarrow \gamma_i^+ = \top$ it respectively follows that $[\vec{c}\cdot\vec{\gamma}^-] = [\vec{a}\cdot\vec{\gamma}^-]$ and $[\vec{c}\cdot\vec{\gamma}^+] = [\vec{a}\cdot\vec{\gamma}^+]$. Thus, since $\vec{c}\cdot\vec{\gamma}^- \sqsubseteq \vec{c}\cdot\vec{\gamma}^+$ due to $\gamma_i^- \sqsubseteq \gamma_i \sqsubseteq \gamma_i^+$ for all $i \in \{1, \ldots, n-1\}$ we obtain $[\vec{a}\cdot\vec{\gamma}^-] \sqsubseteq [\vec{b}\cdot\vec{\gamma}^+]$.

Finally, choosing $\vec{a}$ arbitrary in $\mathrm{MChain}(A^*_{0,1})$ and setting $\vec{\gamma} = (\bot, \ldots, \bot)$, we obtain $[\vec{a}\cdot\vec{\gamma}]$ as the bottom of $\overline{[A^*_{0,1}, \Gamma]}$. □

Problem 6.2 *A categorical variant of this construction would be desirable.*

Lemma 6.3 $\mathrm{MChain}(\mathrm{P}(\{1, \ldots, n\})^*_{0,1}) \cong S(n)$ *as sets.*

Proof. The sets $\mathrm{MChain}(\mathrm{P}(\{1, \ldots, n\})^*_{0,1})$ and $S(n)$ are in bijective correspondence via $\forall i \in \{1, \ldots, n-1\} : a_i = \bigvee\{e_j \mid j \in \sigma[\{1, \ldots, i\}]\}$. □

Theorem 6.4 (Construction of classical states) *Let* $n \geq 2$.

$$\left(\overline{[\mathrm{P}(\{1, \ldots, n\})^*_{0,1}, [0,1]]}, \sqsubseteq\right) \cong (\Delta^n, \sqsubseteq)$$

Proof. Assume $\xi : [0,1] \to [1,\infty]$ to be a fixed order isomorphism. Let $\vec{a}\cdot\vec{\gamma} \in [\mathrm{P}(\{1, \ldots, n\})^*_{0,1}, [0,1]]$. We can define a set $\mathcal{C}^{\vec{a}\cdot\vec{\gamma}}$ of coordinates as follows. For each $a_i \in \vec{a}$ such that $i \in I(\vec{\gamma}) \cup \{\iota(\vec{\gamma})\}$ define $c(i) \in \mathrm{Coord}(\Delta^n)$ such that $\mathcal{I}^{c(i)} = (I^i, \{1, \ldots, n\} \setminus I^i)$ where I^i is implicitly defined by $a_i = \bigvee\{e_j \mid j \in I^i\}$, and by setting $c^i_1/c^i_2 = \xi(\gamma_i)$ whenever $\gamma_i \neq 1$ and $c^i_2 = 0$ otherwise. The set $\mathcal{C}^{\vec{a}\cdot\vec{\gamma}} = \{c^i \mid i \in I(\vec{\gamma}) \cup \{\iota(\vec{\gamma})\}\}$ satisfies the conditions in Proposition 4.4 and as such $\mathcal{C}^{\vec{a}\cdot\vec{\gamma}} = \mathcal{C}^x$ for $x = \bigsqcup \mathcal{C}^{\vec{a}\cdot\vec{\gamma}}$. For $\vec{a}\cdot\vec{\gamma}, \vec{b}\cdot\vec{\varphi} \in [\mathrm{P}(\{1, \ldots, n\})^*_{0,1}, [0,1]]$ we have $\mathcal{C}^{\vec{a}\cdot\vec{\gamma}} = \mathcal{C}^{\vec{b}\cdot\vec{\varphi}}$ iff $\vec{a}\cdot\vec{\gamma} \sim \vec{b}\cdot\vec{\varphi}$ in the above defined equivalence relation on $[\mathrm{P}(\{1, \ldots, n\})^*_{0,1}, [0,1]]$. Due to uniqueness of the decomposition in coordinates (Theorem 4.3) we obtain an injective correspondence between $\overline{[\mathrm{P}(\{1, \ldots, n\})^*_{0,1}, [0,1]]}$ and Δ^n and by Proposition 4.4 it follows that it is also surjective.

We now show that this correspondence also preserves the order. It follows from the definition of $\sqsubseteq$ that for $[\vec{a}\cdot\vec{\gamma}], [\vec{b}\cdot\vec{\varphi}] \in \overline{[\mathrm{P}(\{1, \ldots, n\})^*_{0,1}, [0,1]]}$ we have $[\vec{a}\cdot\vec{\gamma}] \sqsubseteq [\vec{b}\cdot\vec{\varphi}]$ iff there exists $\vec{c} \in \mathrm{MChain}(\mathrm{P}(\{1, \ldots, n\})^*_{0,1})$ such that $\vec{c}\cdot\vec{\gamma} \in [\vec{a}\cdot\vec{\gamma}]$ and $\vec{c}\cdot\vec{\varphi} \in [\vec{b}\cdot\vec{\varphi}]$ and such that $\vec{c}\cdot\vec{\varphi} \sqsubseteq \vec{b}\cdot\vec{\varphi}$. Moreover,

(i) Existence of $\vec{c} \in \mathrm{MChain}(\mathrm{P}(\{1, \ldots, n\})^*_{0,1})$ with $\vec{c}\cdot\vec{\gamma} \in [\vec{a}\cdot\vec{\gamma}]$ and $\vec{c}\cdot\vec{\varphi} \in [\vec{b}\cdot\vec{\varphi}]$ coincides with existence of $\sigma \in S(n)$ which monotonizes both $x = \bigsqcup \mathcal{C}^{\vec{a}\cdot\vec{\gamma}}$ and $y = \bigsqcup \mathcal{C}^{\vec{b}\cdot\vec{\varphi}}$, extending the isomorphism in Lemma 6.3.

(ii) Due to $c^i_1/c^i_2 = \xi(\gamma_i)$ for $\gamma_i \neq 1$ and $c^i_2 = 0$ for $\gamma_i = 1$, the pointwisely defined order for $\vec{\gamma}$ and $\vec{\varphi}$ induces eq.(1) for $x = \bigsqcup \mathcal{C}^{\vec{a}\cdot\vec{\gamma}}$ and $y = \bigsqcup \mathcal{C}^{\vec{b}\cdot\vec{\varphi}}$.

Explicit verification of the above completes the proof. □

Remark 6.5 It should be clear to the reader that the metric on $[0,1]$ doesn't play any role, i.e., $[0,1]$ should be read as an order-theoretic abstraction.

Remark 6.6 The alternative representation of classical states in Proposition 3.4 incarnates as an instance of an alternative formulation of this construction. It simplifies the definition of the set $\overline{[A^*_{0,1},\Gamma]}$ but one looses lucidity w.r.t. the pointwise nature of the induced order. Explicitly, let $\mathrm{Chain}(A^*_{0,1})$ be all chains in $A^*_{0,1}$, let $\Gamma_{\perp,\top} := \Gamma\setminus\{\perp,\top\}$, let $\Gamma_\perp := \Gamma\setminus\{\perp\}$, let

$$\mathrm{Cl}_\top(\Gamma_\perp^{n-1}) := \{(\gamma_1,\ldots,\gamma_k) \mid k\leq n-1; \gamma_1,\ldots,\gamma_{k-1}\in\Gamma_{\perp,\top}; \gamma_k\in\Gamma_\perp\}\ ,$$

and denoting by $|-|$ the length of a list we obtain

$$\overline{[A^*_{0,1},\Gamma]} \cong \left\{\vec{a}\cdot\vec{\gamma} \mid \vec{a}\in\mathrm{Chain}(A^*_{0,1})\,;\vec{\gamma}\in\mathrm{Cl}_\top(\Gamma_\perp^{n-1})\,;|\vec{a}|=|\vec{\gamma}|\right\}\ .$$

Theorem 6.7 (Construction of quantum states) *Let $n\geq 2$.*

$$\left(\overline{\left[(\mathbb{L}^n)^*_{0,1}\,,[0,1]\right]}\,,\sqsubseteq\right)\cong(\Omega^n,\sqsubseteq)\,.$$

We omit the proof here. We do want to expose a remarkable fact. Contrary to a Boolean algebra where orthogonality is captured by the order via

$$a\perp b\Leftrightarrow a\wedge b=0\,,$$

the lattice $\mathbb{L}^n$ admits many different orthocomplementations. [9] Mixed quantum states, due to the particular status measurements have in quantum theory, are measures $\omega:\mathbb{L}^n\to[0,1]$ which satisfy

$$a\perp b\ \Rightarrow\ \omega(a\vee b)=\omega(a)+\omega(b)\,. \tag{5}$$

By Gleason's theorem [6] these are in bijective correspondence with the density matrices (the set which we denoted in [3] by Ω^n). We can envision a constructor $\underline{\mathrm{Val}}[-]$, acting on all posets D that go equipped with an orthogonality relation $\perp$, which assigns to each $(D,\perp)$ the (monotone) measures $\omega: D\to[0,1]$ that satisfy (5), ordered along the lines of [3]. [10] We have

$$\underline{\mathrm{Val}}\,[(\mathrm{P}(\{1,\ldots,n\}),(-)^c)]\cong(\Delta^n,\sqsubseteq)\quad\&\quad\underline{\mathrm{Val}}\,[(\mathbb{L}^n,(-)')]\cong(\Omega^n,\sqsubseteq)\,,$$

with $(-)^c$ the Boolean complement and $(-)'$ any orthocomplementation on $\mathbb{L}^n$. The above *entropic geometry construction* however enables to produce an isomorphic copy of $(\Omega^n,\sqsubseteq)$ without the requirement of specification of an orthocomplementation on $\mathbb{L}^n$. Indeed, we obtain the constructor $\underline{\mathrm{EntGeom}}[-]$ which acts on any poset and satisfies

$$\underline{\mathrm{EntGeom}}\,[\mathrm{P}(\{1,\ldots,n\})]\cong(\Delta^n,\sqsubseteq)\quad\&\quad\underline{\mathrm{EntGeom}}\,[\mathbb{L}^n]\cong(\Omega^n,\sqsubseteq)\,.$$

A detailed exposition and elaboration on this matter is in preparation [4].

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[9] An orthocomplementation on a lattice L is an antitone involution $(-)': L\to L$ which satisfies $a\wedge a'=0$ and $a\vee a'=1$. It provides an orthogonality relation via $a\perp b\Leftrightarrow a\leq b'$.

[10] Besides domain-theoretic differences, a sharp distinction between $(\Delta^n,\sqsubseteq)$ and the Jones–Plotkin probabilistic powerdomain [8] is the fact that the Bayesian order is a relation on probability measures *contra* the Jones–Plotkin construction which builds a probabilistic universe on top of a pre-existing order-theoretic structure; we claim that the epistemic nature of probability has a primal mathematical structure on its own which is order-theoretic.

As a third example let D be a $(n+1)$-element chain with $n \geq 2$. Then

$$\left(\overline{\left[D^*_{0,1}\,,[0,1]\right]}\,,\sqsubseteq\right) \cong (\Lambda^n, \sqsubseteq)\,.$$

This **construction of monotone states** constitutes a fragment of both the classical and the quantum states construction; it constitutes the *atom* of the entropic geometry construction.

Interpretation. The Boolean logic $A \cong \mathrm{P}(\{1,\ldots,n\})$ can be generated by introducing disjunction on its atomic properties $\{e_1,\ldots,e_n\}$. These atomic properties provide *total specification* of the system. A disjunction $e_i \vee \ldots \vee e_j$ only provides *partial specification* of the system. It however still provides *total knowledge* on truth of the property $e_i \vee \ldots \vee e_j$. We could emphasize this by writing $(e_i \vee \ldots \vee e_j, \top)$ standing for "total knowledge on truth of $e_i \vee \ldots \vee e_j$".

Rather than only providing total knowledge on properties, we can increase expressiveness by making *partiality of knowledge* explicit: We will write $(e_i \vee \ldots \vee e_j, \gamma)$ with $\gamma \in \Gamma_{\perp,\top}$ the degree of partiality of our knowledge. This for example allows to refine $(e_i \vee e_j, \top)$ to $((e_i,\gamma),(e_i \vee e_j,\top))$ standing for "most likely the state of the system is e_i, with certainty it is either e_i or e_j, and the degree to which it is rather in e_i than in e_j is γ". The list

$$((a_1 := e_i, \gamma_1), \ldots, (a_{k-1}, \gamma_{k-1}), (a_k := a_{k-1} \vee e_j, \top))$$

with $\gamma_1,\ldots,\gamma_{k-1} \in \Gamma_{\perp,\top}$ then expresses that *most likely* the system is in pure state e_i, with certainty it is either in one of the states that span a_k, and the degree to which a_i is more likely than a_{i+1} is encoded as γ_i; any occurence of $(a_j,\perp)$ should be conceived as a *void* statement — their explicit ommitance exactly provides the alternative construction of Remark 6.6; we can extend the list with a superfluous tail, or, if it has lenght n, delete $(1,\top)$ from it, in order to obtain a maximal chain $\vec{a} = (a_1,\ldots,a_{n-1})$. Such a list provides full specification of our knowledge about the system. This explains why we can reproduce all classical states by means of this construction.

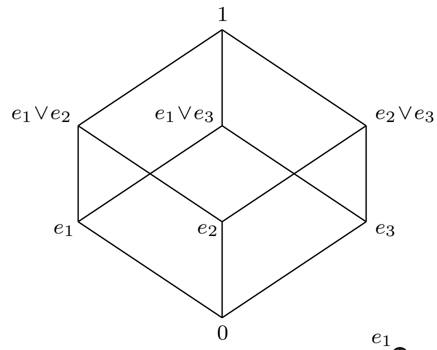
An order relation arises naturally. We compare $\vec{a}\cdot\vec{\gamma}$ and $\vec{a}\cdot\vec{\varphi}$ by pointwisely comparing $\vec{\gamma}$ and $\vec{\varphi}$; we have $\vec{a}\cdot\vec{\gamma} \sqsubseteq \vec{a}\cdot\vec{\varphi}$ iff each property in $\vec{a}$ is less likely to be true for $\vec{a}\cdot\vec{\gamma}$ than it is for $\vec{a}\cdot\vec{\varphi}$. The void statements then cause an equivalence relation on the set of all possible specifications of this kind.

Note that we do *not* have to require $i \leq j \Rightarrow \gamma_i \sqsubseteq \gamma_j$ since $\gamma_i, \gamma_j \in \Gamma_{\perp,\top}$ encode ratios of decrease of likelyness of the newly added atomic property in the next list element as compared to the remaining head of the list; on the other hand whenever $i \leq j$ then $\gamma_i = \top \Rightarrow \gamma_j = \top$ has to be fulfilled since in that case we have $a_i \Rightarrow a_j$. The bounds $\perp$ and $\top$ indeed play a distinct role in the construction, one is void and the other captures truth.

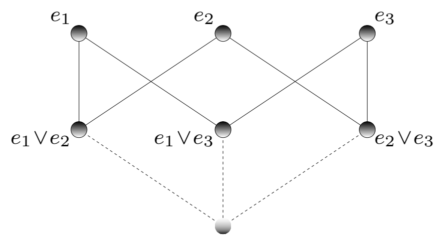
This reasoning also extends to chains in arbitrary posets when envisioned as algebras of properties of a system: Whenever we have (a_i,γ_i) with $\gamma_i \neq \top$, we add a weaker property $a_{i+1} \in A$ which is such that $a_i \Rightarrow a_{i+1}$, untill we obtain a_k such that $(a_k,\top)$ — this a_k can of course be 1. The construction of quantum states illustrates this claim.

The geometric picture. We illustrate the above for the case of $n = 3$.

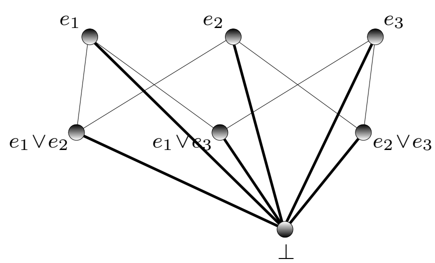
$A =$



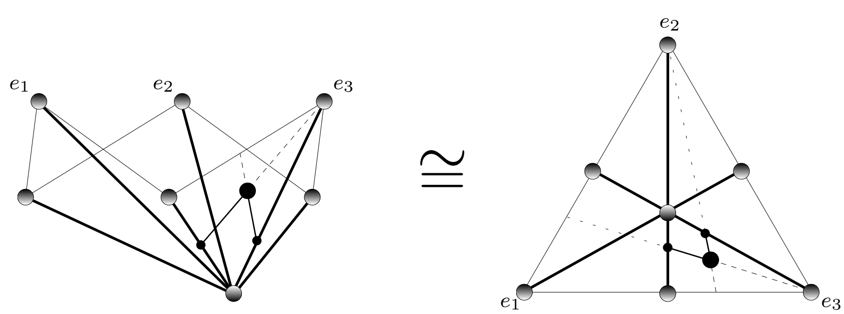
$A^*_{0,1} =$



Pairing elements of $A^*_{0,1}$ with those of Γ creates increasing "lines" which all rise from a common source, namely the "void" statement (denoted as $\perp$).



Finally, the formation of lists for all chains in MChain($A^*_{0,1}$) fills the regions enclosed by the corresponding lines resulting in a triangle.



Note how the formation of lists of pairs (= conjunctive) corresponds with the generation of points as joins of coordinates ($\sim$ reversed order).

Entropic geometry is not merely a geometry of lines but one of directed lines. The triangle or the tetrahedron are not merely convex geometric objects. For example, the center of the triangle is a special point from which directed lines emerge, which stand for the decrease of entropy. In a dynamic perspective where the lines Γ obtain the connotation of *flow*, the bounds $\perp$ and $\top$ obtain the connotation of *initiation* and *termination*. The fact that the 4-tuple $(A, \Gamma, \perp, \top)$ generates an entropic geometry by the above presented systematic formal procedure can then be interpreted as

$$\text{Entropic Geometry} = \text{Logic} + \text{Flow} + \text{Initiation} + \text{Termination}\,.$$

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